Regularized Determinants and Some Applications.
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Abstract
In these series of lectures we will do the simplest computations done in String Theory. We will compute in the simplest case the partition function of the free bosonic field theory on circle. These computations led to the notion of Rational Conformal Field Theories, T-duality and mirror symmetry. We will give two different methods of computation of the partition function of the free bosonic theory realized as a sigma model of maps of Riemann surface to a circle. The first approach is based on Feynman path integral of the action of the free bosonic theory which is the energy of the map defined on the space of maps of an elliptic curve to a circle with radius $R$. The other one is based on the representation theory and the operator formalism.

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INTRODUCTION
Among the Conformal Field Theories there is a class of Conformal Field Theories that play the same role as the compact groups among Lie groups. These conformal field Theories are called Rational Conformal Field Theories. The Rational Conformal Field Theories (RCFT) were introduced in 1984 by Freedman, Qiu and Shenker. They are particularly nice class of Conformal Field Theories which have more structures and a hope that they can be classified. RCFT's are characterized by having a symmetric algebra extending Virasoro algebra, in therm of which the Hilbert space can be decomposed into finite irreducible representations.

Recently Vafa and Gukov proved that the sigma model of maps of Riemann surfaces to an elliptic curves is a Rational Conformal Field Theory if and only if the elliptic curve is with complex multiplication. See [3]. Narain and Witten computed the partition function of the sigma model of maps of Riemann surface...
to real torus. See [5]. Hosono, Bong Lian and Shing-Tung Yau found a necessary and sufficient condition for the sigma model with target real torus to be a Rational Conformal Field Theory. See [6].

The purpose of the seminar is to see when the Conformal Field Theories that correspond to sigma models of maps of elliptic curve to circle is a RCFT. Later we will try to understand the papers [3], [5] and [6]. We will discuss some conjectures of Gukov and Vafa about RCFT concerning sigma models of maps of Riemann surfaces to K3 or Calabi-Yau manifolds.

The problem that we will discuss in the seminar is for which values of the radius of the circle the theory of the maps from the elliptic curves to the circle we will get RCFT. The model that is obtained from the sigma model of maps of elliptic curve to a circle is called the model of free bosonic field on a circle. The partition function of this model is given by

\[ Z(q, \bar{q}) = \frac{1}{|\eta|^2} \sum_{m,n} q^{\frac{1}{2} (n^2 + mr)} \bar{q}^{\frac{1}{2} (n^2 - mr)}, \tag{1} \]

where \( q = \exp (2\pi \sqrt{-1} \tau) \) and \( \eta(q) \) is the Dedekind eta function defined as follows:

\[ \eta(q) = q^{\frac{1}{24}} \prod_{n=1}^\infty (1 - q^n). \]

By definition, the theory is rational if one can represent the partition function \( Z(q, \bar{q}) \) as a finite sum of the form:

\[ Z(q, \bar{q}) = \sum_{i,j} M_{i,j} \chi_j(q) \overline{\chi_j(q)}, \]

where \( M_{i,j} \) are positive integers and \( \chi_j(q) \) are holomorphic characters:

\[ \chi_j(q) = \text{tr}_{\nu_j} q^{L_0 - c/24}. \]

We will explain the meaning of the operator \( L_0 \) and will show that the free bosonic theory is rational if \( r^2 = \frac{p}{q} \), where \( p, q \) are mutually prime integers. In this case \( \chi_j(q) \) will be twisted theta functions.

1 Lecture I - Regularizations and Regularized Determinants of Laplacians

1.1 Example

Let us try to "regularize"

\[ \sum_{n=1}^\infty \frac{1}{n^2} = 1 + \ldots + 1 + \ldots \]
For this reason we will use Riemann zeta function:
\[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \]
where we will assume that \( \Re s > 1 \). We know that \( \zeta(s) \) has meromorphic continuation in \( \mathbb{C} \) with only pole at 1. We have
\[ \zeta(s) = \frac{1}{1-s} + \gamma + O(1-s), \]
and \( \gamma \) is the Euler constant. Then formally
\[ 1 + \ldots + 1 + \ldots = \zeta(0). \]
So we need to compute \( \zeta(0) =? \)

Let us regularized \( \sum_{k=1}^{\infty} k \) by using Riemann zeta function:
\[ \sum_{k=1}^{\infty} k = \zeta(-1) = -\frac{1}{12}. \]

Next we will describe the basic properties of Riemann zeta function.

### 1.2 Poisson Summation Formula

**Theorem I.2.1** Suppose that \( f(x) \) is a function on \( \mathbb{R}^n \) which rapidly decreases at \( \pm \infty \) along each line in \( \mathbb{R}^n \). Let \( \Lambda \) be a non degenerate lattice in \( \mathbb{R}^n \). Then we have the following formula
\[ \sum_{\lambda \in \Lambda} f(\lambda) = \frac{1}{\text{vol}(\Lambda)} \sum_{\lambda \in \Lambda^*} \hat{f}(\lambda), \]
where \( \Lambda^* \) is the dual lattice of \( \Lambda \).

**Proof:** Let us consider the function
\[ g(x) = \sum_{\lambda \in \Lambda} f(x + \lambda). \] (2)
Clearly \( g(x) \) is a periodic function. We have
\[ g(0) = \sum_{\lambda \in \Lambda} f(\lambda). \] (3)
Since \( g(x) \) is a period function then it will have a Fourier series assuming that \( \Lambda = \mathbb{Z}^n \):
\[ g(x) = \sum_{\lambda \in \mathbb{Z}^n} a_{\lambda} \exp(2\pi i \langle x, \lambda \rangle), \]
where
\[ a_\lambda = \int_0^1 \ldots \int_0^1 g(x) \exp (-2\pi i \langle x, \lambda \rangle). \]

Thus we get
\[ g(x) = \sum_{\lambda \in \mathbb{Z}^n} \left( \int_0^1 \ldots \int_0^1 g(x) \exp (-2\pi i \langle x, \lambda \rangle) \right) \exp (2\pi i \langle x, \lambda \rangle). \] (4)

Substituting (2) in (4) we get that
\[ g(x) = \sum_{\lambda \in \mathbb{Z}^n} \left( \sum_{\mu \in \mathbb{Z}^n} \int_0^1 \ldots \int_0^1 f(x + \mu) \exp (-2\pi i \langle x, \lambda \rangle) \right) \exp (2\pi i \langle x, \lambda \rangle) = \]
\[ \sum_{\lambda \in \mathbb{Z}^n} \left( \sum_{\mu \in \mathbb{Z}^n} \int_0^1 \ldots \int_0^1 f(x + \mu) \exp (-2\pi i \langle x, \lambda \rangle) \right) \exp (2\pi i \langle x, \lambda \rangle). \] (5)

Notice that
\[ \left( \sum_{\mu \in \mathbb{Z}^n} \int_0^1 \ldots \int_0^1 f(x + \mu) \exp (-2\pi i \langle x, \lambda \rangle) \right) = \int_{\mathbb{R}^n} f(x) \exp (-2\pi i \langle x, \lambda \rangle) dx = \hat{f}(\lambda). \] (6)

Substituting (6) in (5) we deduce that
\[ g(x) = \sum_{\lambda \in \mathbb{Z}^n} \hat{f}(\lambda) \exp (2\pi i \langle x, \lambda \rangle). \]

Let us substitute \( x = 0 \) in the last formula. We get
\[ g(0) = \sum_{\lambda \in \mathbb{Z}^n} \hat{f}(\lambda). \] (7)

Combining (3) and (7) we get the Poisson summation formula. ■

1.3 Gamma Function

Definition I.3.1 We will define Gamma function \( \Gamma(s) \) as follows:
\[ \Gamma(s) = \int_0^\infty \exp(-t)t^{s-1}dt. \]

Let \( \gamma \) be the Euler constant:
\[ \gamma = \lim_{n \to \infty} \left( 1 + \ldots + \frac{1}{n} - \log n \right). \]
Proposition I.3.2 The following formulas hold:

\[
\frac{1}{\Gamma(s)} = s \exp(\gamma s) \prod_{n \geq 1} \left(1 + \frac{s}{n}\right) \exp\left(-\frac{s}{n}\right), \quad \frac{1}{\Gamma(s)} = s + \gamma s^2 + O(s^3), \quad (8)
\]

\[
\Gamma(s + 1) = s\Gamma(s), \quad \Gamma(s)\Gamma(1 - s) = \frac{\pi}{\sin(\pi s)} \quad \text{and} \quad 2^{2s-1}\Gamma(s)\Gamma(s + \frac{1}{2}) = \pi^{\frac{1}{2}}\Gamma(2s).
\]

For the proof of the properties of \(\Gamma(s)\) see [8].

1.4 Theta Function

Definition I.4.1 Let \(\Lambda\) be an Euclidean lattice in \(\mathbb{R}^n\). We define the Theta function related to \(\Lambda\) as follows:

\[
\theta_{\Lambda}(\tau) = \sum_{l \in \Lambda} \exp\left(\pi \sqrt{-1} \tau \langle l, l \rangle\right),
\]

where \(\tau \in \mathbb{C}\) and \(\text{Im} \tau > 0\).

Theorem I.4.2 The following transformation law holds for \(\theta_{\Lambda}(\tau)\)

\[
\theta_{\Lambda}(\sqrt{-1}t) = \frac{1}{t^{n/2}\text{vol}(\Lambda)^{\frac{1}{2}}} \theta_{\Lambda}\left(\frac{-1}{t}\right).
\]

Proof: Theorem I.4.2 follows from the Poison summation formula which states that if \(f(t)\) is a function that rapidly declines at \(\pm \infty\) then we have for any Euclidean lattice \(\Lambda\)

\[
\sum_{\lambda \in \Lambda} f(\lambda) = \frac{1}{\text{vol}(\Lambda)} \sum_{\Lambda \in \Lambda} \hat{f}(\lambda), \quad (9)
\]

where \(\hat{f}(p)\) is the Fourier transform of \(f(t)\):

\[
\hat{f}(p) = \int_{-\infty}^{\infty} \exp\left(-2\pi \sqrt{-1} \langle p, t \rangle\right) f(t) dt.
\]

Theorem IV.I follows directly from (9). Indeed we are considering the function

\[
f = \exp(-\pi \sqrt{-1}x^2).
\]

The Fourier transform \(\hat{f}\) of \(f\) is given by

\[
\hat{f}(p) = \int_{-\infty}^{\infty} \exp\left(-2\pi \sqrt{-1}px - \pi tx^2\right) dt = \int_{-\infty}^{\infty} \exp\left(\pi \sqrt{-1}\left(-2\sqrt{-1}pt + \left(\sqrt{t}\right)^2\right)\right) dt =
\]

taking into account that we are considering the lattice \(\sqrt{t}\Lambda\) and its dual lattice is \(\frac{1}{\sqrt{t}}\Lambda\). \(\blacksquare\)
1.5 Zeta Function

We will assume that the following properties of Riemann zeta function are well known:

\[
\zeta(s) = \frac{1}{s-1} + \gamma + O(s). \tag{10}
\]

**Theorem I.5.1** Riemann Zeta function satisfies the following functional equation:

\[
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s). \tag{11}
\]

**Proof:** Notice that

\[
2\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty (\theta(\sqrt{-1}t) - 1)t^{\frac{s}{2}} \frac{dt}{t}.
\]

We can rewrite this expression as follows:

\[
2\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^1 (\theta(\sqrt{-1}t) - 1)t^{\frac{s}{2}} \frac{dt}{t} + \int_1^\infty (\theta(\sqrt{-1}t) - 1)t^{\frac{s}{2}} \frac{dt}{t}.
\]

On the other hand

\[
\int_0^1 (\theta(\sqrt{-1}t) - 1)t^{\frac{s}{2}} \frac{dt}{t} = \\
\int_0^1 (\theta(\sqrt{-1}t) t^{\frac{s-1}{2}} dt - \int_0^1 t^{\frac{s-1}{2}} dt = \int_0^1 (\theta(\sqrt{-1}t) t^{\frac{s-1}{2}} dt - \frac{2}{t}.
\]

By using the transformation law for theta function we get that

\[
\int_0^1 (\theta(\sqrt{-1}t) t^{\frac{s-1}{2}} dt = \int_0^1 (\theta(\sqrt{-1}t) \frac{1}{t} t^{\frac{s-1}{2}} dt.
\]

By changing the variables \(\frac{1}{t} = y\) we get

\[
\int_0^1 (\theta(\sqrt{-1}t) t^{\frac{s-1}{2}} dt = \int_1^\infty (\theta(\sqrt{-1}y) y^{\frac{3-s}{2}} + y^{-2+\frac{1}{2}} dy = \int_1^\infty (\theta(\sqrt{-1}y) y^{\frac{1-s}{2}} - 1) dy.
\]

Thus we get

\[
\int_0^1 (\theta(\sqrt{-1}t) t^{\frac{s-1}{2}} dt = \int_1^\infty ((\theta(\sqrt{-1}y) - 1) y^{\frac{1-s}{2}} - 1) dy + \int_1^\infty y^{\frac{1-s}{2}} dy =
\]

7
\[
\int_{1}^{\infty} (\theta(\sqrt{-1}y) - 1) y^{-\frac{s}{2}} dy - \frac{2}{1 - s}.
\]

Combining all these computations we get

\[
2\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) =
\]

\[
-\frac{2}{s} - \frac{2}{1 - s} + \left(\int_{1}^{\infty} (\theta(\sqrt{-1}t) - 1)t^{\frac{s}{2}} \frac{dt}{t} + \int_{1}^{\infty} (\theta(\sqrt{-1}t) - 1)t^{\frac{s}{2}} \frac{dt}{t}\right).
\]

From here the functional equation for Riemann zeta function follows. ■

**Corollary I.5.1.1** \(\zeta(0) = -\frac{1}{2}, \zeta(-1) = -\frac{1}{12}\).

From (11) we obtain that we have that

\[
\log \zeta(1 - s) = \log s + \gamma s + O(s^2)
\]

and thus we obtain that

\[
\frac{\zeta'(1 - s)}{\zeta(1 - s)} = \frac{1}{s} + \gamma.
\]  

(12)

## 2 Regularizations

### 2.1 Sterling formula

**First Approach:** Let us consider the Sterling formula:

\[
n! = n^n \sqrt{n} \sqrt{2\pi} e^{-n} \left(1 + O\left(\frac{1}{n}\right)\right),
\]

i.e.

\[
\log (n!) = n \log n + \frac{1}{2} \log n - n + \log \sqrt{2\pi}.
\]

We define \(\log (\infty!) = \sqrt{2\pi}\).

### 2.2 Zeta Function Regularization

**Second Approach.** We start with Riemann Zeta function:

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},
\]

\(\text{Re } s > 1\). Zeta function has a meromorphic continuation to the whole complex plane, with a single pole at 1. More precisely

\[
\zeta(s) = \frac{1}{s - 1} + \gamma + O(1 - s).
\]
Zeta function satisfies the functional equation:
\[
\pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma \left( \frac{1-s}{2} \right) \zeta(1-s).
\]
Since
\[
\zeta'(s) = -\sum n^{-s} \log n,
\]
when \( \text{Re} \, s > 1 \), then we define
\[
1 \times 2 \ldots n \times \ldots = e^{-\zeta'(0)} = \sqrt{2\pi}.
\]

2.3 \( \Theta \)-Function Regularization

Third Approach. We start with \( \Theta \)-function
\[
\Theta(t) = \sum_{n \geq 1} e^{-n^2 t}.
\]
Take Mellin transform for \( \text{Re} \, s > 1 \)
\[
\zeta(2s) = \frac{1}{\Gamma(s)} \int_0^\infty \Theta(t) t^{s-1} dt.
\]
Naive approach to compute \( \zeta'(0) \) would be to exchange the order of differentiation and integration on the right side:
\[
2\zeta(0)'' = \int_0^\infty \Theta(t) \left( \frac{t^s}{\Gamma(t)} \right)' \frac{dt}{t} = \int_0^\infty \frac{\Theta(t) dt}{t}.
\]
The integral \( \int_0^\infty \frac{\Theta(t) dt}{t} \) does not exist and diverges at \( t = 0 \). One can prove that there exists \( a_0 \) and \( a_{-1} \) such that
\[
\int_\varepsilon^\infty \Theta(t) \frac{dt}{t} + a \log \varepsilon - a_{-1} \sqrt{\varepsilon}
\]
has a limit, called the finite part of the integral. This suggests that
\[
\log (\infty!) = \text{" finite part of the integral } \int_0^\infty \Theta(t) \frac{dt}{t}.\]

In these lectures we use the second and third approach to regularized sequences \( \lambda_i \) such that: \( \Lambda : 0 < \lambda_1 \leq \lambda_2 \leq \ldots \) and the zeta function attached to them is defined by:
\[
\zeta_\Lambda(s) = \sum_{n \geq 1} \frac{1}{\lambda_n^s}.
\]
We will make the following three assumptions about \( \zeta_\Lambda(s) \):
1. $\zeta_\Lambda(s)$ converges for $\text{Re } s \gg 0$.

2. $\zeta_\Lambda(s)$ has a meromorphic continuation to the whole $\mathbb{C}$.

3. $\zeta_\Lambda(s)$ has no pole at 0.

**Definition II.3.1** $\lambda_1 \times \ldots \times \lambda_n \times \ldots = e^{-\zeta_\Lambda'(0)}$.

**Lemma II.3.2** For any integer $N \geq 1$ we have

$$(\lambda_1 \times \ldots \times \lambda_N) \times (\lambda_{N+1} \times \ldots),$$

where by $(\lambda_{N+1} \times \ldots)$ we mean the regularization of $(\lambda_{N+1} \times \ldots)$.

**Lemma II.3.3** For any positive real number $a$, we have

$$(a\lambda_1) \times (a\lambda_2) \times \ldots = (\lambda_1 \times \ldots \times \lambda_n \times \ldots) a^{\zeta_\Lambda'(0)}.$$  

For the third approach of the regularization of the product of increasing positive numbers $\Lambda : 0 < \lambda_1 \leq \lambda_2 \leq \ldots$ we will use

$$\Theta_\Lambda(t) := \sum_{n \geq 1} e^{-\lambda_n t}.$$  

We will assume:

1. $\Theta_\Lambda(t)$ converges for $t > 0$.

2. For every $k \in \mathbb{N}$, there are real numbers $a_i, i \in \mathbb{Z}$, with $a_i = 0$ for $i < -d$, such that

$$\Theta_\Lambda(t) = \sum_{n=-d}^{k} a_n t^n + O(t^{k+1}) \text{ for } t \to 0.$$  

3. $\Theta_\Lambda(t)$ is $O(e^{-ct})$ for $t \to \infty$, for some positive number $c$.

**Theorem II.3.4** If $\Theta_\Lambda(t)$ satisfies the above two first conditions, then

$$\zeta_\Lambda(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \Theta_\Lambda(t) t^{s-1} dt$$  

and satisfies the three conditions that we required about $\zeta_\Lambda(s)$. Furthermore we have:

$$\zeta_\Lambda(0) = a_0$$

and

$$\zeta_\Lambda'(0) = \text{finite part of } \left( \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \Theta_\Lambda(t) \frac{dt}{t} \right) + \gamma a_0.$$  

$$\det \left( -\frac{\beta}{2\pi} \frac{d^2}{d\phi^2} \right) =$$
2.4 Mellin Transform

**Theorem II.4.1** Let \( f(t) \) be a function defined on the positive real numbers such that it decays exponentially on the infinite and near zero it has the following asymptotic expansion:

\[
f(t) \sim \sum_{k=1}^{N} \frac{a_k}{t^k} + a_0 + O(t).
\]

Then the limit \( \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} f(t) t^{s-1} dt \) exists and

\[
M(f)(s) = \frac{1}{\Gamma(s)} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} f(t) t^{s-1} dt.
\]

is well defined. \( M(f)(s) \) has a meromorphic continuation in \( \mathbb{C} \) and \( M(f)(0) = a_0 \).

**Proof:** We will consider the following expression for \( M(f)(s) \):

\[
M(f)(s) = \frac{1}{\Gamma(s)} \int_{0}^{1} \left( f(t) - \sum_{k=1}^{N} \frac{a_k}{t^k} + a_0 \right) t^{s-1} dt + \\
\frac{1}{\Gamma(s)} \left( \int_{0}^{1} \left( \sum_{k=1}^{N} \frac{a_k}{t^k} + a_0 \right) t^{s-1} dt + \int_{1}^{\infty} f(t) t^{s-1} dt \right).
\]

Clearly the expression

\[
\frac{1}{\Gamma(s)} \int_{1}^{\infty} f(t) t^{s-1} dt
\]

defines an entire function on \( \mathbb{C} \). We have

\[
\int_{0}^{1} \left( \sum_{k=1}^{N} \frac{a_k}{t^k} + a_0 \right) t^{s-1} dt = \frac{a_0}{s} + \sum_{k=1}^{N} \frac{a_k}{s-k}.
\]

Thus the function

\[
\int_{0}^{1} \left( \sum_{k=1}^{N} \frac{a_k}{t^k} + a_0 \right) t^{s-1} dt
\]

has a meromorphic continuation on \( \mathbb{C} \). In the same manner we can prove that the function

\[
\int_{0}^{1} \left( f(t) - \sum_{k=1}^{N} \frac{a_k}{t^k} + a_0 \right) t^{s-1} dt = \int_{0}^{1} \left( \sum_{k=1}^{N} b_k t^k \right) t^{s-1} dt
\]
has a meromorphic continuation for any positive integer \( N \). This proves that \( M(f)(s) \) has a meromorphic continuation in \( \mathbb{C} \). From (8) and (13) we deduce that \( M(f)(0) = a_0 \).■

3 Hodge Theory and Regularization of Laplacians

3.1 Hodge Theory

Definition III.2.1 Let \( M \) be a Riemannian manifold. Then on the \( m \)-forms we can define \( d^* \) the conjugate of \( d \) as follows:

\[
\langle d\omega_1 , \omega_2 \rangle = \langle \omega_1 , d^* \omega_2 \rangle .
\]

It is easy to see that \( d^* = - * d * \), where * is the Hodge operator.

Definition III.2.2 We will define the Laplacian \( \Delta_k \) acting on \( k \)-forms as follows:

\[
\Delta_k = d \circ d^* + d^* \circ d .
\]

Definition III.2.3 A \( k \)-form \( \omega \) on \( M \) will be called a Harmonic form if

\[
\Delta_k \omega = 0 .
\]

This condition is equivalent to

\[
d\omega = d^* \omega = 0 .
\]

Theorem III.2.1 (Hodge) Let \( M \) be a compact Riemannian manifold. Let \( C^\infty ( M , \Omega^k_M ) \) be the space of \( C^\infty \) \( k \)-forms. Then we have the following orthogonal decomposition

\[
C^\infty ( M , \Omega^k_M ) = \mathbb{H}_k \oplus \text{Im } d \oplus \text{Im } d^* .
\]

Corollary III.2.1.1 Let \( Z^k(M) \) be the space of closed \( k \)-forms on \( M \). Then

\[
Z^k(M) = \mathbb{H}_k \oplus \text{Im } d \text{ and } H^k(M, \mathbb{R}) = \mathbb{H}_k .
\] (14)

3.2 Gauss Integrals and Regularization

The results in this Subsection are based on the following integral: Let \( A \) be a symmetric non degenerate strictly positive matrix, then

\[
\int_{\mathbb{R}^n} \exp - (Ax, x) \, dx = \frac{(\pi)^{\frac{n}{2}}}{\sqrt{\det A}} .
\] (15)

Based on (15) we can represent

\[
\int_{f \in C^\infty_0 ( M , \mathbb{R})} df \exp ( - \langle \Delta f , f \rangle ) = \lim_{n \to \infty} \frac{(\pi)^{\frac{n}{2}}}{\sqrt{\lambda_1 \times ... \times \lambda_n}} ,
\] (16)
where \( \lambda_i \) are the eigen values of the Laplacian \( \Delta \). We know that \( \Delta \) is a self adjoint positive operator. Therefore all of its eigen values are non negative. In order to make sense of (16) we will present different schemes of regularization of \( (\sqrt{\pi}) \times \ldots \times (\sqrt{\pi}) \times \ldots = \prod_{i=1}^{\infty} \sqrt{\pi} \) and \( \prod_{\lambda_i > 0} \lambda_i \).

3.3 Zeta Function Regularization of the Determinants of the Laplacians.

Let \( \sigma(\Delta) \) be the spectrum of the Laplacian \( \Delta \). In defining \( \det \Delta \) it would then be reasonable to write:

\[
\det \Delta = \prod_{\lambda \in \sigma(\Delta)} \lambda.
\]

This has at least two fatal flaws. The first is that in general there are harmonic functions/formes, so then we would have \( \det \Delta = 0 \). We evade this problem by then defining the determinant to be the product of all non-zero eigenvalues of \( \Delta \). So we have:

\[
\det \Delta = \prod_{\lambda \in \sigma(\Delta)}' \lambda
\]

where the ' denotes that the product is over non-zero \( \lambda \). The second flaw is that this infinite product is, in general, divergent. The cleverness in this definition is in how this problem is dealt with.

We first define the "zeta function for \( \Delta \)" to be:

\[
\zeta_{\Delta}(s) = \sum_{\lambda \in \sigma(\Delta)}' \lambda^{-s}.
\]

While this function is not defined for \( s = 0 \) we note that, formally, we have:

\[
\frac{d}{ds} \zeta_{\Delta}(s) = -\sum_{\lambda \in \sigma(\Delta)}' (\log \lambda) \lambda^{-s}
\]

and thus

\[
\frac{d}{ds} \zeta_{\Delta}(s)|_{s=0} = -\sum_{\lambda \in \sigma(\Delta)} (\log \lambda).
\]

From here we obtain that formally we can state that

\[
\prod_{\lambda \in \sigma(\Delta)}' \lambda = \exp (-\zeta'_{\Delta}(0)).
\]
We will prove that $\zeta_\Delta(s)$ has a meromorphic continuation in $\mathbb{C}$ and it is well defined at 0. Thus we will assume that
\[
\det \Delta = \exp (-\zeta'_\Delta(0)).
\]

### 3.4 Computation of the Regularized Determinant of Laplacian of a Flat metric on the Circle with Radius $R$

#### Theorem II.4.1
The following formula holds
\[
\left. \frac{d}{ds} \zeta(s) \right|_{s=0} = -\frac{1}{2} \left( \log 2 - \log \pi \right).
\]

**Proof:** From the functional equation for $\zeta(s)$ we get that
\[
\zeta(s) = \pi^{s-\frac{1}{2}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{1-s}{2}\right)^{-1} \zeta(1-s).
\]

Thus we get that
\[
\frac{d}{ds} \log \zeta(s) = \frac{\zeta'(s)}{\zeta(s)} = \log \pi + \frac{1}{2} \Gamma'\left(\frac{1}{2}\right) - \frac{1}{2} \Gamma'\left(\frac{1-s}{2}\right) + \frac{\zeta'(1-s)}{\zeta(1-s)}.
\]

Substituting in (18) the expression from (12) we obtain
\[
\frac{\zeta'(s)}{\zeta(s)} = \log \pi + \frac{1}{2} \Gamma'\left(\frac{1}{2}\right) - \frac{1}{2} \Gamma'\left(\frac{1-s}{2}\right) - \frac{1}{s} - \gamma + O(s).
\]

**Proposition III.4.1.1** We have
\[
\frac{\Gamma'(s)}{\Gamma(s)} = -\frac{1}{s} - \gamma + O(s).
\]

**Proof:** From the formula $\Gamma(s+1) = s\Gamma(s)$ we get that $\Gamma(s) = \frac{1}{s} \Gamma(s+1)$. Thus
\[
\frac{\Gamma'(s)}{\Gamma(s)} = \frac{1}{s} + \frac{\Gamma'(s+1)}{\Gamma(s+1)}.
\]

We need to show that
\[
\lim_{s \to 0} \frac{\Gamma'(s+1)}{\Gamma(s+1)} = \frac{\Gamma'(1)}{\Gamma(1)} = -\gamma.
\]

This follows directly from formula (8). ■

Combining formula (18) and (19) we get that
\[
\lim_{s \to 0} \frac{\zeta'(s)}{\zeta(s)} = \frac{\zeta'(0)}{\zeta(0)} = \log \pi - \frac{1}{2} \Gamma'\left(\frac{1}{2}\right) - \frac{1}{2} \gamma.
\]
Proposition III.4.1.2 We have
\[-\frac{1}{2} \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} = \log 2 + \frac{\gamma}{2}. \quad (21)\]

Proof: Formula (8) implies that
\[\Gamma(s + 1) = \pi^{-\frac{1}{2}} 2^s \Gamma\left(\frac{s + 1}{2}\right) \Gamma\left(1 + \frac{s}{2}\right).\]
Thus we get that
\[\frac{\Gamma'(1)}{\Gamma(1)} = \log 2 + \frac{1}{2} \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} + \frac{1}{2} \frac{\Gamma'(1)}{\Gamma(1)}. \quad (22)\]
From (22) we derive formula (21). ■

Substituting into (20) formula (21) we get that
\[\zeta'(0) = \zeta(0)(\log \pi + \log 2). \quad (23)\]
Thus
\[\zeta'(0) = \zeta(0)(\log \pi + \log 2) = -\frac{\log \pi}{2} - \frac{\log 2}{2}\]
On the other hand \(\zeta(0) = -\frac{1}{2}\) implies (17). ■

Corollary III.4.1.3 The regularized determinant of the Laplacian \(-\frac{\beta}{2\pi} \frac{d^2}{d\theta^2}\) on the circle of radius \(R\) is given by
\[\det \Delta_R = \frac{4\pi R^2}{\beta}.\]

Proof: It is easy to see that the eigen functions of the Laplacian
\[-\frac{\beta}{2\pi} \frac{d^2}{d\theta^2}\]
are \(\psi_n = \exp\left(\frac{2\pi i \theta n}{R}\right)\). Thus the eigen values are given by
\[\lambda_n = \left(\frac{2\pi \beta}{2\pi R}\right)^2 n^2 = \left(\frac{\beta}{R}\right)^2 n^2.\]
Thus we get that
\[\det \left(-\frac{\beta}{2\pi} \frac{d^2}{d\theta^2}\right) = \prod \left(\frac{\beta}{R}\right)^2 n^2.\]
So
\[\zeta_{S_R}^{s}(s) = \left(\frac{\beta}{R}\right)^{-2s} \sum_{k \geq 1} \frac{1}{n^{2k}}.\]
Thus
\[ \zeta'_S R(0) = 2 \zeta'(0) + \zeta(0) + \frac{d}{ds} e^{-2s \log(\frac{s}{2})} \bigg|_{s=0} = 2 \zeta'(0) + \zeta(0) - 2 \log \left( \frac{\beta R}{2} \right) \]

From \( \zeta(0) = -\frac{1}{2} \) we obtain that \( \zeta_R(0) = -\log \beta + \log R - \log 2 - \frac{1}{2} \). Thus
\[ \exp(-\zeta_R(0)) = \frac{4\pi R^2}{\beta} \]

Cor. III.4.1.3 is proved. ■

4 Sigma Models

4.1 Basic Definitions

Definition IV.1.1 Let \((M, g)\) and \((N, h)\) be two Riemannian manifolds. We will consider the set of all \( C^\infty \) maps \( \text{Map}(M, N) \). We will define the energy \( S(\phi) \) of \( \phi \in \text{Map}(M, N) \) as follows; We know that \( d\phi \in C^\infty(M, T_M^* \otimes \phi^*(T_N)) \).

The metrics \( g \) and \( h \) induce metrics on the bundles \( M, T^*_M \) and \( \phi^*(T_N) \). Thus we can define pointwise the norm \( \|d\phi\|^2 \) of \( d\phi \).

Definition IV.1.2 Let \( \phi \in \text{Map}(M, N) \). We will define \( S(\phi) \) as follows:
\[ S(\phi) := \int_M \|d\phi\|^2 \text{vol}(g), \]

where \( d\phi \in \text{Hom}(T_M, T_N) = T_M^* \otimes T_N \) and \( \|d\phi\|^2 \) is the norm of with respect to the induced metric on \( T_M^* \otimes T_N \) defined pointwise.

4.2 Computation of a Feynman Integral Defined on the Space of Maps with Target Circle

Theorem IV.2.1 Let \( C^\infty_0(M, \mathbb{R}) \) be the space of all \( C^\infty \) functions on \( M \) such that
\[ \int_M f \text{vol}(g) = 0. \] (24)

Then we have
\[ \text{Map}(M, S^1) = \bigcup_{\alpha \in H^1(M, \mathbb{Z}), \ f \in C^\infty_0(M, \mathbb{R})} (\alpha + df). \]

Proof: Suppose that \( \phi : M \to S^1 \). Then on \( S^1 \) with a radius \( R \), we have a standard one form \( d\theta \) such that
\[ \int_{S^1} \frac{d\theta}{2\pi R} = 1. \]
Clearly $\phi^*(\frac{d\theta}{2\pi R}) \in H^1(M, \mathbb{Z})$. Suppose that $\alpha \in H^1(M, \mathbb{Z})$. Let $x_0$ be a fixed point on $M$. Let us consider the function on $M$:

$$
\phi(x) = R \exp \left( 2\pi i \int_{x_0}^x \alpha \right).
$$

The condition $\alpha \in H^1(M, \mathbb{Z})$ implies that $\phi(x)$ is a well defined complex valued function on $M$. Clearly $\phi(x) \in \text{Map}(M, S^1)$. We have the following formula for $\alpha$:

$$
\alpha = d \log(\phi(x)).
$$

Since there is one to one correspondence between exact one forms and functions on $M$ that satisfy (24) we conclude the proof of Theorem IV.2.1 ■

**Cor. IV.2.2** Let $\phi \in \text{Map}(M, S^1_\mathbb{R})$. Let $\phi : M \to S^1$ corresponds to the one form $\alpha_\phi$. Then

$$
S(\phi) = \|\alpha_\phi\|^2 \left\| \frac{d\theta}{2\pi R} \right\|^2 = \|\alpha_\phi\|^2.
$$

Let $E_\tau = \mathbb{C}/(\tau, 1)$ be an elliptic curve obtained from the lattice in $\mathbb{C}$ spanned by $(\tau, 1)$. We will describe how in String Theory was computed the following Feynman integral:

$$
\int D\phi e^{-\frac{1}{2\pi R} S(\phi)}.
$$

The measure $D\phi$ is not mathematically defined.

"Theorem" IV.2.3(not rigorous) We have the following expression for (26):

$$
\int D\phi e^{-\frac{1}{2\pi R} S(\phi)} = \sum_{\alpha \in H^1(E_\tau, \mathbb{Z})} \exp \left( -\frac{1}{2\pi R} \|\alpha_h\|^2 \right) \int_{f \in C^\infty_c(M, \mathbb{R})} Df \exp \left( -\frac{1}{2\pi R} (\Delta f, f) \right),
$$

(27)

$\alpha_h$ is a harmonic representative of $\alpha \in H^1(M, \mathbb{Z})$ and $\Delta$ is the Laplacian of some metric on $M$.

**Physical Proof:** Theorem IV.2.3 implies that $\phi$ corresponds to a closed form $\alpha_\phi \in H^1(M, \mathbb{Z})$. Hodge Theorem and (14) implies that $\alpha_\phi = \alpha_{h, \phi} + df$. Thus $\alpha_\phi = \alpha_{h, \phi} + df$ and $\langle \theta, \theta \rangle = \frac{1}{2\pi R}$ we get

$$
S(\phi) = \langle \alpha_\phi, \alpha_\phi \rangle \langle \theta, \theta \rangle = \frac{1}{2\pi R} \left( \|\alpha_{h, \phi}\|^2 + \langle df, df \rangle \right) =
$$

$$
2\pi R \left( \|\alpha_{h, \phi}\|^2 + \langle f, d^* \circ df \rangle \right) = \frac{1}{2\pi R} \left( \|\alpha_{h, \phi}\|^2 + \langle \Delta f, f \rangle \right).
$$

(28)

From (28) we derive (27). ■
4.3 Harmonic Forms in $H^1(S^1, \mathbb{Z})$

Lemma III.5.1 The harmonic forms in $H^1(S^1, \mathbb{Z})$ with respect to the flat metric on $S^1_R$ in are given by

$$\alpha_{h,n} = \frac{nd\theta}{2\pi R}.$$ 

Proof: If $\alpha_h$ is a harmonic form, then $*\alpha_h = \text{const.}$ So any harmonic form $\alpha_h = cd\theta$, where $c$ is a constant and $*d\theta = 1$ since $d\theta \wedge *d\theta = \text{vol}(g) = d\theta$. If the period of the harmonic form $\alpha_{h,n}$ is $n$, then we must have

$$\int_0^{2\pi} c\alpha_{h,n} = \int_0^{2\pi} cd\theta = 2cn\pi R.$$ 

So we get that $c = \frac{1}{2\pi R}$. ■

Lemma II.4.1 The norms of the harmonic forms with respect to the flat metric on $S^1$ in are given by

$$\|\alpha_{h,n}\|^2 = \int_0^{2\pi} \alpha_{h,n} \wedge *\alpha_{h,n} = \int_0^{2\pi} \left(\frac{n}{2\pi R}\right)^2 d\theta = \frac{1}{2\pi} \frac{n^2}{R^2}.$$ 

Proof: Obvious. ■

4.4 Computation of the Partition Function of the Sigma Model of Maps of the Circle to the Circle

Theorem II.4.1 The partition function $Z(R, 1)$ of the Sigma Model of maps of the Circle of Radius $r$ to a circle of radius 1 is given by the formula:

$$Z(R, 1) = Z(R, 1) = \left(\sum_n \exp \left(-\frac{n^2}{2\pi R^2}\right)\right) \frac{\pi \beta}{2\pi R}.$$ 

Proof: We need to use the results of Section I. Thus we have according to (27)

$$Z(R, 1) = \sum_n \exp \left(-\frac{1}{2\pi} \|\alpha_h\|_{L^2}^2\right) \frac{2\pi R}{\det \left(-\frac{\beta}{2\pi} \frac{d^2}{d\theta^2}\right)}.$$ 

We have $\alpha_h = \frac{n}{2\pi R} d\theta$. Therefore $\|\alpha_h\|_{L^2}^2 = \frac{n^2}{2\pi R^2}$. Thus we get

$$Z(R, 1) = \left(\sum_n \exp \left(-\frac{1}{2\pi} \|\alpha_h\|_{L^2}^2\right)\right) \frac{2\pi R}{\det \left(-\frac{\beta}{2\pi} \frac{d^2}{d\theta^2}\right)} = \left(\sum_n \exp \left(-\frac{n^2}{2\pi R^2}\right)\right) \frac{2\pi R}{\det \left(-\frac{\beta}{2\pi} \frac{d^2}{d\theta^2}\right)}.$$
We know that \( \det \left( -\frac{\beta}{2\pi} \frac{d^2}{d\theta^2} \right) = 2\pi R \) and so

\[
Z(R, 1) = \left( \sum_n \exp \left( -\frac{n^2}{2\pi R^2} \right) \right) \frac{\pi \beta}{2\pi R}.
\]

The Theorem is proved. ■

Theorem II.4.2 We have

\[
Z(R, 1) = \left( \sum_n \exp \left( -\frac{n^2}{2\pi R^2} \right) \right) \frac{\pi \beta}{2\pi R} = Z(R, 1) = \sum_n \exp \left( -\frac{\pi R n^2}{\beta} \right).
\]

Proof: Use Poisson summation. ■

4.5 Mirror Symmetry

We have \( Z(R, 1) = Z\left(\frac{2\pi}{R}, 1\right) \). This follows from the Poisson summation formula.

5 Elliptic Curves; Theta, Zeta and Explicit Formula for the Regularized Determinant of the Laplacian on the Circle

5.1 Plane Curves of Degree Three-Algebraic Theory

Definition III.1.1 Let \( E \) be a non-singular curve of degree three in \( \mathbb{P}^2 \). This means that is given by

\[ F_3(x_0 : x_1 : x_3) = 0. \]

where is a homogeneous polynomial of degree three. Then we will call an elliptic curve.

Theorem III.1.2 Any elliptic curve by change of variables by polynomials can be represented as follows:

\[ x_0 x_1^2 = 4x_2^3 - g_2 x_2 x_0^2 - g_3 x_0^3, \quad (3.1.1) \]

where if we devide the above equation by \( x_0^3 \) and make the substitutions

\[ y = \frac{x_1}{x_0}, \text{ and } x = \frac{x_2}{x_0} \]

we will get a polynomial

\[ y^2 = 4x^3 - g_2 x - g_3 \quad (3.1.2) \]

and the polynomial does not have multiple roots.

It is easy to see that the topological type of the surface defined by (3.1.1) is a product of two circles. This is so since the Riemann surface defined (3.1.1) is obtained by glueing two planes with two cuts.
5.2 Analytic Theory of Elliptic Curves

Theorem III.2.1 Let

\[ \rho_\Lambda(z) = \frac{1}{z^2} + \sum_{\gamma \in \Lambda - 0} \left( \frac{1}{(z - \gamma)^2} - \frac{1}{\gamma^2} \right) . \]

Then the map defined by \( z \in \mathbb{C}/\Lambda \rightarrow (\rho_\Lambda(z), \frac{d}{dz} \rho_\Lambda(z)) \) is a one to one map between \( E_\Lambda \) and the plane curve given by the equation \( y^2 = 4x^3 - g_2x - g_3 \), where

\[ g_2 = 60 \sum \frac{1}{(mc_2 + nc_2)^4} \quad \text{and} \quad g_3 = 140 \sum \frac{1}{(mc_2 + nc_2)^6}. \]

5.3 Teichmüller Theory of Elliptic Curves

Definition III.3.1 Suppose that \( \Lambda \) is a free abelian group in \( \mathbb{C} \) generated by the complex numbers \( c_1 \) and \( c_2 \) such that \( \text{Im} \frac{c_2}{c_1} > 0 \).

Definition III.3.2 Suppose that \( \Lambda \) is a free abelian group in \( \mathbb{C} \). Then \( E_\Lambda = \mathbb{C}/\Lambda \) has a natural complex structure since \( \Lambda \) acts without fixed points on \( \mathbb{C} \). We will call \( E_\Lambda = \mathbb{C}/\Lambda \) an elliptic curve.

Theorem III.3.3 Let \( E_{\Lambda_1} = \mathbb{C}/\Lambda_1 \) and \( E_{\Lambda_2} = \mathbb{C}/\Lambda_2 \) be two different elliptic curves. Then there exists one to one complex analytic correspondence between \( E_{\Lambda_1} \) and \( E_{\Lambda_2} \) if and only if \( \Lambda_1 = c\Lambda_2 \), where \( c \) is a non zero complex number.

Proof: The complex structure of an elliptic curve \( \mathbb{C}/\Lambda \) descends from the complex structure of the universal cover \( \mathbb{C} \) of the elliptic curve. Hence any holomorphic map \( f : \mathbb{C}/\Lambda_1 \rightarrow \mathbb{C}/\Lambda_2 \) can be lifted to the universal cover of the elliptic curves. Thus we have the following commuting diagram:

\[ \begin{array}{ccc}
\tilde{f} : \mathbb{C} & \rightarrow & \mathbb{C} \\
\downarrow & & \downarrow \\
 f : \mathbb{C}/\Lambda_1 & \rightarrow & \mathbb{C}/\Lambda_2 
\end{array} \]

If \( f \) is an isomorphism then \( \tilde{f} \) will be a complex analytic isomorphism of \( \mathbb{C} \). Thus \( \tilde{f} \) will be a linear map. This implies that \( \Lambda_1 = c\Lambda_2 \). ■

Definition III.3.4 Let \( E_\Lambda \) be an elliptic curve. We will call the triple \( (E_\Lambda, (e_1, e_2) \& \text{the natural orientation of } \Lambda) \) a marked elliptic curve.

Theorem III.3.5 The set \( T \) of all marked elliptic curves with an orientation modulo isomorphisms is in one to one correspondence with the homogeneous space \( \text{GL}_2^+(\mathbb{R})/\mathbb{C}^* \), where \( \mathbb{C}^* \) is identified with the set of matrices

\[ \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \]

b) \( \text{GL}_2^+(\mathbb{R})/\mathbb{C}^* \) is complex analytically isomorphic to the upper half plane \( h = \{ \tau \in \mathbb{C} | \text{Im} \tau > 0 \} \).
Proof of a): The lattice $\Lambda$ and the basis $(e_1 = a_{11} + \sqrt{-1}a_{12}, e_2 = a_{21} + \sqrt{-1}a_{22})$ of $\Lambda$ are in one to one correspondence with all two by two matrices $(a_{ij})$. The condition that $\frac{e_1}{e_2} \notin \mathbb{R}$ imply that $\det(a_{ij}) \neq 0$. The orientation of $\Lambda$ imply that $\det(a_{ij}) > 0$.

Proof of b): In the orbit of the pair of non zero complex numbers $(e_1, e_2)$ under the action of the group $\mathbb{C}^*$ we can normalize the second number to be 1. This element is unique in the orbit. Thus we have that

$$(e_1, e_2) \sim \left(\frac{e_1}{e_2}, 1\right) = (\tau, 1).$$

By fixing the orientation we will assume that

$$\tau = \frac{e_1}{e_2} \in \mathfrak{h}.$$

Our Theorem is proved. ■

Remark. Theorem III.3.5 describes the Teichmüller space of elliptic curves. The Teichmüller space is defined as the space of all complex structures on the elliptic curves modulo the action of the group of diffeomorphisms of the two dimensional real torus.

5.4 Moduli of Elliptic Curves

Theorem III.4.2 The set of all elliptic curves modulo isomorphism is isomorphic to $\text{PSL}_2(\mathbb{Z}) \backslash \mathfrak{h}$. The action of the group $\text{PSL}_2(\mathbb{Z})$ on $\mathfrak{h}$ is given by

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

Proof: Let us denote by $E_\tau$ the elliptic curve corresponding to the lattice span by $(\tau, 1)$. It is easy to see that $E_{\tau_0}$ and $E_{\tau_1}$ are isomorphic if and only if

$$\tau_1 \mapsto \frac{a\tau_0 + b}{c\tau_0 + d}.$$

Indeed the condition that $E_{\tau_0}$ and $E_{\tau_1}$ are isomorphic means that $E_{\tau_1}$ is generated by the same lattice as the lattice generated by $(\tau_0, 1)$ multiplied by a non zero complex number. This means that the lattice $(\tau_1, 1)$ after a multiplication by some non zero complex number is the same as the lattice generated by $(\tau_0, 1)$. This means that the elliptic curve $E_{\tau_1}$ as a marked curve has the same lattice as the lattice generated by generated by $(\tau_0, 1)$ but with a different marking. This means that the new marking of $E_{\tau_0}$ must be of the type $(a\tau_0 + b, c\tau_0 + d)$, where

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1.$$

We know that the lattice spanned by $(a\tau_0 + b, c\tau_0 + d)$ and $\left(\frac{a\tau_0 + b}{c\tau_0 + d}, 1\right)$ define isomorphic elliptic curves. ■
5.5 Harmonic Forms on the Elliptic Curves

**Definition III.5.1** One form $\alpha$ on an elliptic curve $E_\Lambda = \mathbb{C}/\Lambda$ is called harmonic if $\alpha = adz + b\overline{dz}$, where $a$ and $b$ are some complex numbers.

**Theorem III.5.2** Let $E_\tau$ be an elliptic curve that corresponds to a lattice spanned by $(\tau,1)$. Then the harmonic forms that correspond to closed forms $\alpha_{m,n}$ such that

$$
\int_0^1 \alpha_{n,m} = n \quad \text{and} \quad \tau \int_0^\tau \alpha_{n,m} = m
$$

are the forms

$$
\alpha_{h,m,n} = \frac{1}{2\sqrt{-1}\Im \tau} \left( mzd - n\tau zd \right) + \frac{1}{2\sqrt{-1}\Im \tau} \left( -md\overline{z} + n\tau \overline{d}z \right).
$$

**Proof:** It is well known fact that the holomorphic and anti holomorphic forms on Kähler manifolds are harmonic. Therefore on $E_\tau = \mathbb{C}/(\tau,1)$ the holomorphic forms are $adz$ and the anti holomorphic forms are $b\overline{dz}$. From here it follows that $\alpha_{h,m,n}$ as defined by (29) are harmonic forms. Direct computations show that

$$
\frac{1}{2\sqrt{-1}\Im \tau} \int_0^1 \left( m (d - d\overline{z}) + n(\tau \overline{d} - \tau d) \right) = n
$$

and

$$
\frac{1}{2\sqrt{-1}\Im \tau} \int_0^\tau \left( m (d - d\overline{z}) + n(\tau \overline{d} - \tau d) \right) = m.
$$

Theorem III.5.3 is proved. ■

**Corollary III.5.4** We have the following formulas for the $L^2$ norms of $\alpha_{h,m,n}$:

$$
\|\alpha_{h,m,n}\|^2 = \left( \frac{1}{2\Im \tau} \right)^2 (m - \tau n)(m - \tau n).
$$

6 Computation of the Partition Function of Maps of an Elliptic Curve to the Circle

6.1 Application of the Poisson Summation Formula

References

We are planning to "prove" the following Theorem:
Theorem IV.1.1" The partition function of the free boson model based on the sigma model of maps of the elliptic curve $E_\tau$ to a circle of radius $r$ as defined by (27) is given by:

$$Z(q, \bar{q}) = \frac{1}{|\eta|^2} \sum_{m,n} q^{\frac{1}{24}} (\frac{m}{2\pi} + mr) q^{\frac{1}{24}} (\frac{m}{2\pi} - mr),$$

where $q = \exp (2\pi \sqrt{-1} \tau)$ and $\eta(q)$ is the Dedekind eta function defined as follows:

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

Proof: We will need the following Lemma:

Lemma IV.1.2 Let $\phi_{n',n} : E_\tau \to S^1_r$ be a harmonic map such that the restriction of $\phi_{n',n}$ on the one cycle on $E_\tau$ defined by $[0,1]$ has degree $n'$ and the restriction on the cycle $[\varnothing, \tau]$ has a degree $n$. Then the pullback of $\phi_{n',n}$ on the universal cover $\mathbb{C}$ of $E_\tau$ is given by the function:

$$X_{n',n}(z, \bar{z}) = \frac{\pi r}{\sqrt{-1} \tau_2} (n'(z - \bar{z}) + n(\tau z - \bar{\tau} z)), \quad (31)$$

where $\tau = \tau_1 + \sqrt{-1} \tau_2$.

Proof: The map given by (31) will be harmonic if the differentials one forms $\partial X_{n',n}(z, \bar{z})$ and $\overline{\partial} X_{n',n}(z, \bar{z})$ are harmonic. Clearly the one forms $\partial X_{n',n}(z, \bar{z})$ and $\overline{\partial} X_{n',n}(z, \bar{z})$ are holomorphic and antiholomorphic. They are harmonic.

Lemma IV.1.3 The energy $S(\phi_{n',n})$ of the map given by (31) is expressed as follows:

$$S(\phi_{n',n}) = \frac{1}{2\pi} \int_{E_\tau} \partial X_{n',n}(z, \bar{z}) \wedge \overline{\partial} X_{n',n}(z, \bar{z}) = \frac{2\pi}{\tau_2} \left( (n'r - nr\tau_1)^2 + (nr\tau_2)^2 \right),$$

where we assumed that $\int_{E_\tau} dx \wedge dy = 4\tau_2$.

Proof: The proof is obvious.

Theorem IV.1.4 We have the following equality:

$$\sum_{n',n=-\infty}^{\infty} \exp \left( -2\pi \left( \frac{1}{\tau_2} \right) (n'r - nr\tau_1)^2 + \tau_2 n^2 r^2 \right) = \frac{1}{r} \sqrt{\frac{\tau_2}{2}} \sum_{n,m=-\infty}^{\infty} q^{\frac{1}{24} (\frac{m}{2\pi} + nr)^2} q^{\frac{1}{24} (\frac{m}{2\pi} - nr)^2},$$

where $q = \exp (2\pi i \tau)$.
**Proof:** The proof of Theorem IV.1.1 is based on the following Lemma and Poisson summation formula:

**Lemma IV.1.6** The Fourier transform $\hat{f}(p)$ of the function

$$f(n'r) = \exp \left( -\frac{2\pi}{\tau_2} (n'r - nr\tau_1)^2 \right)$$

is given by the formula:

$$\hat{f}(p) = \exp \left( -\frac{1}{2} \pi \tau_2 p^2 + 2\pi \sqrt{-1} nr p \right).$$

**Proof:** We know that the definition of the Fourier transform implies

$$\hat{f}(p) = \sqrt{\frac{\tau_2}{2}} \int_{-\infty}^{\infty} \exp \left( -\frac{2\pi}{\tau_2} (n'r - nr\tau_1)^2 - 2\sqrt{-1} \pi pn'r \right) d(n'r). \quad (32)$$

We have the following equality

$$\left( \frac{2\pi}{\tau_2} (n'r - nr\tau_1)^2 \right) + 2\pi pn'r =$$

$$\left( \sqrt{\frac{2\pi}{\tau_2}} (n'r - nr\tau_1) + \left( \sqrt{\frac{-1}{2} \pi \tau_2} \right) p \right)^2 + \frac{1}{2} \pi \tau_2 p^2 - 2\pi \sqrt{-1} nr p. \quad (33)$$

Substituting (33) in (32) we get

$$\hat{f}(p) = \int_{-\infty}^{\infty} \exp \left( -\frac{2\pi}{\tau_2} (n'r - nr\tau_1)^2 - 2\sqrt{-1} \pi pn'r \right) d(n'r) =$$

$$\int_{-\infty}^{\infty} \exp \left( -\left( \sqrt{\frac{2\pi}{\tau_2}} (n'r - nr\tau_1) + \left( \sqrt{\frac{-1}{2} \pi \tau_2} \right) p \right)^2 - \frac{1}{2} \pi \tau_2 p^2 + 2\pi i\tau_1 nr p \right) d(n'r) =$$

$$\sqrt{\frac{\tau_2}{2}} \exp \left( -\frac{1}{2} \pi \tau_2 p^2 + 2\pi inrp \right).$$

Lemma IV.1.6 is proved. ■

**Lemma IV.1.7** We have the following formula:

$$\sum_{n'=-\infty}^{\infty} \exp \left( -\frac{2\pi}{\tau_2} (n'r - nr\tau_1)^2 \right) =$$

$$\frac{1}{r} \sqrt{\frac{\tau_2}{2}} \sum_{m'=-\infty}^{\infty} \exp \left( -\frac{1}{2} \pi \tau_2 \left( \frac{m'}{r} \right)^2 + 2\pi imn \right). \quad (34)$$
Proof: The formula (34) follows from the Poisson summation formula:

\[
\sum_{n' = -\infty}^{\infty} f(n'r) = \frac{1}{r} \sum_{m = -\infty}^{\infty} \hat{f}(\frac{m}{r}).
\]

Lemma IV.1.7 is proved. ■

The end of the proof of Theorem IV.1.4:
We have by Lemma IV.1.2

\[
\sum_{n', n = -\infty}^{\infty} \exp\left(-2\pi \left(\frac{1}{\tau_2} (n'r - nr\tau_1)^2 - \tau_2 n^2 r^2\right)\right) =
\frac{1}{r} \sqrt{\frac{\tau_2}{2}} \sum_{n, m = -\infty}^{\infty} \exp\left(-\frac{1}{2} \pi \tau_2 \left(\frac{m}{r}\right)^2 + 2\pi imn - \tau_2 n^2 r^2\right).
\]

Direct computations show that

\[
\frac{1}{r} \sqrt{\frac{\tau_2}{2}} \sum_{n, m = -\infty}^{\infty} \exp\left(-\frac{1}{2} \pi \tau_2 \left(\frac{m}{r}\right)^2 + 2\pi imn - \tau_2 n^2 r^2\right) =
\frac{1}{r} \sqrt{\frac{\tau_2}{2}} \sum_{m, n} q^{\frac{1}{2} \tau_2 (n + mr)} q^{-\frac{1}{2} \tau_2 (n - mr)}.
\]

Our Physical Theorem IV.1.4 is proved. ■

6.2 Computation of the Regularized Determinant of the Laplacian of the Flat Metric on the Elliptic Curve $E_\zeta$

Let

\[
\zeta(s) = \sum_{n > 0} n^{-s}
\]

be the Riemann zeta function. Then the functional equation

\[
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).
\]

implies

\[
\zeta(1) = -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12} \quad \text{and} \quad \zeta'(0) = -\frac{1}{2} \ln 2\pi.
\]

Lemma IV.2.1 We have the following formula for the regularized product

\[
\prod_{n = -\infty}^{\infty} (n + a) = 2i \sin \pi a.
\]  (35)

Proof: Recall that

\[
\sin \pi a = a \prod_{n > 0}^{\infty} \left(1 - \frac{a^2}{n^2}\right).
\]
By using the above regularization formulas
\[ \lambda_1 \times \ldots \times \lambda_n \times \ldots = e^{-\zeta'(0)} \]
and
\[ (a\lambda_1) \times (a\lambda_2) \times \ldots = (\lambda_1 \times \ldots \times \lambda_n \times \ldots) a^{\zeta'(0)}. \]
and the product formula for \( \sin \pi a \) we get that
\[ \prod_{n=-\infty}^{\infty} (n + a) = a \prod_{n>0} (-n^2) \prod_{n>0} \left(1 - \frac{a^2}{n^2}\right) = 2i \sin \pi a. \]

Lemma IV.2.1 is proved. ■

**Theorem IV.2.2** Let \( \Delta_{1,\tau} \) be the regularized determinant of the Laplacian of the flat metric on the elliptic curve \( E_\tau = \mathbb{C}/(\tau, 1) \). Then
\[ \Delta_{1,\tau} = 4\tau^2 \left(q\bar{q}\right)^{\frac{1}{12}} \prod_{m>0} (1 - q^m)^2 (1 - \bar{q}^m)^2 = 4\tau^2 \eta(\tau)\bar{\eta}(\tau) \]

where \( q = \exp(2\pi i \tau) \).

**Proof:** The proof of Theorem IV.2.1 is based on the following Lemma:

**Lemma IV.2.1.1** The eigen values of the Laplacian of the flat metric on the elliptic curve \( E_\tau = \mathbb{C}/(\tau, 1) \) are
\[ \left(\frac{\pi}{\tau_2}\right)^{2} (n - m\tau) (n - m\bar{\tau}). \] (36)

**Proof:** Direct check shows that the eigen functions of the Laplacian of the flat metric on the elliptic curve \( E_\tau = \mathbb{C}/(\tau, 1) \) are
\[ \psi_{nm} = \exp\left(\left(\frac{\pi}{\tau_2}\right) (n (z - \bar{z}) + m (\tau \bar{z} - \tau z))\right). \]

From this formula we derive directly formula (36). ■

From (36) and
\[ n \prod_{n\neq 0} 2^n = 2\zeta(-2) = (2\pi)^2 \]
formally we get
\[ \Delta_{1,\tau} = \prod_{(m,n) \neq (0,0)} \left(\frac{\pi}{\tau_2}\right)^{2} (n - m\tau) (n - m\bar{\tau}) = \]
\[ \frac{\pi^2}{\tau_2} \prod_{n\neq 0} n^2 \prod_{m\neq 0, n \in \mathbb{Z}} (n - m\tau) (n - m\bar{\tau}) = \]
\[ \frac{\pi^2}{\tau_2} (2\pi)^{2} \prod_{m>0, n \in \mathbb{Z}} (n - m\tau) (n + m\tau) (n - m\bar{\tau}) (n + m\bar{\tau}) = \]
\[
\frac{\pi^2}{\tau_2^2} (2\pi)^2 \prod_{m>0} \left( \prod_{n \in \mathbb{Z}} (n - m\tau)(n + m\tau)(n - m\tau) \right).
\]

(37)

By using formula (35) we get from (37)

\[
\Delta_{1,\tau} = \left( \frac{\pi}{\tau_2} \right)^2 (2\pi)^2 \prod_{(m,n) \neq (0,0)} (n - m\tau)(n - m\tau) = \\
\frac{\tau_2^2}{\pi^2} (2\pi)^2 \prod_{m>0} \left( (\exp(-\pi im\tau) - \exp(\pi im\tau))^2 (\exp(-\pi im\tau) - \exp(\pi im\tau))^2 \right) = \\
4\tau_2^2 \prod_{m>0} ((q\bar{q})^m (1 - q^m)^2 (1 - \bar{q}^m)^2).
\]

Now taking into account that

\[
\zeta(-1) = -\frac{1}{12}
\]

we get that

\[
\frac{\tau_2^2}{\pi^2} (2\pi)^2 \prod_{m>0} (q\bar{q})^m \prod_{m>0} (1 - q^m)^2 (1 - \bar{q}^m)^2 = \\
\frac{\tau_2^2}{\pi^2} (2\pi)^2 \left( \sum_{q\bar{q}^{m \geq 1}} \prod_{m>0} (1 - q^m)^2 (1 - \bar{q}^m)^2 = \\
4\tau_2^2 (q\bar{q})^{\frac{1}{12}} \prod_{m>0} (1 - q^m)^2 (1 - \bar{q}^m)^2 = 4\tau_2^2 \eta(\tau)\bar{\eta}(\tau).
\]

Theorem IV.2.1 is proved. ■

References


