p-adic families of modular forms

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Plan

- Lecture 1
  - Background and Motivation
  - Overconvergent p-adic modular forms
  - The canonical subgroup and the U_p operator
  - Families of p-adic modular forms - Strategies of Hida and Coleman
- Lecture 2
  - Completely continuous operators - Theories of Serre and Coleman
  - Families of overconvergent p-adic modular forms
  - Overconvergent p-adic modular forms of small slope are classical
  - Generalizations to Hilbert modular forms
  - Other approaches

Modular groups

$\mathrm{SL}_2(\mathbb{Z}) = \left\{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\ ,\ a,b,c,d \in \mathbb{Z}\ ,\ \det\gamma = 1\right\}$
$\mathrm{SL}_2(\mathbb{Z}) \supset \Gamma_0(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z})\ ,\ c \equiv 0 \pmod{N}\}$
$\Gamma_0(N) \supset \Gamma_1(N) = \{\gamma \in \Gamma_0(N)\ ,\ a \equiv d \equiv 1 \pmod{N}\}$
$\mathrm{SL}_2(\mathbb{Z})$, hence $\Gamma_0(N)$ and $\Gamma_1(N)$ act on the complex upper half space $\mathbb{H}$.

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d}$$

For the next few slides, $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$.
Cusps $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$. e.g., for $\Gamma_0(p)$, p prime two cusps, $0, \infty$

Modular forms

There is an action of Γ on functions $f : \mathbb{H} \to \mathbb{C}$,

$$(\langle\gamma\rangle_k f)(\tau) = (c\tau + d)^{-k} f(\gamma(\tau)) \ , \ \tau \in \mathbb{H} \ ,$$

Consider $f : \mathbb{H} \to \mathbb{C}$, holomorphic and satisfies $\langle\gamma\rangle_k f = f$ for all $\gamma \in \Gamma$.
$\gamma_1 := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, $\gamma_1(\tau) = \tau + 1$ hence $\langle\gamma_1\rangle_k f = f(\tau + 1)$ so

$$f(\tau + 1) = f(\tau)$$

q-expansion (at the cusp ∞)

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n \ , \ q = e^{2\pi i \tau}$$

Similar expansions at other cusps.

Definition

- f as above is holomorphic at a cusp, (resp. and vanishes at a cusp) if $a_n = 0$ for $n < 0$ (resp. $n \leq 0$).
- f is a modular form (resp. a cusp form) of weight k if it is holomorphic (resp. and vanishes) at all the cusps.
- $M_k(N) =$ space of all modular forms for $\Gamma_0(N)$.
- $S_k(N) =$ space of all cusps forms on $\Gamma_0(N)$.

Similarly we will have spaces for $\Gamma_1(N)$, $M_k^1(N)$, $S_k^1(N)$.

Example: Eisenstein series

$$G_k(\tau) = C \cdot \sum_{m,n} (m\tau + n)^{-k}$$

with q-expansion

$$\frac{\zeta(1-k)}{2} + \sum_{q=1}^{\infty} \sigma_{k-1}(n) q^n$$

with

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha$$

$$E_k = G_k / a_0(G_k)$$

Hecke and diamond operators

Hecke operators

$$\sum a_n q^n | T_\ell = \sum a_{\ell n} q^n + \ell^{k-1} \sum a_n q^{\ell n} \ , \ \ell \text{ prime}$$

Diamond operators (For $M_k^1(N)$)

$$\langle d \rangle = \langle \left(\begin{smallmatrix} d & 0 \\ 0 & d^{-1} \end{smallmatrix} \right) \rangle \ , \ d \in (\mathbb{Z}/N)^\times$$

Serre's theory of p-adic modular forms

J.P. Serre, *Formes modulaires et fonctions zeta p-adiques*, in *modular forms of one variable* SLN 350)
$N = 1$, $\mathbb{Q}_p$ coefficients.

Definition

A p-adic modular form is a p-adic q-expansion which is a uniform limit of q-expansions of classical modular forms

We use the norm (valuation)

$$v(\sum a_n q^n) = \inf v(a_n)$$

Theorem

If $f_i \in S_{k_i}$, $i = 1, 2$, and $v(f_1 - f_2) \geq v(f_1) + m$, then $k_1 \equiv k_2 \pmod{(p-1)p^m}$.

Corollary

p-adic modular forms have well defined weights in $X := \varprojlim_m \mathbb{Z}/(p-1)p^m = \mathbb{Z}/(p-1) \times \mathbb{Z}_p$

Notation: M_k = space of p-adic modular forms of weight $k \in X$.

Exercise

1. Show that the Theorem also holds for p-adic modular forms.
2. Show that if $f \in M_k$, $(p-1)p^m \nmid k$, and $v(a_n) \geq 0$ for $n > 0$, then $v(a_0) \geq -m$.
3. Show that if $f_i \in M_{k_i}$ is a sequence of p-adic modular forms and if $k_i \longrightarrow k$ and all the coefficients $a_n(f_i)$ converge for $n > 0$, then so does the $a_0(f_i)$.

Example: p-adic Eisenstein series

$k_i \in \mathbb{Z} \to k \in X$ while $k_i \to \infty$ in $\mathbb{R}$.

$$a_n(G_{k_i}) = \sigma_{k_i}(n) = \sum_{d|n} d^{k_i-1} \to \sum_{p\nmid d|n} d^{k-1} := \sigma_k^*(n)$$

so

$$G_{k_i} \to G_k^* := a_0(G_k^*) + \sum_{n=1}^{\infty} \sigma_k^*(n) q^n \in M_k$$

Exercise

1. Show that the $\sigma_k^*(n)$ are well defined continuous functions of $k \in X$.
2. Deduce that the $a_0(G_k^*)$ are also continuous.

This is Serre's idea of interpolating the special values of zeta, modulo that we need to see the connection with the actual values.

Operators on p-adic modular forms

T_ℓ for $\ell \neq p$ are defined on M_k.
T_p is not, but we get 2 new operators

$$V_p(\sum a_n q^n) = \sum a_n q^{np}$$
$$U_p(\sum a_n q^n) = \sum a_{np} q^n$$

Exercise

1. Show that the T_ℓ for $\ell \neq p$ preserve M_k and find a formula for T_ℓ on q-expansions
2. Show that the V_p and U_p preserve M_k.

Modular curves

$$Y_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathbb{H}$$
$$Y_1(N)(\mathbb{C}) = \Gamma_1(N)\backslash\mathbb{H}$$

Quotients have models smooth over $\mathbb{Z}_{(N)}$.
$\tau \in \mathbb{H}$ gives an elliptic curve over $\mathbb{C}$

$$E_\tau = \mathbb{C}/\langle 1, \tau\rangle \ , \ \langle 1, \tau\rangle = \{\mathbb{Z} + \mathbb{Z}\tau\}$$

with a point of order N $x_\tau = 1/N$
and a cyclic subgroup of order N $P_\tau = \langle x_\tau\rangle$.
$Y_0(N) =$ moduli space of elliptic curves E together with a cyclic subgroup of order N.
$Y_1(N) =$ moduli space of elliptic curves E together with a point of order N.

For $N \geq 3$ exits $\mathcal{E} \xrightarrow{\pi} Y$ - universal elliptic curve.
$\boldsymbol{\omega} := \pi_* \Omega^1_{\mathcal{E}/Y}$
$X = Y \cup$ Cusps - compactification
X classifies semi-elliptic curves.
$\boldsymbol{\omega}$ has a canonical extension to X.
Modular forms of weight $k = \boldsymbol{\omega}^k(X)$
Cusp forms of weight $k + 2 = \Omega^1_{X/?} \otimes \boldsymbol{\omega}^k(X)$.
R a $\mathbb{Z}_{(N)}$ algebra, $\otimes R$ - R-valued modular forms

Abstract definition of modular forms

A rule: $f(E/S, \omega, \text{level structure}) \in S$, S an R algebra s.t.:

- It depends only on the S-isom class of the data
- It commutes with base change
- $f(E, \lambda\omega, ?) = \lambda^{-k} f(E, \omega, ?)$, $\lambda \in S^{\times}$.

q-expansion - evaluate at the Tate curve "$\mathbb{C}^{\times}/q^{\mathbb{Z}}$"
$= (y^2 = 4x^3 - \frac{E_4}{12}x + \frac{E_6}{216})/\mathbb{Z}((q)) \otimes R$ with differential dz/z.
Note: If R is a field the Tate curve can actually be interpreted as the rigid space $R((q))/q^{\mathbb{Z}}$.
q-expansion principle: If the q-expansion vanishes, so does the form.
$\mathbb{Q}_p$-valued modular forms are not p-adic modular forms

The Hasse invariant

Definition

S a $\mathbb{Z}/p$-algebra, E/S elliptic, $\phi : E \to E^{(p)}$ the Frobenius, $\omega \in \Omega^1_{E/S}$ invariant. The Hasse invariant $A = A(E,\omega) \in S$ is defined by

$$A \cdot \omega^{(p)} = \check{\phi}^* \omega$$

For $\lambda \in S^\times$ $(\lambda\omega)^{(p)} = \lambda^p \omega^{(p)}$ while $\check{\phi}^*(\lambda\omega) = \lambda\check{\phi}^*\omega$ so

$$A(E, \lambda\omega) = \lambda^{1-p} A(E,\omega) \Rightarrow A \text{ is modular of weight } p-1$$

Exercise

Show that the q-expansion of A is 1.

Remark

In particular, there are forms modulo p with same expansion and weights congruent modulo $p-1$.

Convergent and overconvergent p-adic modular forms

Well known: Zero locus of A is the supersingular locus $:= \mathcal{SS}$.
Geometric interpretation of p-adic modular forms (Katz)

Definition

Let Ord be the ordinary locus in $X_0(N) \otimes \mathbb{Z}/p$ and let $W_1 W_1(N) =]\mathrm{Ord}[$ be its tube inside the formal completion of $X_n(N)$ along the special fiber.

- The space of convergent modular forms of weight k and level N is
$$M_k(N,1) = \omega(W_1)$$
- The space of overconvergent modular forms of weight k level N and growth condition r is
$$M_k(N,r) = \omega(W_r)$$

Here, $W_r = W_r(N)$ is the strict neighborhood of W_1 defined by the condition $|E_{p-1}| \geq r$.
We will often drop the r and talk simply on overconvergent modular forms
$M_k^\dagger(N) = \omega(W^\dagger(N))$.

The canonical subgroup

$\mathrm{char}(F) = p$, E/F elliptic , $\phi : E \to E^{(p)}$ the Frobenius.
$\mathrm{Ker}\,\phi \subset E$ a finite flat group scheme of order p "contained in 0".
$\mathrm{Ker}\,\phi \subset \mathrm{Ker}[p]$.

Theorem

For $E/\mathbb{Z}_p$ with ordinary reduction there is a unique finite flat group scheme Can *of order p lifting* $\mathrm{Ker}\,\phi$.

Furthermore, this canonical subgroup "overconverges" in s precise sense if $|E_{p-1}(E)| > p^{-p/(p+1)}$.

Applications of the canonical subgroup

Injection $M_k(Np) \to M_k^\dagger(N)$ - Pullback via the map

$$W^\dagger(N) \to X_0(Np)\ ,\ (E, \text{level} N) \to (E, \text{level} N + \text{Can})$$

$V_p = \mathit{Frob}$ operator on $M_k^\dagger(N)$
Define $\Phi : W^\dagger(N) \to W^\dagger(N)$ by $\Phi(E) = (E/\text{Can})$
Φ is a lift of Frobenius.
pullback via $E \to E/\text{Can}$ gives $\Phi^*\omega \to \omega$. Taking global sections

$$\mathit{Frob} : M_k^\dagger(N) \to M_k^\dagger(N)$$

On q expansions it is just V_p.
$U_p = \frac{1}{p}$ trace of Frob.

Families of modular forms

Problem: Given an eigenform f for $\Gamma_1(Np)$, find a p-adic family of eigenforms passing through f.
This means: Can find eigenforms (of other, though congruent weights) approximating f p-adically to any given precision.
Strategy: Try to make a "bundle of p-adic modular forms"
Problem: dimensions of spaces of cusp forms vary
Hida's solution: Cut down to the slope 0 part.
Coleman's solution: Embed in the infinite dimensional space of overconvergent modular forms. Use spectral theory to cut down the varying slope parts.