Symmetric Solutions of Some Production Economies

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Abstract: A symmetric n-person game (n,k) (for positive integer k) is defined in its characteristic function form by $v(S) = [|S|/k]$, where $|S|$ is the number of players in the coalition S and $[x]$ denotes the largest integer not greater than x, (i.e., any k players, but not less, can "produce" one unit). It is proved that in any imputation in any symmetric von Neumann-Morgenstern solution of such a game, a blocking coalition of $p = n - k + 1$ players who receive the largest payoffs is formed, and their payoffs are always equal. Conditions for existence and uniqueness of such symmetric solutions with the other $k - 1$ payoffs equal too are proved; other cases are discussed thereafter.

1. Introduction

The purpose of this paper is the determination of symmetric von Neumann-Morgenstern solutions to a family of symmetric n-person games.

Let n and k be positive integers with $2 \leq k \leq n$, and let q be the unique integer satisfying

$$qk \leq n < (q+1)k. \tag{1}$$

Define an n-person *game* (n,k) by

$$v(S) = \begin{cases} 0, & |S| < k, \\ 1, & k \leq |S| < 2k, \\ 2, & 2k \leq |S| < 3k, \\ \vdots & \\ j, & jk \leq |S| < (j+1)k, \\ \vdots & \\ q, & qk \leq |S|, \end{cases} \tag{2}$$

where $v(S)$ is the characteristic function of the game, and $|S|$ denotes the number of players in S.

The motivation for studying this kind of game comes from economics. Consider an economy consisting of n persons, each of them initially owning one unit of some raw material. This raw material may be used to construct a certain consumer product, but it can only be produced in batches. For the production of one unit of the consumer product, k units of the raw material are needed; but with less than k units of the raw material, nothing can be produced.

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This kind of economy embodies in perhaps the most elemental form the problems of production in which the returns to scale are not constant or decreasing, and more specifically, the problems of set-up costs. Such economies have defied treatment by more conventional concepts of game theory and economics, such as the core, the competitive equilibrium, and so on. Though, of course, the games considered here are far from being the most general such production economies, we feel that they embody some of their most typical features, and that the study of these games may well lead to insights that will generalize to much larger classes of production economies.

For $q = 1$, we get a class of games whose unique symmetric solutions were found by BOTT [1953]. Hence we are interested by the games (n,k) with $q \geq 2$.

Let $p = n - k + 1$. Then a p-player coalition is a "minimal blocking coalition" (i.e., the smallest one which is large enough to block its complement from getting anything). The solution of Bott was based on such coalitions: all the players divide into disjoint p-player coalitions; the remaining players get nothing, and all the members of a particular blocking coalition receive the same amount. For $q \geq 2$, there is only one[2]) blocking coalition, and it has, too, an essential role in the symmetric solutions.

Without loss of generality, we will denote the players $1, 2, \ldots, n$, and let $N = \{1, 2, \ldots, n\}$ be the set of all players. An *imputation* a is an n-dimensional vector $(a_1, a_2, \ldots, a_n)$ satisfying

(i) $a_i \geq 0 = v(\{i\})$ for all $i (1 \leq i \leq n)$, and

(ii) $\sum_{i=1}^{n} a_i \leq q = v(N)$.

A set U of imputations is called *symmetric* if it contains all imputations arising from permutations of the indices of any imputation in U. A symmetric set which is a solution is a *symmetric solution*.

In Theorem 1 we prove that in any symmetric solution to a game (n,k) (with $q \geq 2$), the largest p coordinates of any imputation are equal.

This motivates us to pay special attention to imputations with the p largest coordinates equal and the other $k - 1$ coordinates also equal. Let V be the symmetric set[3]) consisting of all such imputations whose largest coordinate is $\geq 1/k$. In Theorem 2 we prove that V is a symmetric solution to the game (n,k) (with $q \geq 2$) if and only if $n \geq (q + 1)(k - 1)$.

Moreover, whenever $n \geq (q + 1)k - 3$, the set V is also the unique symmetric solution – this is proved in Theorem 3.

To illustrate these theorems, consider the two conditions

(i) $n \geq (q + 1)(k - 1)$,

(ii) $n \geq (q + 1)k - 3$.

[2]) i.e., the maximal number of *disjoint* blocking coalitions that can be formed is one.

[3]) From now on, when we refer to V we will always mean this particular set.

Note that (i) holds for $q \geq k - 1$ (since $n \geq qk$ by (1)), and (i) implies (ii) always (since $q \geq 2$); moreover (i) is equivalent to (ii) for $q = 2$. Thus, if we take a constant k, for $n \geq (k - 1)^2$, V is always a symmetric solution, and this is true also for some smaller n. E.g., if $k = 7$, then V is a symmetric solution for the n that are underlined, and is known to be unique for the n that are double underlined in the following table:

$$\begin{array}{ll}
14, 15, 16, 17, \underline{\underline{18}}, \underline{\underline{19}}, \underline{\underline{20}}, & (q = 2) \\
21, 22, 23, \underline{24}, \underline{\underline{25}}, \underline{\underline{26}}, \underline{\underline{27}}, & (q = 3) \\
28, 29, \underline{30}, \underline{31}, \underline{\underline{32}}, \underline{\underline{33}}, \underline{\underline{34}}, & (q = 4) \\
35, \underline{36}, \underline{37}, \underline{38}, \underline{\underline{39}}, \underline{\underline{40}}, \underline{\underline{41}}, & (q = 5) \\
\underline{42}, \underline{43}, \underline{44}, \underline{45}, \underline{\underline{46}}, \underline{\underline{47}}, \underline{\underline{48}}, & (q = 6) \\
\underline{49}, \underline{50}, \underline{51}, \underline{52}, \underline{\underline{53}}, \underline{\underline{54}}, \underline{\underline{55}}, & (q = 7) \\
\quad\quad\vdots & \quad\vdots
\end{array}$$

We will note that in the case (n,k) is a zero-sum game, i.e. when $n = (q + 1)k - 1$, it was already known that V is a symmetric solution (see GELBAUM [1959], Theorem 3.1); but its uniqueness is proved only here.

2. Preliminaries

An *ordered imputation a* is an imputation satisfying

$$a_1 \geq a_2 \geq \cdots \geq a_n .$$

Thus a symmetric set is characterized by its ordered imputations. We will redefine the concept of domination for ordered imputations. Let a, b be ordered imputations. Then $a \underset{S|T}{\succ} b$ (*a dominates b via S over T*) where $S = \{i_1, i_2, \ldots, i_m\} \subset N$ and $T = \{j_1, j_2, \ldots, j_m\} \subset N$, if

(i) $a_{i_r} > b_{j_r}$ for $1 \leq r \leq m$, and
(ii) $\sum_{i \in S} a_i \leq v(S)$ ("efficiency").

We write $a \succ b$ (*a dominates b*) if there are S and T such that $a \underset{S|T}{\succ} b$.

In all our proofs we need to decide whether or not at least one member of a symmetric set U of imputations dominates a given imputation b. This is equivalent to the problem of deciding whether some *ordered* imputation a in U dominates[4]) the *ordered* imputation b' that we get from b.

Hence we can deal only with ordered imputations. From now on, domination will always be according to the above definition.

We will first prove a simple but useful lemma:

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[4]) Domination between ordered imputations.

Lemma:

Let a, b be ordered imputations, such that $a \succ b$ via $S|T$.

Then there are coalitions S^* and T^* such that $a \succ b$ via $S^*|T^*$, and $|S^*| = |T^*| = k$.

Proof:

Let $|S| = |T| = m$, then $jk \leq m < (j+1)k$ (for some $j \geq 1$), hence

$$\frac{v(S)}{|S|} = \frac{j}{m} \leq \frac{j}{jk} = \frac{1}{k}.$$

Let $S = \{i_1, i_2, \ldots, i_m\}$ and $T = \{j_1, j_2, \ldots, j_m\}$ (with the correspondence $i_r \leftrightarrow j_r$ by $a_{i_r} > b_{j_r}$). Without loss of generality let $i_1 > i_2 > \cdots > i_m$, hence $a_{i_1} \leq a_{i_2} \leq \leq \cdots \leq a_{i_m}$.

Let $S^* = \{i_1, i_2, \ldots, i_k\}$ and $T^* = \{j_1, j_2, \ldots, j_k\}$. Then $a_{i_r} > b_{j_r}$ for $1 \leq r \leq k$, and S^* is effective for a since

$$\sum_{i \in S^*} a_i \leq \frac{k}{m} \sum_{i \in S} a_i \leq \frac{k}{|S|} v(S) \leq 1 = v(S^*).$$

Q.E.D.

3. The Theorems and their Proofs

We are ready now to state exactly and prove our results.

Theorem 1:

Let U be a symmetric solution to the game (n,k) for $q \geq 2$. Let a be an ordered imputation in U. Then

$$a_1 = a_2 = \cdots = a_p,$$

where $p = n - k + 1$.

Proof:

Suppose that $a_{i_0} > a_{i_0+1}$ for some $1 \leq i_0 \leq p-1$. Let b be the ordered imputation given by

$$b_i = a_i + \varepsilon, \quad \text{for} \quad i \neq i_0, \quad \text{and} \quad b_{i_0} = a_{i_0} - (n-1)\varepsilon,$$

where $\varepsilon > 0$ is small enough so that $b_{i_0} > b_{i_0+1}$. Let $S = T = \{p, p+1, \ldots, n\}$. Then $b \succ a$ via $S|T$ ($|S| = |T| = k$, and S is effective for b since $\sum_{i \in S} b_i \leq \frac{k}{n} \sum_{i=1}^{n} b_i = \frac{k}{n} \cdot q \leq 1 = v(S)$). Hence $b \notin U$, and therefore there exists another ordered imputation c in U such that $c \succ b$.

By the previous lemma, the domination is via k-player coalitions. For b we can take without loss of generality the smallest k coordinates $b_p, b_{p+1}, \ldots, b_n$. As for c, let $\{i_1, i_2, \ldots, i_k\}$ be the dominating set:

$$c_{i_r} > b_{p+r-1} \quad \text{for} \quad r = 1, 2, \ldots, k.$$

But $b_{p+r-1} = a_{p+r-1} + \varepsilon$ for all r $(1 \leq r \leq k)$ since $i_0 < p$, hence

$$c_{i_r} > a_{p+r-1} \quad \text{for} \quad r = 1, 2, \ldots, k.$$

I.e., $c \succ a$ via $\{i_1, i_2, \ldots, i_k\} | \{p, p+1, \ldots, n\}$, which contradicts the fact that both a and c are members of the solution U.

Q.E.D.

The next theorem gives the necessary and sufficient condition for V to be a solution.

Theorem 2:

The symmetric set V generated by all the ordered imputations a satisfying

$$a_1 = a_2 = \cdots = a_p = \alpha_1, \tag{3}$$

$$a_{p+1} = a_{p+2} = \cdots = a_n = \alpha_2, \tag{4}$$

and

$$\alpha_1 \geq 1/k \tag{5}$$

is a symmetric solution to the game (n, k) (with $q \geq 2$), if and only if

$$n \geq (q+1)(k-1). \tag{6}$$

Proof:

Unless we indicate otherwise, a symbol of the form $(\alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_2)$ will denote

$$(\underbrace{\alpha_1, \ldots, \alpha_1}_{p}, \underbrace{\alpha_2, \ldots, \alpha_2}_{k-1}),$$

i.e. the imputation whose first p coordinates are α_1 and whose last $k-1$ coordinates are α_2.

First we will prove that whenever (6) is satisfied, V is a symmetric solution to the game.

Suppose a and b are ordered imputations in V, and $a \succ b$. From the lemma, it follows that the domination is via k-player coalitions. Let $a = (\alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_2)$ and $b = (\beta_1, \ldots, \beta_1, \beta_2, \ldots, \beta_2)$. Then at least one of the dominated coordinates is β_1, hence $\alpha_1 > \beta_1$. Since $p\alpha_1 + (k-1)\alpha_2 = q = p\beta_1 + (k-1)\beta_2$, we must have $\alpha_2 < \beta_2$, hence all the dominating coordinates are α_1. By effectiveness $k\alpha_1 \leq 1$ or $\alpha_1 \leq 1/k$, so $\beta_1 < 1/k$, which contradicts (5) for b. Therefore, the set V is internally consistent.

To prove that the set V is extradominative, we will use the fact that for any $a = (\alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_2)$ in V, a coalition of $k-1$ α_1's and one α_2 is efficient:

$$(k-1)\alpha_1 + \alpha_2 \leq 1. \tag{7}$$

For $q \leq k-1$, this follows from (6): $n \geq (q+1)(k-1)$ hence $p = n - (k-1) \geq q(k-1)$ and

$$(k-1)\alpha_1 + \alpha_2 = [q(k-1)\alpha_1 + q\alpha_2]/q \leq [p\alpha_1 + (k-1)\alpha_2]/q = 1.$$

For $q > k - 1$ we get

$$\begin{aligned} q(k-1)\alpha_1 + q\alpha_2 &= q(k-1)\alpha_1 + (q-k+1)\alpha_2 + (k-1)\alpha_2 \\ &\leq (qk-k+1)\alpha_1 + (k-1)\alpha_2 \leq p\alpha_1 + (k-1)\alpha_2 = q \end{aligned}$$

and so (7) is proved.

Now let b be any imputation not in V. We will show that it is dominated by some imputation a in V. We have two cases:

Case 1:

$b_p < 1/k$. Define $\alpha_1 = 1/k$ and α_2 such that $p\alpha_1 + (k-1)\alpha_2 = q$, then $a = (\alpha_1,\ldots,\alpha_1,\alpha_2,\ldots,\alpha_2) \in V$ dominates b via k α_1's.

Case 2:

$b_p \geq 1/k$. Since $b \notin V$, (3) or (4) are not satisfied, hence

$$pb_p + (k-1)b_n < \sum_{i=1}^{n} b_i = q\,.$$

Let $0 < \varepsilon = [q - (pb_p + (k-1)b_n)]/n$, and define $\alpha_1 = b_p + \varepsilon$ and $\alpha_2 = b_n + \varepsilon$. Then $a = (\alpha_1,\ldots,\alpha_1,\alpha_2,\ldots,\alpha_2)$ is an ordered imputation in V, and $a \succ b$ via $k - 1$ α_1's (which dominate $b_p, b_{p+1},\ldots,b_{n-1}$) and one α_2 (which dominates b_n), the effectiveness being ensured by (7). Hence V is a symmetric solution.

Now we prove the converse. It will follow from the fact that if (6) is not satisfied, the set V does not dominate all imputations outside V.

Let b be the ordered imputation defined by

$$\begin{aligned} &b_1 = b_2 = \cdots = b_p = \frac{1}{k-1}, \\ &b_{p+1} = b_{p+2} = \cdots = b_{n-1} = \frac{1}{k-2}\left[q - \frac{p}{k-1}\right], \quad \text{and} \\ &b_n = 0 \end{aligned}$$

($k > 2$ otherwise (6) holds). But $q - \dfrac{p}{k-1} > 0$ (this is equivalent to the converse of (6)), i.e. $b_{n-1} > b_n$, hence $b \notin V$.

Let $a = (\alpha_1,\ldots,\alpha_1,\alpha_2,\ldots,\alpha_2)$ be an ordered imputation in V such that $a \succ b$; without loss of generality, let $b_p, b_{p+1},\ldots,b_n$ be the k dominated coordinates. Then $\alpha_1 > b_p = \dfrac{1}{k-1}$ hence $(k-1)\alpha_1 > 1$, which implies that there are at most $k - 2$ α_1's in the dominating coalition. Hence there are at least two α_2's, which dominate b_n, b_{n-1}, therefore $\alpha_2 > b_{n-1}$. But this leads to a contradiction, for

$$\begin{aligned} q &= p\alpha_1 + (k-1)\alpha_2 > pb_p + (k-1)b_{n-1} \\ &= \frac{p}{k-1} + \frac{k-1}{k-2}\left[q - \frac{p}{k-1}\right] \end{aligned}$$

$$= q + \frac{1}{k-2}\left[q - \frac{p}{k-1}\right] > q$$

the last inequality again following from $n < (q+1)(k-1)$. Hence b is not dominated by any member of V, which proves that V cannot be a solution.

Q.E.D.

Now we come to the uniqueness theorem.

Theorem 3:

Let V be the symmetric set generated by all ordered imputations satisfying (3)–(5). If

$$n \geq (q+1)k - 3 \tag{8}$$

then V is the *unique* symmetric solution to the game (n,k) (with $q \geq 2$).

Proof:

Let U be a symmetric solution to the game. By Theorem 1, if a is any ordered imputation in U, then $a_1 = a_2 = \cdots = a_p = \alpha_1$. Suppose $a_{i_0} > a_{i_0+1}$ for $p + 1 \leq i_0 \leq n - 1$. Define an ordered imputation b by

$$b_i = a_i + \varepsilon\,, \quad \text{for} \quad i \neq i_0\,, \quad \text{and}$$
$$b_{i_0} = a_{i_0} - (n-1)\varepsilon\,,$$

where $\varepsilon > 0$ is small enough so that $b_{i_0} > b_{i_0+1}$.

Then $b \succ a$ via $S|T$, where $S = T = \{p-1, p, \ldots, i_0 - 1, i_0 + 1, \ldots, n\}$. The effectiveness of this coalition for b follows from

$$\begin{aligned}
&b_1 + b_2 + \cdots + b_k \\
&\geq b_{k+1} + b_{k+2} + \cdots + b_{2k} \\
&\quad\vdots \\
&\geq b_{(q-2)k+1} + b_{(q-2)k+2} + \cdots + b_{(q-1)k-1} + b_{i_0} \\
&\geq b_{p-1} + b_p + \cdots + b_{i_0-1} + b_{i_0+1} + \cdots + b_n\,.
\end{aligned}$$

The rows are disjoint (the last two because $(q-1)k - 1 \leq n - k - 1 = p - 2 < p - 1$); the sum of all q rows is at most $\sum_{i=1}^{n} b_i = q$, hence the last one is ≤ 1.

Therefore, $b \notin U$, and there exists an ordered imputation c in U such that $c \succ b$. By Theorem 1, $c_1 = c_2 = \cdots = c_p = \gamma_1$. From (8), we get $qk - 2 \leq p = n - k + 1$, hence

$$\begin{aligned}
(k-1)\gamma_1 + c_{p+2} &\leq [q(k-1)\gamma_1 + 2c_{p+2}]/q \\
&\leq \left[(qk-2)\gamma_1 + \sum_{i=p+1}^{n} c_i\right]/q \leq 1\,.
\end{aligned} \tag{9}$$

Let i be the largest index in the dominating coalition. There are then two possibilities:

Case 1:

$i \leq p + 1$. Then the dominating coordinates must be $\gamma_1, \ldots, \gamma_1, c_i$, and hence

$$\gamma_1 > b_p = \alpha_1 + \varepsilon > \alpha_1$$
$$\gamma_1 > b_{p+1} = a_{p+1} + \varepsilon > a_{p+1}$$
$$\vdots$$
$$\gamma_1 > b_{i_0} = a_{i_0} - (n-1)\varepsilon$$
$$\vdots$$
$$c_i > b_n = a_n + \varepsilon > a_n .$$

Instead of b_{i_0} we can take $b_{p-1} = \alpha_1 + \varepsilon$ (because of the inequality in the first line), and we get $c \succ a$ (via the same coalition for c, and $\{p-1, p, \ldots, i_0 - 1, i_0 + 1, \ldots, n\}$ for a), which is a contradiction because $a, c \in U$.

Case 2:

$i \geq p + 2$. Then in the dominating coalition, we can change the first $k - 1$ coordinates to γ_1 (since $c_i \leq c_{p+2}$, it follows from (9) that it is still efficient), and proceed as in Case 1.

In any case we got a contradiction, hence $a_{p+1} = a_{p+2} = \cdots = a_n = \alpha_2$, i.e. any ordered imputation in U satisfies (3) and (4).

Let b be an ordered imputation in V, i.e. satisfying (3), (4) and (5). Then no ordered imputation a satisfying (3) and (4) can dominate it (see the proof of the internal consistency of V in Theorem 2: if $a \succ b$, where a and b satisfy (3) and (4), then $\alpha_1 > \beta_1 \geq 1/k$ because (5) is fulfilled by b; hence both a and b are in V and this contradicts $a \succ b$). Hence no a in U dominates b, therefore $b \in U$, and $V \subset U$. But V is a solution, so every $a \in U \backslash V$ is dominated by some $b \in V \subset U$, hence $U \backslash V$ must be empty (otherwise $a \succ b$, where $a, b \in U$).

Hence $U = V$, and V is the unique symmetric solution.

Q.E.D.

4. Discussion

Two problems are still open:

(i) what are the symmetric solutions, if any, to the game (n,k) when (6) is not satisfied, and

(ii) is V the unique symmetric solution when (6) but not (8) are satisfied.

The following two examples indicate that the answers are much more complicated than the theorems proved here.

First, let $n = 17$ and $k = 7$ (hence V is not a solution). It can be easily checked that the unique symmetric solution is the symmetric set generated by all the ordered imputations

$$(\underbrace{\alpha_1, \ldots, \alpha_1}_{11}, \underbrace{\alpha_2, \ldots, \alpha_2}_{5}, 0), \quad \text{where} \quad \frac{3}{19} \leq \alpha_1 \leq \frac{2}{11}, \quad \text{and}$$

$$(\underbrace{\alpha_1,\ldots,\alpha_1}_{11},\underbrace{\alpha_2,\ldots,\alpha_2}_{6}), \quad \text{where} \quad \frac{1}{7} \leq \alpha_1 \leq \frac{32}{209}.$$

Even more interesting is the second example: $n = 23$ and $k = 10$. Then all the ordered imputations

$$(\underbrace{\alpha_1,\ldots,\alpha_1}_{14},\underbrace{\alpha_2,\ldots,\alpha_2}_{7},0,0), \quad \text{where} \quad \frac{3}{28} \leq \alpha_1 \leq \frac{1}{7},$$

$$\Big(\underbrace{\frac{5}{49},\ldots,\frac{5}{49}}_{14},\underbrace{\frac{1}{14},\ldots,\frac{1}{14}}_{8},0\Big), \quad \text{and}$$

$$\Big(\underbrace{\frac{19}{196},\ldots,\frac{19}{196}}_{14},\underbrace{\frac{1}{14},\ldots,\frac{1}{14}}_{9}\Big),$$

generate a symmetric solution to the game. Note that in the last imputation, the largest coordinate is less than $1/k$.

By a similar method to that used in the proof of Theorem 3, it can be proved that whenever $n = (q + 1)k - m$ (for $4 \leq m \leq k$), all the smallest $k - 1$ coordinates but the last $m - 3$ must be equal in any imputation in any symmetric solution. Further results were also obtained regarding the uniqueness of V (and of other solutions, as above), but they still do not cover all the cases.

As to the economic meaning of the results, two facts are important.

First – the p-player coalition that is formed, whose members receive the largest payoffs. This illustrates the "exploitation" process of the small and powerless economic agents (here, $k - 1$ players who a priori can get nothing) by the powerful ones, and indicates that some kind of cartel is formed.

Second – the strong competition between the p largest players, that implies the equality of their payoffs (and this is true for any symmetric solution). It is interesting that most of the players (p) receive the same payoff, which is different from the case of the games of BOTT [1953]. This leads to the conclusion that the competition is strong when there are at least two units, and this is similar to other results in the theory of economic games (e.g., SHITOVITZ and SHAPLEY [1961], etc.).

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