MOD p GALOIS REPRESENTATIONS, (φ, Γ)-MODULES AND REPRESENTATIONS OF $\mathrm{B}_2(\mathbf{Q}_p)$

by

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Abstract. — These are the notes for my lectures at the Minerva school on p-adic methods in arithmetic algebraic geometry in Jerusalem.

Contents

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Introduction

The goal of these lectures is to give a construction of the (φ, Γ)-modules associated to irreducible mod p Galois representations, and then to explain how (φ, Γ)-modules can be used to construct representations of $\mathrm{B}_2(\mathbf{Q}_p)$, the Borel subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$. Some of these representations extend to $\mathrm{GL}_2(\mathbf{Q}_p)$ which then gives rise to the mod p version of the p-adic Langlands correspondence.

1. Mod p representations of the local Galois group

In this chapter, we consider E-linear representations of $\mathcal{G}_{\mathbf{Q}_p} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ where E is a finite extension of $\mathbf{F}_p$. We first recall the construction of Serre's fundamental characters after [**Ser72**], and then prove that every irreducible representation of $\mathcal{G}_{\mathbf{Q}_p}$ is a twist of an induction of these characters.

1.1. Serre's fundamental characters. — Using ramification theory, we can break down the extension $\overline{\mathbf{Q}}_p/\mathbf{Q}_p$ into $\overline{\mathbf{Q}}_p \supset \mathbf{Q}_p^{\mathrm{tame}} \supset \mathbf{Q}_p^{\mathrm{unr}} \supset \mathbf{Q}_p$. The group $\mathcal{I}_{\mathbf{Q}_p} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p^{\mathrm{unr}})$ is the inertia subgroup of $\mathcal{G}_{\mathbf{Q}_p}$ and $\mathcal{G}_{\mathbf{Q}_p}/\mathcal{I}_{\mathbf{Q}_p} = \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{unr}}/\mathbf{Q}_p)$ is topologically cyclic generated by the frobenius σ, the group $I_t = \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{tame}}/\mathbf{Q}_p^{\mathrm{unr}})$ is isomorphic to $\prod_{\ell \neq p} \mathbf{Z}_\ell(1)$ (where the "(1)" means that if $\tau \in \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{tame}}/\mathbf{Q}_p^{\mathrm{unr}})$ and $\tilde{\sigma} \in \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{tame}}/\mathbf{Q}_p)$ maps to σ then $\tilde{\sigma}\tau\tilde{\sigma}^{-1} = \tau^p$) and $I_p = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p^{\mathrm{tame}})$ is a pro-p-group, the pro-p-Sylow subgroup of $\mathcal{I}_{\mathbf{Q}_p}$. We say that a character $\eta : \mathcal{G}_{\mathbf{Q}_p} \to E^\times$ is unramified if it is trivial on $\mathcal{I}_{\mathbf{Q}_p}$. It is then determined by $\eta(\sigma)$ and if $\lambda \in E^\times$, we denote by μ_λ the unique unramified character $\mu_\lambda : \sigma \mapsto \lambda^{-1}$.

For each $n \geqslant 1$, choose $\pi_n \in \overline{\mathbf{Q}}_p$ such that $\pi_n^{p^n-1} = -p$. We then have $\mathbf{Q}_p^{\mathrm{tame}} = \cup_{n \geqslant 1} \mathbf{Q}_p^{\mathrm{unr}}(\pi_n)$. Furthermore, if $g \in \mathcal{G}_{\mathbf{Q}_p}$ then $g(\pi_n)/\pi_n$ is a (p^n-1)-th root of unity, so that there exists a character $\omega_n : \mathcal{G}_{\mathbf{Q}_{p^n}} \to \mathbf{F}_{p^n}^\times$ such that $g(\pi_n) = [\omega_n(g)] \cdot \pi_n$ and this character does not depend on the choice of π_n. In exercise 1, you will show that ω_1 is the mod p cyclotomic character which we also denote by ω. Note that ω_n is trivial on I_p and it is therefore a character of I_t. If $d \mid n$ then $\omega_n^{(p^n-1)/(p^d-1)} = \omega_d$. If n is given and $h \in \mathbf{Z}$, we say that h is primitive if it is not divisible by $(p^n-1)/(p^d-1)$ for any $d < n$. Every mod p character of I_t is then of the form ω_n^h for some well-defined n and some primitive h; such a character is said to be of level n.

1.2. Irreducible mod p representations. — If $g \in \mathcal{G}_{\mathbf{Q}_p}$, then $\tau \mapsto \omega_n^h(g\tau g^{-1})$ is a character of $\mathcal{G}_{\mathbf{Q}_{p^n}}$ which is of the form $\omega_n^{p^j h}$ and if h is primitive, then the characters $\omega_n^h, \omega_n^{ph}, \ldots, \omega_n^{p^{n-1}h}$ are pairwise distinct so that by Mackey's criterion, the representation $\mathrm{ind}_{\mathcal{G}_{\mathbf{Q}_{p^n}}}^{\mathcal{G}_{\mathbf{Q}_p}} \omega_n^h$ is irreducible. The determinant of this representation is ω^h on $\mathcal{G}_{\mathbf{Q}_{p^n}}$ and hence we can twist it by an unramified character so that its determinant is ω^h on $\mathcal{G}_{\mathbf{Q}_p}$. The resulting representation is denoted by $\mathrm{ind}(\omega_n^h)$. We will see later on that $\mathrm{ind}(\omega_n^h)$ is defined on $\mathbf{F}_p$ (see also exercise 2 for a proof).

Theorem 1.2.1. — *If W is an absolutely irreducible E-linear representation of $\mathcal{G}_{\mathbf{Q}_p}$ of dimension n, then there exists $\lambda \in \overline{\mathbf{F}}_p^\times$ such that $\lambda^n \in E^\times$ and $1 \leqslant h \leqslant p^n - 2$ primitive such that $W = \mathrm{ind}(\omega_n^h) \otimes \mu_\lambda$.*

Proof. — Since a p-group acting on an E-vector space always has nontrivial fixed points, $W^{I_p} \neq \{0\}$ and since W is irreducible, $W = W^{I_p}$. The restriction of W to $\mathcal{I}_{\mathbf{Q}_p}$ is therefore a representation of I_t and I_t is an abelian group of pro-order prime to p, so that $W|_{I_t}$ is a direct sum of characters after possibly extending scalars. If χ is one of those characters, then it is of level m for some m and it then extends to $\mathcal{G}_{\mathbf{Q}_{p^m}}$ so that $W|_{\mathcal{G}_{\mathbf{Q}_{p^m}}}$ contains χ and by Frobenius reciprocity W contains $\mathrm{ind}_{\mathcal{G}_{\mathbf{Q}_{p^m}}}^{\mathcal{G}_{\mathbf{Q}_p}} \chi$. Since W is irreducible, we have $m = n$ and $\chi = \omega_n^h$ times some unramified character. One concludes by exercise 3. Note that $\det(W) = \omega^h \mu_{\lambda^n}$ so that $\lambda^n \in E^\times$. □

Using exercise 2, we then see that $\mathrm{ind}(\omega_n^h)$ is the only n-dimensional representation of $\mathcal{G}_{\mathbf{Q}_p}$ whose determinant is ω^h and whose restriction to $\mathcal{I}_{\mathbf{Q}_p}$ is $\omega_n^h \oplus \omega_n^{ph} \oplus \cdots \oplus \omega_n^{p^{n-1}h}$.

1.3. Exercises

1. Using Wilson's theorem, show that $(\zeta_p - 1)^{p-1}/p = -1 \bmod (\zeta_p - 1)$ and hence that ω_1 is the mod p cyclotomic character.
2. Choose $\lambda \in \overline{\mathbf{F}}_p^\times$ such that $\lambda^n \in \mathbf{F}_p^\times$, and let $W_\lambda = \{\alpha \in \overline{\mathbf{F}}_p \text{ such that } \alpha^{p^n} = \lambda^{-n}\alpha\}$
   (a) Prove that $\mathrm{Gal}(\mathbf{Q}_p^{\mathrm{unr}}(\pi_n)/\mathbf{Q}_p) \simeq \mathbf{F}_{p^n}^\times \rtimes \hat{\mathbf{Z}}$;
   (b) Prove that the map $\mathbf{F}_{p^n}^\times \rtimes \hat{\mathbf{Z}} \to \mathrm{End}_{\mathbf{F}_p}(W_\lambda)$ given by $(x, 0) \mapsto m_x^h$ (where m_x is the multiplication by x map) and by $(1,1) \mapsto (\alpha \mapsto \alpha^p)$ gives an n-dimensional $\mathbf{F}_p$-linear representation of $\mathcal{G}_{\mathbf{Q}_p}$ which is isomorphic to $(\mathrm{ind}_{\mathcal{G}_{\mathbf{Q}_{p^n}}}^{\mathcal{G}_{\mathbf{Q}_p}} \omega_n^h) \otimes \mu_\lambda$ after extending scalars and whose determinant is $\omega^h \mu_{-1}^{n-1} \mu_\lambda^n$;
   (c) Prove that $\mathrm{ind}(\omega_n^h)$ is defined over $\mathbf{F}_p$.
3. Let F be a finite separable extension of a field E and suppose that W_1 and W_2 are two absolutely irreducible E-linear representations of a group G such that $F \otimes_E W_1 \simeq F \otimes_E W_2$. Prove that $W_1 \simeq W_2$.
4. Let W be a continuous $\overline{\mathbf{F}}_p$-linear representation of a profinite group G. Prove that there exists a finite extension E of $\mathbf{F}_p$ and an E-linear representation W_E of G such that $W = \overline{\mathbf{F}}_p \otimes_E W_E$.

2. Period rings in characteristic p

In this chapter, we define and prove some properties of the ring $\widetilde{\mathbf{E}}$ used in [**FW79**] and in [**Win83**] by Fontaine and Wintenberger in the theory of the field of norms.

2.1. The field $\widetilde{\mathbf{E}}$ and its subrings. — The first ring which we define is $\widetilde{\mathbf{E}}^+ = \{(x_0, x_1, \dots)$ where $x_i \in \mathcal{O}_{\mathbf{C}_p}/p$ and $x_{i+1}^p = x_i\}$, addition and multiplication being componentwise. We denote by θ_i the map $x \mapsto x_i$ so that $\theta_i(x) = \theta_{i+1}(x)^p$. If $x \in \widetilde{\mathbf{E}}^+ \setminus \{0\}$ and if $\hat{x}_i$ denotes any lift of x_i in $\mathcal{O}_{\mathbf{C}_p}$ then $p^i \mathrm{val}_p(\hat{x}_i)$ is eventually constant and we define $\mathrm{val}_{\mathbf{E}}(x)$ to be that limit. The ring $\widetilde{\mathbf{E}}^+$ is a perfect ring of characteristic p, and the function $\mathrm{val}_{\mathbf{E}} : \widetilde{\mathbf{E}}^+ \setminus \{0\}$ is a valuation for which it is complete (see exercise 1).

If we denote by $\varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbf{C}_p}$ the set of sequences $(x^{(0)}, x^{(1)}, \dots)$ such that $(x^{(i+1)})^p = x^{(i)}$ then there is a natural multiplicative map $\varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbf{C}_p} \to \widetilde{\mathbf{E}}^+$ which is a bijection (see exercise 3 below). We use either description of $\widetilde{\mathbf{E}}^+$ in the sequel.

Let $\varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \widetilde{\mathbf{E}}^+$ and $X = \varepsilon - 1$ so that $\mathrm{val}_{\mathbf{E}}(X) = p/(p-1)$. We define $\widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}^+[1/X]$ so that $\widetilde{\mathbf{E}}$ is a perfect field of characteristic p, which contains $\mathbf{F}_p(\!(X)\!)$. We denote by φ the map $x \mapsto x^p$ on $\widetilde{\mathbf{E}}$. Note that $\overline{\mathbf{F}}_p$ maps into $\widetilde{\mathbf{E}}^+$ by $\alpha \mapsto ([\alpha^{1/p^n}])_{n \geqslant 0}$ and $\widetilde{\mathbf{E}}^{\varphi^f = 1} = \mathbf{F}_{p^f}$ (see exercise 5).

Theorem 2.1.1. — *The field $\widetilde{\mathbf{E}}$ is algebraically closed.*

Proof. — Our proof follows Coleman's course notes [**Col97**]. Let $P(T) \in \widetilde{\mathbf{E}}^+[T]$ be a monic polynomial of degree d and let $P_n(T)$ denote $\theta_n(P) \in (\mathcal{O}_{\mathbf{C}_p}/p)[T]$. For each n, choose some lift $\tilde{P}_n(T) \in \mathcal{O}_{\mathbf{C}_p}[T]$ of P_n and denote its roots in $\mathcal{O}_{\mathbf{C}_p}$ by $\alpha_{1,n}, \dots, \alpha_{d,n}$. If $k \geqslant 1$, define $S_{n,k} \subset \mathcal{O}_{\mathbf{C}_p}/p$ to be $\{\alpha_{i,n+k}^{p^k} \bmod p\}$. We claim that if $p^k \geqslant d$ then $S_{n,k}$ does not depend on the choice of $\tilde{P}_n(T)$. Indeed if $\alpha \in \mathcal{O}_{\mathbf{C}_p}$ is such that $\tilde{P}_{n+k}(\alpha) \in p\mathcal{O}_{\mathbf{C}_p}$ then $\prod_{i=1}^d (\alpha - \alpha_{i,n+k}) \in p\mathcal{O}_{\mathbf{C}_p}$ so that there is some i such that $\mathrm{val}_p(\alpha - \alpha_{i,n+k}) \geqslant 1/d$ and therefore $\mathrm{val}_p(\alpha^{p^k} - \alpha_{i,n+k}^{p^k}) \geqslant \min(p^k/d, 1) = 1$. The claim follows from applying this to a root $\alpha = \alpha'_{i,n+k}$ of another lift $\tilde{P}'_{n+k}(T)$ of $P_{n+k}(T)$ so that $S'_{n,k} \subset S_{n,k}$ and we have equality by symmetry. Likewise, we have $S_{n,k+1} \subset S_{n,k}$ and since $S_{n+1,k}^p = S_{n,k+1}$ this tells us that the sets $\{S_{n,k}\}_{n \geqslant 0}$ form a compatible system of nonempty sets of cardinal $\leqslant d$ so that their projective limit is nonempty. Since $P_n(\alpha_{i,n+k}^{p^k}) = P_{n+k}(\alpha_{i,n+k})^{p^k}$ in $\mathcal{O}_{\mathbf{C}_p}/p$, an element of that projective limit is a root of $P(T)$ which proves that $\widetilde{\mathbf{E}}$ is algebraically closed. □

Because $\widetilde{\mathbf{E}}$ is algebraically closed, it contains $\mathbf{F}_p(\!(X)\!)^{\mathrm{alg}}$ and by a theorem of Ax (see [**Ax70**] or exercise 6 below), the field $\mathbf{E} = \mathbf{F}_p(\!(X)\!)^{\mathrm{sep}}$ is a dense subfield of $\mathbf{F}_p(\!(X)\!)^{\mathrm{alg}}$.

Theorem 2.1.2. — *The field $\widetilde{\mathbf{E}}$ contains $\mathbf{E}$ as a dense subfield.*

The usual way of proving this theorem uses the theory of the field of norms of Fontaine and Wintenberger (see [**FW79**] and [**Win83**]) and we only give a sketch of the main ideas. By Ax's theorem, it is enough to prove that $\mathbf{F}_p(\!(X)\!)^{\mathrm{alg}}$ is dense in $\widetilde{\mathbf{E}}$. If K is a finite extension of $\mathbf{Q}_p$ let $K_\infty = K(\mu_{p^\infty})$ and let I be the set of elements of valuation

$\geqslant 1/p$. We set $\widetilde{\mathbf{E}}_K^+ = \{x = (x_0, x_1, \ldots)$ where $x_n \in \mathcal{O}_{K_\infty}/I$ and $x_{n+1}^p = x_n\}$. Using some ramification theory, we can prove that the map $x \mapsto x^p$ from $\mathcal{O}_{K_\infty}/I \to \mathcal{O}_{K_\infty}/I$ is surjective. We have $\mathcal{O}_{\mathbf{F}_p(\!(X)\!)^{\mathrm{perf}}} \subset \widetilde{\mathbf{E}}_{\mathbf{Q}_p}^+$ and the map $\theta_n : \mathcal{O}_{\mathbf{F}_p(\!(X)\!)^{\mathrm{perf}}} \to \mathcal{O}_{(\mathbf{Q}_p)_\infty}/I$ is easily seen to be surjective so that $\mathcal{O}_{\mathbf{F}_p(\!(X)\!)^{\mathrm{perf}}}$ is dense in $\widetilde{\mathbf{E}}_{\mathbf{Q}_p}^+$ by an analogue of exercise 2. Furthermore, if L/K is a finite extension then so is $\widetilde{\mathbf{E}}_L/\widetilde{\mathbf{E}}_K$ where $\widetilde{\mathbf{E}}_K = \widetilde{\mathbf{E}}_K^+[1/X]$. Finally, $\cup_{K/\mathbf{Q}_p}\widetilde{\mathbf{E}}_K$ is dense in $\widetilde{\mathbf{E}}$ by exercise 2 and therefore $\mathbf{F}_p(\!(X)\!)^{\mathrm{alg}}$ is dense in $\widetilde{\mathbf{E}}$.

2.2. The action of $\mathcal{G}_{\mathbf{Q}_p}$ on $\widetilde{\mathbf{E}}$. — The group $\mathcal{G}_{\mathbf{Q}_p}$ acts on $\mathcal{O}_{\mathbf{C}_p}$ by continuity and this action extends to a continuous action on $\widetilde{\mathbf{E}}$. For example, $g(f(X)) = f((1+X)^{\chi_{\mathrm{cycl}}(g)} - 1)$ so that if $\mathcal{H}_{\mathbf{Q}_p} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p(\mu_{p^\infty})) = \ker \chi_{\mathrm{cycl}}$ then the action of $\mathcal{H}_{\mathbf{Q}_p}$ on $\mathbf{F}_p(\!(X)\!)$ is trivial; furthermore, if $h \in \mathcal{H}_{\mathbf{Q}_p}$ and if K is a separable extension of $\mathbf{F}_p(\!(X)\!)$ then $h(K)$ is another separable extension of $\mathbf{F}_p(\!(X)\!)$ so that we get a map $\mathcal{H}_{\mathbf{Q}_p} \to \mathrm{Gal}(\mathbf{E}/\mathbf{F}_p(\!(X)\!))$.

Theorem 2.2.1. — *The map $\mathcal{H}_{\mathbf{Q}_p} \to \mathrm{Gal}(\mathbf{E}/\mathbf{F}_p(\!(X)\!))$ is an isomorphism.*

Proof. — We first check that the map is injective. If $h \in \mathcal{H}_{\mathbf{Q}_p}$ acts trivially on $\mathbf{E}$ then by continuity and theorem 2.1.2, h acts trivially on $\widetilde{\mathbf{E}}$ and therefore on $\mathcal{O}_{\mathbf{C}_p}$ so that $h = 1$. We now check that the map is surjective. If α is an automorphism of $\mathbf{E}$ then it extends by continuity to an automorphism of $\widetilde{\mathbf{E}}$ trivial on $\mathbf{F}_p(\!(X)\!)^{\mathrm{perf}}$ and the proof of theorem 2.1.2 shows that the finite extensions of $\mathbf{F}_p(\!(X)\!)^{\mathrm{perf}}$ are of the form $\widetilde{\mathbf{E}}_K$ where $K/\mathbf{Q}_p$ is finite and $\widetilde{\mathbf{E}}_K$ only depends on K_∞ so that α is the image of some element $h \in \mathcal{H}_{\mathbf{Q}_p}$. □

We now give a "characteristic p" construction of ω_n. In order to do so, we work with $\widetilde{\mathbf{E}}^+$ viewed as $\varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbf{C}_p}$. Let $Y \in \mathbf{E}$ be an element such that $Y^{(p^n-1)/(p-1)} = X$. If $g \in \mathcal{G}_{\mathbf{Q}_p}$, then $f_g(X) = [\omega(g)]X/g(X)$ depends only on the image of g in $\Gamma = \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$ and since mod p we have $f_g(X) \in 1 + X\mathbf{F}_p[\![X]\!]$, the formula $f_g^s(X)$ makes sense if $s \in \mathbf{Z}_p$.

Proposition 2.2.2. — *If $g \in \mathcal{G}_{\mathbf{Q}_{p^n}}$ then $g(Y) = Y\omega_n^p(g) f_g^{-\frac{p-1}{p^n-1}}(X)$.*

Proof. — Recall that $X \in \widetilde{\mathbf{E}}^+ = \varprojlim \mathcal{O}_{\mathbf{C}_p}$ is equal to $\varepsilon - 1$ where $\varepsilon = (\zeta_{p^j})_{j\geqslant 0}$ and where $(\zeta_{p^j})_{j\geqslant 0}$ is a compatible sequence. If $j \geqslant 1$, pick $\pi_{n,j} \in \mathcal{O}_{\mathbf{C}_p}$ such that

$$\pi_{n,j}^{\frac{p^n-1}{p-1}} = \zeta_{p^j} - 1.$$

If $g \in \mathcal{G}_{\mathbf{Q}_{p^n}}$, then $g(\zeta_{p^j} - 1) = [\omega(g)](\zeta_{p^j} - 1) f_g^{-1}(\zeta_{p^j} - 1)$ and so there exists $\omega_{n,j}(g) \in \mathbf{F}_{p^n}^\times$ such that

$$\frac{g(\pi_{n,j})}{\pi_{n,j}} = [\omega_{n,j}(g)] f_g^{-\frac{p-1}{p^n-1}}(\zeta_{p^j} - 1),$$

where $[\cdot]$ is the Teichmüller lift from $\mathbf{F}_{p^n}^\times$ to $\mathbf{Q}_{p^n}^\times$. The map $g \mapsto \omega_{n,j}(g)$ is a character of $\mathcal{G}_{\mathbf{Q}_{p^n}}$ which does not depend on the choice of $\pi_{n,j}$. In addition, we have

$$\begin{cases}(\zeta_{p^{j+1}}-1)^p = (\zeta_{p^j}-1)\cdot(1+\mathrm{O}(p^{1/p})) & \text{if } j \geqslant 1,\\ (\zeta_p-1)^{p-1} = -p\cdot(1+\mathrm{O}(p^{1/p})) \end{cases}$$

so that $\omega_{n,j+1}^p = \omega_{n,j}$ if $j \geqslant 1$ and $\omega_{n,1} = \omega_n$. This also tells us that we may choose the $\pi_{n,j}$ so that $\pi_{n,j+1}^p/\pi_{n,j} \in 1 + p^{1/p}\mathcal{O}_{\mathbf{C}_p}$. If we write $Y = (y^{(i)}) \in \varprojlim \mathcal{O}_{\mathbf{C}_p}$, then we have $y^{(i)} = \lim_{j\to+\infty} \pi_{n,i+j}^{p^j}$ since the $\pi_{n,j}$ are compatible in the sense that $\pi_{n,j+1}^p/\pi_{n,j} \in 1 + p^{1/p}\mathcal{O}_{\mathbf{C}_p}$ so that if $g \in \mathcal{G}_{\mathbf{Q}_{p^n}}$, then

$$\frac{g(y^{(i)})}{y^{(i)}} = [\omega_{n,i}(g)] \cdot \lim_{j\to\infty} (f_g^{-\frac{p-1}{p^n-1}}(\zeta_{p^{i+j}}-1))^{p^j},$$

and therefore we have $g(Y) = Y\omega_n^p(g) f_g^{-\frac{p-1}{p^n-1}}(X)$ in $\widetilde{\mathbf{E}}$. □

2.3. Exercises

1. Prove that $\widetilde{\mathbf{E}}^+$ is a perfect ring of characteristic p and that the function $\mathrm{val}_{\mathbf{E}} : \widetilde{\mathbf{E}}^+ \setminus \{0\}$ is a valuation for which it is complete.
2. What is the topology defined on $\mathcal{O}_{\mathbf{C}_p}/p$ by val_p? Show that a subset P of $\widetilde{\mathbf{E}}^+$ is dense if and only if $\theta_n : P \to \mathcal{O}_{\mathbf{C}_p}/p$ is surjective for all $n \geqslant 0$.
3. If $x = (x_0, x_1, \ldots) \in \widetilde{\mathbf{E}}^+$, let $\hat{x}_i \in \mathcal{O}_{\mathbf{C}_p}$ denote an arbitrary lift of x_i.
   (a) Prove that the sequence $\{\hat{x}_{i+j}^{p^j}\}_{j\geqslant 0}$ converges in $\mathcal{O}_{\mathbf{C}_p}$ and that if we call $x^{(j)}$ its limit, then $(x^{(0)}, x^{(1)}, \ldots)$ is an element of $\varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbf{C}_p}$ which maps to $x \in \widetilde{\mathbf{E}}^+$.
   (b) If $x, y \in \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbf{C}_p}$, what is the formula for $(x+y)^{(j)}$?
4. How does one divide elements in $\widetilde{\mathbf{E}}$?
5. Prove that $\widetilde{\mathbf{E}}^{\varphi^f=1} = \mathbf{F}_{p^f}$.
6. Let k be a perfect field of characteristic p and let $K = k(\!(X)\!)$. The valuation val_X defined on K extends uniquely to K^{sep} and to K^{alg}.
   (a) Let $\alpha \in K^{\mathrm{sep}}$ be such that $\mathrm{val}_X(\alpha) \geqslant 0$ and let $n \geqslant 0$. If $\beta \in K^{\mathrm{alg}}$ is the p-th root of α in K^{alg} and if β_n is a root of the polynomial $T^p - X^nT - \alpha = 0$, what can you say about $\mathrm{val}_X(\beta - \beta_n)$?
   (b) Using the above, prove the following result which is a particular case of a theorem of Ax (see [**Ax70**]) : the field K^{sep} is dense in K^{alg}.
7. This is a project rather than an exercise. Prove proposition 2.2.2 by using the interpretation of ω_n in class field theory and Laubie's result from [**Lau88**] on the compatibility of class field theories in the field of norms.

3. Fontaine's (φ, Γ)-modules

In this chapter, we use the properties of $\mathbf{E}$ seen above to state and prove Fontaine's equivalence of categories from [**Fon90**] between Galois representations and (φ, Γ)-modules. After that, we compute some explicit examples.

3.1. Fontaine's equivalence of categories. — If W is an $\mathbf{F}_p$-linear representation of $\mathcal{G}_{\mathbf{Q}_p}$ then the $\mathbf{F}_p(\!(X)\!)$-vector space $\mathrm{D}(W) = (\mathbf{E} \otimes_{\mathbf{F}_p} W)^{\mathcal{H}_{\mathbf{Q}_p}}$ inherits the frobenius φ of $\mathbf{E}$ and the residual action of $\Gamma = \mathcal{G}_{\mathbf{Q}_p}/\mathcal{H}_{\mathbf{Q}_p}$.

Definition 3.1.1. — A (φ,Γ)-*module* over $\mathbf{F}_p(\!(X)\!)$ is a finite dimensional $\mathbf{F}_p(\!(X)\!)$-vector space endowed with a semilinear frobenius φ such that $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathbf{F}_p(\!(X)\!))$ and a continuous and semilinear action of Γ commuting with φ.

For example, $\mathbf{F}_p(\!(X)\!)$ itself is a (φ,Γ)-module. We'll see that $\mathrm{D}(W)$ is a (φ,Γ)-module over $\mathbf{F}_p(\!(X)\!)$. If E is a finite extension of $\mathbf{F}_p$, we endow it with the trivial φ and the trivial action of Γ so that we may talk about (φ,Γ)-modules over $E(\!(X)\!) = E \otimes_{\mathbf{F}_p} \mathbf{F}_p(\!(X)\!)$ and we then have the following result which is proved in §1.2 of [**Fon90**].

Theorem 3.1.2. — *The functor $W \mapsto \mathrm{D}(W)$ gives an equivalence of categories between the category of E-representations of $\mathcal{G}_{\mathbf{Q}_p}$ and the category of (φ,Γ)-modules over $E(\!(X)\!)$.*

Proof. — Given the isomorphism $\mathcal{H}_{\mathbf{Q}_p} \simeq \mathrm{Gal}(\mathbf{E}/\mathbf{F}_p(\!(X)\!))$ from theorem 2.2.1, Hilbert's theorem 90 tells us that $\mathrm{H}^1_{\mathrm{discrete}}(\mathcal{H}_{\mathbf{Q}_p}, \mathrm{GL}_d(\mathbf{E})) = \{1\}$ (where we only consider cocycles which are trivial on an open subgroup of $\mathcal{H}_{\mathbf{Q}_p}$) so that if W is an $\mathbf{F}_p$-linear representation of $\mathcal{H}_{\mathbf{Q}_p}$ then $\mathbf{E} \otimes_{\mathbf{F}_p} W \simeq \mathbf{E}^{\dim(W)}$ as representations of $\mathcal{H}_{\mathbf{Q}_p}$. The $\mathbf{F}_p(\!(X)\!)$-vector space $\mathrm{D}(W) = (\mathbf{E} \otimes_{\mathbf{F}_p} W)^{\mathcal{H}_{\mathbf{Q}_p}}$ is therefore of dimension $\dim(W)$ (in particular, it is a (φ,Γ)-module) and we can recover W from $\mathrm{D}(W)$ by the formula $W = (\mathbf{E} \otimes_{\mathbf{F}_p(\!(X)\!)} \mathrm{D}(W))^{\varphi=1}$.

If D is a (φ,Γ)-module of dimension d over $\mathbf{F}_p(\!(X)\!)$, then let $W(\mathrm{D}) = (\mathbf{E} \otimes_{\mathbf{F}_p(\!(X)\!)} \mathrm{D})^{\varphi=1}$. If we choose a basis $\{d_i\}_{1 \leqslant i \leqslant d}$ of D and if $\mathrm{Mat}(\varphi)^{-1} = (q_{ij})_{1 \leqslant i,j \leqslant d}$ in that basis, then a small computation shows that $\sum_{i=1}^d \lambda_i \otimes d_i \in (\mathbf{E} \otimes_{\mathbf{F}_p(\!(X)\!)} \mathrm{D})^{\varphi=1}$ if and only if $\lambda_k^p = \sum_{i=1}^d q_{ki}\lambda_i$ for all $1 \leqslant k \leqslant d$. The algebra $A = \mathbf{E}[X_1,\ldots,X_d]/(X_k^p - \sum_i q_{ki}X_i)_{1 \leqslant j \leqslant d}$ is an étale $\mathbf{E}$-algebra of rank p^d and since $\mathbf{E}$ is separably closed, it is isomorphic to $\mathbf{E}^{p^d}$, which gives us p^d elements in W so that W is an $\mathbf{F}_p$-vector space of dimension d (see exercise 2 for a more concrete proof of this).

It is then easy to check that the functors $W \mapsto \mathrm{D}(W)$ and $\mathrm{D} \mapsto W(\mathrm{D})$ are inverse of each other. Finally, if $E \neq \mathbf{F}_p$ then one can consider an E-representation as an $\mathbf{F}_p$-representation with an E-linear structure and likewise for (φ,Γ)-modules, so that the equivalence carries over. □

3.2. Examples of (φ,Γ)-modules. — Every character of $\mathcal{G}_{\mathbf{Q}_p}$ is of the form $\omega^h\mu_\lambda$ for some $0 \leqslant h \leqslant p-2$ and some $\lambda \in E^\times$. By exercise 4, the (φ,Γ)-module $\mathrm{D}(\omega^h\mu_\lambda)$ is a one dimensional $E(\!(X)\!)$-vector space with a basis e for which $\varphi(e) = \lambda \cdot e$ and $\gamma(e) = \omega^h(\gamma) \cdot e$.

The structure of the (φ,Γ)-modules associated to irreducible representations of $\mathcal{G}_{\mathbf{Q}_p}$ can be deduced from the theorem below.

Theorem 3.2.1. — *The* (φ,Γ)*-module* $\mathrm{D}(\mathrm{ind}(\omega_n^h))$ *is defined on* $\mathbf{F}_p(\!(X)\!)$ *and admits a basis* $e_0,\dots,e_{n-1}$ *in which* $\gamma(e_j)=f_\gamma(X)^{hp^j(p-1)/(p^n-1)}e_j$ *if* $\gamma\in\Gamma$ *and* $\varphi(e_j)=e_{j+1}$ *for* $0\leqslant j\leqslant n-2$ *and* $\varphi(e_{n-1})=(-1)^{n-1}X^{-h(p-1)}e_0$.

Proof. — Let W be the $\mathbf{F}_p$-representation of $\mathcal{G}_{\mathbf{Q}_p}$ associated to the (φ,Γ)-module described in the statement of the theorem. If $f=X^h\cdot e_0\wedge\ldots\wedge e_{n-1}$, then $\varphi(f)=f$ and $\gamma(f)=\omega(\gamma)^hf$ so that the determinant of W is indeed ω^h and therefore we only need to show that the restriction of $\mathbf{F}_{p^n}\otimes_{\mathbf{F}_p}W$ to $\mathcal{I}_{\mathbf{Q}_p}$ is $\omega_n^h\oplus\omega_n^{ph}\oplus\cdots\oplus\omega_n^{p^{n-1}h}$. To clarify things, let us write $\mathbf{F}_{p^n}^\natural$ for $\mathbf{F}_{p^n}$ when it occurs as a coefficient field, so that the frobenius φ is trivial on $\mathbf{F}_{p^n}^\natural$.

If we write $\mathbf{F}_{p^n}^\natural\otimes_{\mathbf{F}_p}\mathbf{E}$ as $\prod_{k=0}^{n-1}\mathbf{E}$ via the map $x\otimes y\mapsto(\sigma^k(x)y)$ where σ is the absolute frobenius on $\mathbf{F}_{p^n}^\natural$, then given $(x_0,\dots,x_{n-1})\in\prod_{k=0}^{n-1}\mathbf{E}$, we have

$$\varphi((x_0,\dots,x_{n-1}))=(\varphi(x_{n-1}),\varphi(x_0),\dots,\varphi(x_{n-2}))$$
$$g((x_0,\dots,x_{n-1}))=(g(x_0),\dots,g(x_{n-1})),$$

if $g\in\mathcal{G}_{\mathbf{Q}_{p^n}}$ (but not if $g\in\mathcal{G}_{\mathbf{Q}_p}$). Choose $\alpha\in\overline{\mathbf{F}}_p$ such that $\alpha^{p^n-1}=(-1)^{n-1}$ and define

$$v_0=(\alpha Y^h,0,\dots,0)\cdot e_0+(0,\alpha^pY^{ph},\dots,0)\cdot e_1+\cdots(0,\dots,0,\alpha^{p^{n-1}}Y^{p^{n-1}h})\cdot e_{n-1}$$
$$v_1=(0,\alpha Y^h,\dots,0)\cdot e_0+(0,0,\alpha^pY^{ph},\dots,0)\cdot e_1+\cdots(\alpha^{p^{n-1}}Y^{p^{n-1}h},0,\dots,0)\cdot e_{n-1}$$
$$\vdots$$
$$v_{n-1}=(0,\dots,0,\alpha Y^h)\cdot e_0+(\alpha^pY^{ph},0,\dots,0)\cdot e_1+\cdots(0,\dots,0,\alpha^{p^{n-1}}Y^{p^{n-1}h},0)\cdot e_{n-1}.$$

We can check that the vectors $v_0,\dots,v_{n-1}$ give a basis of $\mathbf{F}_{p^n}^\natural\otimes_{\mathbf{F}_p}(\mathbf{E}\otimes_{\mathbf{F}_p(\!(X)\!)}\mathrm{D}(W))$. The formulas for the action of φ imply that $\varphi(v_j)=v_j$ so that $v_j\in\mathbf{F}_{p^n}^\natural\otimes_{\mathbf{F}_p}W$. The formulas for the action of Γ and lemma 2.2.2 imply that $g(v_j)=\omega_n^{hp^{1-j}}v_j$ if $g\in\mathcal{I}_{\mathbf{Q}_p}$ which finishes the proof. □

3.3. Exercises

1. Check that $W\mapsto\mathrm{D}(W)$ commutes with direct sums and tensor products.
2. Let K be some separably closed field of characteristic $p>0$, let $Q=(q_{ij})\in\mathrm{GL}_d(K)$ and let $A=K[X_1,\dots,X_d]/(X_k^p-\sum_i q_{ki}X_i)$.
   (a) Using the fact that Q is invertible, show that the image of the map $x\mapsto x^p$ generates A over K and hence that if $\{b_i\}_{i\in I}$ is a basis of A, then so is $\{b_i^p\}_{i\in I}$.
   (b) Prove that A is a product of field extensions of K.
   (c) Prove that $A\simeq K^{p^d}$.
3. Let D be a (φ,Γ)-module on $\mathbf{F}_p(\!(X)\!)$. Show that there exists a basis of D in which $\mathrm{Mat}(\varphi)\in\mathrm{M}_d(\mathbf{F}_p[\![X]\!])$ and $\mathrm{Mat}(\gamma)\in\mathrm{GL}_d(\mathbf{F}_p[\![X]\!])$ if $\gamma\in\Gamma$.
4. The goal of this exercise is to prove that if $W=E\cdot v$ where $g(v)=(\omega^s\mu_\lambda)(g)v$, then the (φ,Γ)-module $\mathrm{D}(W)$ has a basis e for which $\varphi(e)=\lambda\cdot e$ and $\gamma(e)=\omega^s(\gamma)\cdot e$.
   (a) Why is this true if $\lambda=1$?
   (b) Assume now that $E=\mathbf{F}_p$, and let $\pi\in\overline{\mathbf{F}}_p$ be such that $\pi^{p-1}=\lambda$. If $g\in\mathcal{H}_{\mathbf{Q}_p}$, prove that $g(\pi\otimes v)=\pi\otimes v$ and finish the proof of the statement in this case.

(c) If $E = \mathbf{F}_{p^n}$, use the isomorphism $\mathbf{F}_{p^n} \otimes_{\mathbf{F}_p} \mathbf{E} = \prod_{k=0}^{n-1} \mathbf{E}$ of the proof of theorem 3.2.1 to prove the assertion as in (b).

5. What can you say about the representation attached to D if there exists a basis of D in which $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathbf{F}_p[\![X]\!])$?

4. Construction of representations of $\mathrm{B}_2(\mathbf{Q}_p)$

In this chapter, we study the operator ψ on (φ,Γ)-modules and give Colmez' construction (see [**Col07**]) of a representation of $\mathrm{B}_2(\mathbf{Q}_p)$ starting from the data of a (φ,Γ)-module.

4.1. The operator ψ and Colmez' functor. — The field $E(\!(X)\!)$ is a vector space over $E(\!(X^p)\!)$ which admits $1, X, \ldots, X^{p-1}$ as a basis and hence also $1, (1+X), \ldots, (1+X)^{p-1}$. If $\alpha(X) \in E(\!(X)\!)$, we can therefore write $\alpha(X) = \sum_{j=0}^{p-1}(1+X)^j \alpha_j(X^p)$ in a unique way and we define $\psi(\alpha(X)) = \alpha_0(X)$. If D is a (φ,Γ)-module and if $y \in \mathrm{D}$, then using the above facts and exercise 2 we can write $y = \sum_{j=0}^{p-1}(1+X)^j \varphi(y_j)$ and we set $\psi(y) = y_0$. The operator ψ thus defined commutes with the action of Γ (see exercise 3) and satisfies $\psi(\alpha(X)\varphi(y)) = \psi(\alpha(X))y$ and $\psi(\alpha(X^p)y) = \alpha(X)\psi(y)$.

Lemma 4.1.1. — *Every (φ,Γ)-module* D *admits a $E[\![X]\!]$-lattice stable under ψ.*

Proof. — Let M denote a lattice of D and let $\varphi^*(M)$ denote the $E[\![X]\!]$-module generated by $\varphi(M)$, which is still a lattice of D so that there exists $h \geqslant 0$ with $X^{h(p-1)}M \subset \varphi^*(M)$. This implies that $X^{-h}M \subset \varphi^*(X^{-h}M)$ and hence that $\mathrm{N} = X^{-h}M$ is the sought after lattice. □

If D is a (φ,Γ)-module, let N denote some $E[\![X]\!]$-lattice stable under ψ of D and let $(\varprojlim_\psi \mathrm{D})^b$ denote the set of sequences $y = (y_n)_{n\geqslant 0}$ such that $\psi(y_{n+1}) = y_n$ for all $n \geqslant 0$ and such that $\{y_n\}_{n\geqslant 0}$ is bounded for the X-adic topology, meaning that there exists some j (depending on y) such that $y_n \in X^{-j}\mathrm{N}$ for all $n \geqslant 0$. The set $(\varprojlim_\psi \mathrm{D})^b$ is stable under ψ which is bijective on it, it is stable under the action of Γ (if N is a ψ-stable lattice, then so is $\gamma(\mathrm{N})$ if $\gamma \in \Gamma$) and if $\alpha(X) \in E[\![X]\!]$, then we define $\alpha \cdot y \in (\varprojlim_\psi \mathrm{D})^b$ by $(\alpha y)_n = \varphi^n(\alpha) y_n$.

Let $\mathrm{D}^\sharp$ denote the set of y_0 for all $y \in (\varprojlim_\psi \mathrm{D})^b$; this is a $E[\![X]\!]$-module stable under ψ and Γ and on which ψ is surjective. Furthermore, if N is as above and if $j \geqslant 0$, then $\psi(X^{-j}\mathrm{N}) \subset X^{-\lceil j/p \rceil}\mathrm{N}$ so that $\mathrm{D}^\sharp \subset X^{-1}\mathrm{N}$. The natural map $\varprojlim_\psi \mathrm{D}^\sharp \to (\varprojlim_\psi \mathrm{D})^b$ is therefore an isomorphism, and $\mathrm{D}^\sharp$ is the largest bounded sub-$E[\![X]\!]$-module of D which is stable under ψ and Γ and on which ψ is surjective.

If $\mathrm{D} = \mathrm{D}(\omega^s \mu_\lambda) = E(\!(X)\!) \cdot e$ with $\varphi(e) = \lambda e$ and $\gamma(e) = \omega^s(\gamma)e$, then by exercise 4 we see that $\mathrm{D}^\sharp = X^{-1}E[\![X]\!] \cdot e$ in this case.

By exercise 5, $\mathrm{D}^\sharp$ is always a full lattice in D. We give below an explicit construction of $\mathrm{D}^\sharp(W)$ for $W = \mathrm{ind}(\omega_n^h) \otimes \omega^s \mu_\lambda$. Theorem 3.2.1 above implies that the (φ,Γ)-module $\mathrm{D}(W)$ is defined on $E(\!(X)\!)$ and admits a basis $e_0,\dots,e_{n-1}$ in which $\gamma(e_j) = \omega^s(\gamma) f_\gamma(X)^{hp^j(p-1)/(p^n-1)} e_j$ if $\gamma \in \Gamma$ and $\varphi(e_j) = \lambda e_{j+1}$ for $0 \leqslant j \leqslant n-2$ and $\varphi(e_{n-1}) = (-1)^{n-1} \lambda X^{-h(p-1)} e_0$. Since $\omega_n^{(p^n-1)/(p-1)} = \omega$, we can always modify h (and s accordingly) in order to have $1 \leqslant h \leqslant (p^n-1)/(p-1) - 1$ so that $h(p-1) \leqslant p^n - 2$. Let $i_{n-1} \dots i_1 i_0$ be the expansion of $h(p-1)$ in base p and let $h_k = i_{n-k} + p i_{n-k+1} + \dots + p^{k-1} i_{n-1}$ so that $h_0 = 0$ and $h_n = h(p-1)$.

Lemma 4.1.2. — *If $f_j = X^{h_j} e_j$ and if $\alpha(X) \in E(\!(X)\!)$, then we have*

$$\psi(\alpha(X) f_j) = \begin{cases} \lambda^{-1} \psi(\alpha(X) X^{i_{n-j}}) f_{j-1} & \text{if } j \geqslant 1, \\ \lambda^{-1} (-1)^{n-1} \psi(\alpha(X) X^{i_0}) f_{n-1} & \text{if } j = 0. \end{cases}$$

Proof. — If $j \geqslant 1$, then we can write $\alpha(X) f_j = \lambda^{-1} \alpha(X) X^{h_j} \varphi(e_{j-1})$ and since $h_j = p h_{j-1} + i_{n-j}$, we have

$$\begin{aligned} \psi(\alpha(X) f_j) &= \lambda^{-1} X^{h_{j-1}} \psi(\alpha(X) X^{i_{n-j}}) e_{j-1} \\ &= \lambda^{-1} \psi(\alpha(X) X^{i_{n-j}}) f_{j-1}. \end{aligned}$$

If $j = 0$, then $\alpha(X) f_0 = \alpha(X) e_0 = \alpha(X) (-1)^{n-1} \lambda^{-1} X^{h(p-1)} \varphi(e_{n-1})$ so that

$$\begin{aligned} \psi(\alpha(X) f_0) &= \lambda^{-1} (-1)^{n-1} X^{h_{n-1}} \psi(\alpha(X) X^{i_0}) e_{n-1} \\ &= \lambda^{-1} (-1)^{n-1} \psi(\alpha(X) X^{i_0}) f_{n-1} \end{aligned}$$

which finishes the proof. □

Corollary 4.1.3. — *The $E[\![X]\!]$-module $\mathrm{D}^\sharp(W) = \oplus_{j=0}^{n-1} E[\![X]\!] \cdot f_j$ is stable under ψ and the action of Γ and the map $\psi : \mathrm{D}^\sharp(W) \to \mathrm{D}^\sharp(W)$ is surjective.*

Proof. — Lemma 4.1.2 implies that $\mathrm{D}^\sharp(W)$ is stable under ψ. Furthermore, the formula $\psi(X^{pm+r}) = (-1)^r X^m$ for $0 \leqslant r \leqslant p-1$ implies that the map $\alpha(X) \mapsto \psi(\alpha(X) X^{i_{n-j}})$ is surjective for $j \geqslant 1$, as well as the map $\alpha(X) \mapsto \psi(\alpha(X) X^{i_0})$, which implies that $\psi : \mathrm{D}^\sharp(W) \to \mathrm{D}^\sharp(W)$ is surjective. □

A result of Colmez then implies that $\mathrm{D}^\sharp(W)$ is uniquely determined by the corollary 4.1.3 above.

4.2. Representations of $\mathrm{B}_2(\mathbf{Q}_p)$. — Every element of $\mathrm{B}_2(\mathbf{Q}_p)$ can be written as a product of matrices of the form $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$ with $x \in \mathbf{Q}_p^\times$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p^j \end{smallmatrix}\right)$ with $j \in \mathbf{Z}$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)$ with $a \in \mathbf{Z}_p^\times$ and $\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)$ with $z \in \mathbf{Z}_p$. If W is a representation of $\mathcal{G}_{\mathbf{Q}_p}$, if $\mathrm{D}(W)$ is the associated (φ,Γ)-module and if $\chi : \mathbf{Q}_p^\times \to E^\times$ is a smooth character, we endow $\varprojlim_\psi \mathrm{D}^\sharp(W)$ with an action of $\mathrm{B}_2(\mathbf{Q}_p)$ as follows :

$$\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \cdot y\right)_i = \chi^{-1}(x) y_i;$$
$$\left(\begin{pmatrix} 1 & 0 \\ 0 & p^j \end{pmatrix} \cdot y\right)_i = y_{i-j} = \psi^j(y_i);$$
$$\left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \cdot y\right)_i = \gamma_{a^{-1}}(y_i), \text{ where } \gamma_{a^{-1}} \in \Gamma \text{ is such that } \chi_{\mathrm{cycl}}(\gamma_{a^{-1}}) = a^{-1} \in \mathbf{Z}_p^\times;$$
$$\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot y\right)_i = (1+X)^{p^i z} y_i.$$

We then define $\Omega(W) = (\varprojlim_\psi \mathrm{D}^\sharp(W))^*$ so that $\Omega(W)$ is a smooth representation (see exercise 6 for a proof of this) of $\mathrm{B}_2(\mathbf{Q}_p)$ whose central character is χ^{-1}.

Assume now that $W = \omega^s \mu_\lambda$. We start by observing that contrary to the situation in exercise 9, the representation $\Omega(W)$ is not irreducible. Indeed, if $\mathrm{D}(W) = E(\!(X)\!) \cdot e$ as above so that $\mathrm{D}^\sharp(W) = X^{-1} E[\![X]\!] \cdot e$, then (see exercise 4) $\mathrm{D}^\natural(W) = E[\![X]\!] \cdot e$ is a submodule of $\mathrm{D}^\sharp(X)$ which is stable under the action of ψ (which is surjective on it) and Γ, so that $\varprojlim_\psi \mathrm{D}^\natural(W)$ is a subrepresentation of $\varprojlim_\psi \mathrm{D}^\sharp(W)$. If $y = (y_0, y_1, \ldots) \in \varprojlim_\psi \mathrm{D}^\sharp(W)$ with $y_0 = f_0(X)e$, we denote by $\mathrm{res}(y) \in E$ the residue of f_0. Let ω and μ_λ denote the characters of $\mathbf{Q}_p^\times$ associated to ω and μ_λ by class field theory so that if $x = x_0 p^v \in \mathbf{Q}_p^\times$ then $\omega(x) = \overline{x}_0$ and $\mu_\lambda(x) = \lambda^v$. If χ_1 and χ_2 are two characters of $\mathbf{Q}_p^\times$, let $\chi_1 \otimes \chi_2 : \mathrm{B}_2(\mathbf{Q}_p) \to E^\times$ be the character $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mapsto \chi_1(a)\chi_2(d)$.

Lemma 4.2.1. — *The map* res *induces an exact sequence of representations of* $\mathrm{B}_2(\mathbf{Q}_p)$

$$0 \to \chi\omega^{1-s}\mu_\lambda^{-1} \otimes \omega^{s-1}\mu_\lambda \to \Omega(W) \to (\varprojlim_\psi \mathrm{D}^\natural(W))^* \to 0$$

Proof. — By duality, we need to show that we have an exact sequence

$$0 \to \varprojlim_\psi \mathrm{D}^\natural(W) \to \varprojlim_\psi \mathrm{D}^\sharp(W) \to \chi^{-1}\omega^{s-1}\mu_\lambda \otimes \omega^{1-s}\mu_{\lambda^{-1}} \to 0.$$

If we write $y = (\lambda^n X^{-1} e)_{n \geqslant 0}$, then $y \in \varprojlim_\psi \mathrm{D}^\sharp(W)$ and $\mathrm{res}(y) = 1$ so that we only need compute $\mathrm{res}(b \cdot y)$ for $b \in \mathrm{B}_2(\mathbf{Q}_p)$ and we find

$$\begin{aligned}
&\mathrm{res}\left(\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) \cdot y\right) = \chi^{-1}(x),\\
&\mathrm{res}\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \cdot y\right) = \mathrm{res}((\lambda^{n-1} X^{-1} e)_{n \geqslant 0}) = \lambda^{-1},\\
&\mathrm{res}\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) \cdot y\right) = \mathrm{res}((\omega^{s-1}(\gamma_a^{-1})(\lambda^n + \mathrm{O}(X)) X^{-1} e)_{n \geqslant 0}) = a^{1-s},\\
&\mathrm{res}\left(\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) \cdot y\right) = 1,
\end{aligned}$$

which finishes the proof. □

In the next chapter, we will determine $\Omega(W)$ when W is one dimensional.

4.3. Exercises

1. Prove that if $0 \leqslant r \leqslant p-1$, then $\psi(X^{pm+r}) = (-1)^r X^m$.
2. Prove that a (φ, Γ)-module D has a basis whose elements belong to $\varphi(\mathrm{D})$.
3. Prove that ψ commutes with the action of Γ.
4. Prove the following properties of the action of ψ on $E(\!(X)\!)$.
   (a) $\psi(X^{-j-1}E[\![X]\!]) \subset X^{-j}E[\![X]\!]$ if $j \geqslant 1$,
   (b) $\psi(X^{j+1}E[\![X]\!]) \supset X^{j}E[\![X]\!]$ if $j \geqslant 0$,
   (c) $\psi(1/X) = 1/X$,
   and use them to deduce that the only $E[\![X]\!]$-submodules of rank 1 of $E(\!(X)\!)$ which are stable under ψ and on which ψ is surjective are $E[\![X]\!]$ and $X^{-1}E[\![X]\!]$.
5. Let D be a (φ, Γ)-module and let M be a lattice of D stable under φ (see exercise 3 of §3). Show that $M \subset \mathrm{D}^\sharp$.
6. (see §2.3 of [**Ber08**]) Let E be a finite field, G a topological group, X a profinite E-linear representation of G and X^* its continuous dual.
   (a) Prove that X^* is a smooth representation of G.
   (b) Prove that if X is topologically irreducible, then X^* is irreducible.
7. Prove that if W is one dimensional, then $(\varprojlim_\psi \mathrm{D}^\natural(W))^*$ is an irreducible representation of $\mathrm{B}_2(\mathbf{Q}_p)$.
8. Let $W = \mathrm{ind}(\omega_n^h) \otimes \omega^s \mu_\lambda$. Denote by θ_0 the linear form on $\mathrm{D}^\sharp(W)$ given by

$$\theta_0 : \alpha_0(X) f_0 + \cdots + \alpha_{n-1}(X) f_{n-1} \mapsto \alpha_0(0).$$

   If $y = (y_0, y_1, \ldots)$, then we define $\theta \in \Omega(W)$ to be the linear form $\theta : y \mapsto \theta_0(y_0)$.
   (a) If $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times \cap \mathrm{B}_2(\mathbf{Q}_p)$, then compute $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \cdot \theta$.
   (b) Show that θ is an eigenvector of

$$\sum_{j=0}^{p^n-1} \binom{j}{h(p-1)} \begin{pmatrix} p^n & -j \\ 0 & 1 \end{pmatrix}.$$

9. Let $W = \mathrm{ind}(\omega_n^h) \otimes \omega^s \mu_\lambda$ with $n \geqslant 2$ and let M be a sub-$E[\![X]\!]$-module of $\mathrm{D}(W)$ stable under ψ and Γ and on which ψ is surjective.
   (a) Prove that $\mathrm{M} = \mathrm{D}^\sharp(W)$.
   (b) Prove that $\Omega(W)$ is an irreducible representation of $\mathrm{B}_2(\mathbf{Q}_p)$.

5. Parabolic inductions

In this chapter, we determine $\Omega(W)$ when $W = \omega^s \mu_\lambda$ following the computations in [**Ber05**].

5.1. Locally constant functions. — Let $\mathrm{LC}(\mathbf{Z}_p, E)$ denote the space of locally constant functions. If $n \geqslant 0$, then the function $\binom{X}{n} = \frac{X(X-1)\cdots(X-n+1)}{n!}$ is a continuous function $\mathbf{Z}_p \to \mathbf{Z}_p$ and hence its reduction modulo p belongs to $\mathrm{LC}(\mathbf{Z}_p, E)$. Furthermore, a classical result tells us that $\{\binom{X}{n}\}_{n \geqslant 0}$ is a basis of $\mathrm{LC}(\mathbf{Z}_p, E)$ (see exercise 1). If $\nu : \mathrm{LC}(\mathbf{Z}_p, E) \to E$ is a linear form, we also say that ν is a measure on $\mathbf{Z}_p$ and we write $\int_{\mathbf{Z}_p} f d\nu$ instead of $\nu(f)$. The Amice transform (cf. [**Ami78**]) $\mathcal{A}(\nu)(X) \in E[\![X]\!]$ of a measure ν is defined by

$$\mathcal{A}(\nu)(X)=\sum_{n=0}^{\infty}\nu\left(z\mapsto\binom{z}{n}\right)X^n=\nu(z\mapsto(1+X)^z)=\int_{\mathbf{Z}_p}(1+X)^z d\nu(z).$$

Since the $\{\binom{X}{n}\}_{n\geqslant 0}$ form a basis of $\mathrm{LC}(\mathbf{Z}_p,E)$, the map $\mathcal{A}:\mathrm{LC}(\mathbf{Z}_p,E)^*\to E[\![X]\!]$ is an isomorphism.

If ν is a measure and if U is an open subset of $\mathbf{Z}_p$ then $\mathrm{Res}_U(\nu)$ is the measure defined by $\int_{\mathbf{Z}_p} f d\mathrm{Res}_U(\nu)=\int_{\mathbf{Z}_p}\mathbf{1}_U(z)f(z)d\nu(z)=\int_U f(z)d\nu(z)$. In exercise 4 you will prove that if ν is a measure and if $\psi(\nu)$ is the measure whose Amice transform is $\psi(\mathcal{A}(\nu))$, then

$$\int_{\mathbf{Z}_p} f(z)d\psi(\nu)=\int_{p\mathbf{Z}_p} f(p^{-1}z)d\nu.$$

Let $\mathrm{LC}_0(\mathbf{Q}_p,E)$ denote the space of locally constant compactly supported E-valued functions on $\mathbf{Q}_p$. If $y=(y_0,y_1,\ldots)$ is a sequence of elements of $E[\![X]\!]$ such that $\psi(y_{i+1})=y_i$ and if ν_i is the measure such that $\mathcal{A}(\nu_i)=y_i$ so that we can define a measure ν on $\mathbf{Q}_p$ attached to y by the formula :

$$\int_{\mathbf{Q}_p} f d\nu_y=\int_{\mathbf{Z}_p} f(p^{-i}z)d\nu_i$$

if $f\in\mathrm{LC}_0(\mathbf{Q}_p,E)$ and if $i\gg 0$ is large enough so that f has support in $p^{-i}\mathbf{Z}_p$. This gives us an isomorphism $\mathrm{LC}_0(\mathbf{Q}_p,E)^*=\varprojlim_\psi E[\![X]\!]$.

5.2. Parabolic inductions. — If χ_1 and χ_2 are two smooth characters of $\mathbf{Q}_p^\times$, then the parabolic induction $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi_1\otimes\chi_2)$ is by definition the set of locally constant functions $\sigma:\mathrm{GL}_2(\mathbf{Q}_p)\to E$ such that $\sigma(bg)=(\chi_1\otimes\chi_2)(b)\sigma(g)$ if $b\in\mathrm{B}_2(\mathbf{Q}_p)$ and $g\in\mathrm{GL}_2(\mathbf{Q}_p)$. The space $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi_1\otimes\chi_2)$ is naturally a representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ whose action is given by $(g\cdot\sigma)(h)=\sigma(hg)$ and we consider $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi_1\otimes\chi_2)$ as a representation of $\mathrm{B}_2(\mathbf{Q}_p)$.

If $\sigma\in\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi_1\otimes\chi_2)$, then the map $\sigma\mapsto\sigma(\mathrm{Id})$ gives us a map from $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi_1\otimes\chi_2)$ to $\chi_1\otimes\chi_2$ whose kernel we denote by $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi_1\otimes\chi_2)_0$ so that we have an exact sequence of representations of $\mathrm{B}_2(\mathbf{Q}_p)$

$$0\to\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi_1\otimes\chi_2)_0\to\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi_1\otimes\chi_2)\to\chi_1\otimes\chi_2\to 0.$$

If $\sigma\in\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi_1\otimes\chi_2)_0$, then we associate to σ a function $f_\sigma\in\mathrm{LC}_0(\mathbf{Q}_p,E)$ by the formula $f_\sigma(x)=\sigma\left(\left(\begin{smallmatrix}0&1\\-1&x\end{smallmatrix}\right)\right)$. The Bruhat decomposition $\mathrm{GL}_2(\mathbf{Q}_p)=\mathrm{B}_2(\mathbf{Q}_p)\sqcup\mathrm{B}_2(\mathbf{Q}_p)\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\mathrm{B}_2(\mathbf{Q}_p)$ shows that the map $\sigma\mapsto f_\sigma$ gives a bijection between $\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi_1\otimes\chi_2)_0$ and $\mathrm{LC}_0(\mathbf{Q}_p,E)$ and that since $(b\cdot\sigma)(g)=\sigma(gb)$ if $\sigma\in\mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\chi_1\otimes\chi_2)$ and $b\in B$ and $g\in G$, we have

$$(\clubsuit)\quad f_{\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)\cdot\sigma}(x)=\sigma\left(\left(\begin{smallmatrix}0&1\\-1&x\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)\right)=\sigma\left(\left(\begin{smallmatrix}d&0\\0&a\end{smallmatrix}\right)\begin{pmatrix}0&1\\-1&\frac{dx-b}{a}\end{pmatrix}\right)=\chi_1(d)\chi_2(a)f_\sigma\left(\frac{dx-b}{a}\right).$$

The main result connecting $\Omega(W)$ and parabolic inductions is the theorem below.

Theorem 5.2.1. — *If* $W = \omega^s \mu_\lambda$ *then* $(\varprojlim_\psi \mathrm{D}^\natural(W))^* \simeq \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\omega^s\mu_\lambda \otimes \chi\omega^{-s}\mu_{\lambda^{-1}})_0$.

Proof. — We will define a map $\varprojlim_\psi \mathrm{D}^\natural(W) \to \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\omega^s\mu_\lambda \otimes \chi\omega^{-s}\mu_{\lambda^{-1}})_0^*$, and show that it is a $\mathrm{B}_2(\mathbf{Q}_p)$-equivariant isomorphism.

If $y \in \varprojlim_\psi \mathrm{D}^\natural(W)$, then $y = (y_i)_{i \geqslant 0}$ and we have $y_i = f_i(X)e$ with $f_i(X) \in E[\![X]\!]$. We see that $\psi(\lambda^{-i} f_i) = \lambda^{-(i-1)} f_{i-1}$ and we define a measure $\nu_{y,i}$ on $\mathbf{Z}_p$ by requiring that $\mathcal{A}(\nu_{y,i}) = \lambda^{-i} f_i$ so that we can define a measure ν_y on $\mathbf{Q}_p$ by asking that if $f \in \mathrm{LC}_0(\mathbf{Q}_p, E)$ has support in $p^{-i}\mathbf{Z}_p$, then

$$\int_{\mathbf{Q}_p} f d\nu_y = \int_{\mathbf{Z}_p} f(p^{-i}z) d\nu_{y,i}$$

as in the previous paragraph. The map $y \mapsto \nu_y$ is then a bijection which we can use to identify $\Omega(W)$ with the dual of the parabolic induction via the map $\sigma \mapsto f_\sigma$. In order to finish the proof of the theorem, we need to show that this map is $\mathrm{B}_2(\mathbf{Q}_p)$-equivariant. The action of $\mathrm{B}_2(\mathbf{Q}_p)$ on measures is given by its action on their Amice transforms and we find

$$\begin{aligned}
\int_{\mathbf{Q}_p} f(z) d\nu_{\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)\cdot y} &= \chi^{-1}(x) \int_{\mathbf{Q}_p} f(z) d\nu_y \\
\int_{\mathbf{Q}_p} f(z) d\nu_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)\cdot y} &= \lambda^{-1} \int_{\mathbf{Q}_p} f(p^{-1}z) d\nu_y \\
\int_{\mathbf{Q}_p} f(z) d\nu_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)\cdot y} &= d^{-s} \int_{\mathbf{Q}_p} f(d^{-1}z) d\nu_y \\
\int_{\mathbf{Q}_p} f(z) d\nu_{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\cdot y} &= \int_{\mathbf{Q}_p} f(z+b) d\nu_y.
\end{aligned}$$

By comparing these formulas with (♣), we do find that

$$\int_{\mathbf{Q}_p} f_{g\cdot\sigma}(z) d\nu_{g\cdot y} = \int_{\mathbf{Q}_p} f_\sigma(z) d\nu_y$$

if $g \in \mathrm{B}_2(\mathbf{Q}_p)$, $y \in \varprojlim_\psi \mathrm{D}^\natural(W)$ and $\sigma \in \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\omega^s\mu_\lambda \otimes \chi\omega^{-s}\mu_{\lambda^{-1}})_0$. □

Corollary 5.2.2. — *We have an exact sequence*

$$0 \to \chi\omega^{1-s}\mu_{\lambda^{-1}} \otimes \omega^{s-1}\mu_\lambda \to \Omega(W) \to \mathrm{Ind}_{\mathrm{B}}^{\mathrm{G}}(\omega^s\mu_\lambda \otimes \chi\omega^{-s}\mu_{\lambda^{-1}})_0 \to 0.$$

5.3. Exercises

1. If $f \in \mathrm{LC}(\mathbf{Z}_p, E)$ is some locally constant function, we set $(\Delta f)(x) = f(x+1) - f(x)$.
   (a) Show that there exists some $m \geqslant 1$ such that $f(x)$ only depends on $x \bmod p^m$ and that $\Delta^{p^m}(f)$ is then equal to 0;
   (b) What is $\Delta(\binom{X}{n})$? Compute the kernel of Δ^k;
   (c) Prove that $\{\binom{X}{n}\}_{n \geqslant 0}$ is a basis of the E-vector space $\mathrm{LC}(\mathbf{Z}_p, E)$.
2. Using exercise 1, prove that every continuous function $f : \mathbf{Z}_p \to \mathbf{Q}_p$ can be written as $f(x) = \sum_{n \geqslant 0} a_n \binom{x}{n}$ where $a_n \in \mathbf{Q}_p$ and $a_n \to 0$.
3. Prove that if ν is a measure, then $\int_{\mathbf{Z}_p} f(z) d\varphi(\nu) = \int_{\mathbf{Z}_p} f(pz) d\nu$.

4. If ν is a measure, prove that $\int_{\mathbf{Z}_p} f(z)d\psi(\nu) = \int_{p\mathbf{Z}_p} f(p^{-1}z)d\nu$.
5. If ν is a measure, let $x\nu$ be the measure defined by $\int_{\mathbf{Z}_p} f dx\nu = \int_{\mathbf{Z}_p} zf(z)d\nu(z)$.
   (a) Prove that $\mathcal{A}(x\nu) = (1+X)(d/dX)\mathcal{A}(\nu)$;
   (b) Prove that the map $f(X) \mapsto (1+X)(df/dX)$ is a bijection on $E[\![X]\!]^{\psi=0}$.
6. If μ, ν are measures, define $\mu * \nu$ by $\int_{\mathbf{Z}_p} f d\mu * \nu = \int_{\mathbf{Z}_p}\int_{\mathbf{Z}_p} f(x+y)d\mu(x)d\nu(y)$. Prove that $\mathcal{A}(\mu * \nu) = \mathcal{A}(\mu)\mathcal{A}(\nu)$.
7. Is the exact sequence from corollary 5.2.2 split?

6. The p-adic Langlands correspondence

The mod p version of the p-adic Langlands correspondence is a bijection (see [**Bre03a**] and [**Bre03b**]) between the set of 2-dimensional E-linear representations of $\mathcal{G}_{\mathbf{Q}_p}$ and certain smooth admissible representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ (first studied in [**BL95**] and [**BL94**]). A more general version of the computations we have given above is used to realize this correspondence using (φ,Γ)-modules, following an idea of Colmez (see [**Col08**] and [**Col07**]). The details are worked out in [**Ber05**] and the computations are simplified and extended to representations of $\mathcal{G}_{\mathbf{Q}_p}$ of dimension $\geqslant 2$ in [**Ber08**]. In particular, if W is irreducible of dimension $\geqslant 2$, then the representation $\Omega(W)$ of §4 is completely determined in [**Ber08**]. A detailed introduction to the p-adic Langlands correspondence, especially in characteristic 0, is in Breuil's "introduction générale" to Astérisque 319.

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