VALUED FIELDS, METASTABLE GROUPS

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ABSTRACT. We introduce a class of theories called *metastable*, including the theory of algebraically closed valued fields (ACVF) as a motivating example. The key local notion is of definable types dominated by their stable part. A theory is metastable (over a sort Γ) if every type over a sufficiently rich base structure can be viewed as part of a Γ-parametrized family of stably dominated types. We initiate a study of definable groups in metastable theories of finite rank. Groups with a stably dominated generic type are shown to have a canonical stable quotient. Abelian groups are shown to be decomposable into a part coming from Γ, and a definable direct limit system of groups with stably dominated generic. In the case of ACVF, among affine definable groups we characterize the groups with stably dominated generics.

CONTENTS

1. Introduction

Let V be a variety over a valued field (F, val). By a (val-) *constructible set* we mean a finite Boolean combination of sets of the form

$$\{x \in U : \mathrm{val} f(x) \leq \mathrm{val} g(x)\}$$

where U is an open affine, and f, g are regular functions on U. Two such sets can be identified if they have the same points in any valued field extension of F, or equivalently, by a theorem of Robinson, in any fixed algebraically closed valued field extension K of F.

In many ways these are analogous to constructible sets in the sense of the Zariski topology, and (more closely) to semi-algebraic sets over real fields. However while the latter two categories are closed under quotients by equivalence relations, the valuative constructible sets are not. For instance, the valuation ring $\mathcal{O} = \{x : \mathrm{val}(x) \geq 0\}$, is constructible, and has constructible ideals $\alpha\mathcal{O} = \{x : \mathrm{val}(x) \geq \alpha\}$ and $\alpha\mathcal{M} = \{x : \mathrm{val}(x) > \alpha\}$. For any group scheme G over $\mathcal{O}$, one obtains corresponding congruence subgroups; but the quotients $\mathcal{O}/\alpha\mathcal{O}$ and $G(\mathcal{O}/\alpha\mathcal{O})$ are not constructible. We thus enlarge the category by formally adding quotients, referred to as *imaginary sorts*; the objects of the larger category are called *definable sets* [1]. It is explained in [7] that in place of this abstract procedure, it suffices to add sorts for the homogeneous spaces $G(K)/G(\mathcal{O})$, where $G = GL_n$ for some n, and also $G(K)/I$ for a certain subgroup I; we will not require detailed knowledge of this here. One that will be explicitly referred to is the value group $\Gamma = GL_1(K)/GL_1(\mathcal{O})$; this is a divisible ordered Abelian group, with no additional induced structure. [2]

The paper [8], continuing earlier work, studied the category of quantifier-free definable sets over valued fields, especially with respect to imaginaries. As usual, the direct study of a concrete structure of any depth is all but impossible, if it is not aided by a general theory. We first tried to find a generalization of stability (or simplicity) in a similar format, capable of dealing with valued fields as stability does with differential fields, or simplicity with difference fields. To this we encountered resistance; what we found instead ([8]) was not a new analogue of stability, but a new method of utilizing classical stability in certain unstable structures.

Even a very small stable part can have a decisive effect on the behavior of a quite "large," unstable type. This is sometimes analogous to the way that the (infinitesimal) linear approximation to a variety can explain much about the variety; and indeed in some cases casts tangent spaces and Lie algebras in an unexpected model theoretic role.

Two main principles encapsulate the understanding gained:

(1) Certain types are dominated by their stable parts. They behave "generically" as stable types do.

(2) Uniformly definable families of types make an appearance; they are indexed by the linear ordering Γ of the value group, or by other, piecewise-linear structures definable in Γ. An arbitrary type can be viewed as a definable limit of stably dominated types (from (1)).

A general study of stably dominated types was initiated in [8]; it is summarized in §2. (2) was only implicit in the proofs there. We state a precise version of the principle, and call a theory satisfying (2)*metastable*. We concentrate here on finite rank metastability.

(1) is given a general group-theoretic rendering in Proposition 4.6. Generically stably dominated groups are defined, and it is shown that a group homomorphism into a stable group controls them (generically.) Theorem 5.9 clarifies the second principle in the context of Abelian

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[1] In the literature they are sometimes referred to as *interpretable*. However interpretations include reducts, which we do not allow here.

[2] We will refer to *stable sorts* also. These include first of all the residue field $\mathrm{k} = \mathcal{O}/\mathcal{M}$, but also vector spaces over k of the form $\mathbb{L}/\mathcal{M}\mathbb{L}$, where $\mathbb{L}$ is a lattice.

groups. A metastable Abelian group of finite rank is shown to contain nontrivial generically stably dominated groups T_α, unless it is internal to Γ. Moreover, the groups T_α are shown to form a definable direct limit system, so that the group is described by three ingredients: Γ -groups, generically stably dominated groups, and definable direct limits of groups.

We then apply the theory to groups interpretable over algebraically closed valued fields. Already the case of Abelian varieties is of considerable interest; all three ingredients above occur, and the description beginning with a definable map into a piecewise linear definable group L takes a different aspect than the classsical one. The points of L over a non-archimedean local field will form a finite group, related to the group of connected components in the Neron model. On the other hand over $\mathbb{R}(t)$ the same formulas will give tori over $\mathbb{R}$.

Non-commutative definable groups include the congruence subgroups of algebraic groups, and quotients by them.

Previous work in this direction used other theories, inspired by topology; see [23], [10] regarding the p-adics.

We now describe the results in more detail. Our main notion will be that of a generically metastable group. In a metastable theory, we will try to analyze arbitrary groups, and to some extent types, using them. In the case of valued fields, this notion is related to but distinct from compactness (over those fields where topological notions make sense, i.e. local fields.) For groups defined over local fields, generic metastability implies compactness of the group of points over every finite extension. Abelian varieties with bad reduction show that the converse fails; this failure is explained by another aspect of the theory, definable homomorphic quotients defined over Γ.

Let T be a first order theory. It is convenient to view a projective system of definable sets D_i as a single object, a pro-definable set; similarly for a compatible system of definable maps $\alpha_i : P \to D_i$. We will also use the terminology $*$-definable, following Shelah's $*$-types.

Definition 1.1. *A partial type P over C is* stably dominated *if there exists over C a pro-definable map $\alpha : P \to D$, D stable and stably embedded, such that for any tuple b, $\alpha(a) \downarrow D \cap \mathrm{dcl}(b)$ implies*

$$tp(b/C, \alpha(a)) \models tp(b/Ca)$$

Let Γ be a sort of T. By a *substructure* of a model of T we mean a subset closed under definable functions. We write $A \leq M \models T$ for short. A *parameterically definable* set is a definable set in T_A for $A \leq M \models T$.

In this paper, we will assume:

(1) Γ is stably embedded: every subset of Γ^n defined with parameters in a model M of T is definable from parameters in $\Gamma(M)$. Equivalently, when M is saturated, the natural map $Aut(M) \to Aut(\Gamma(M))$ is surjective. (cf. [3], appendix.)

(2) Γ is orthogonal to the stable part: no infinite definable subset of Γ^{eq} is stable.

In the case of algebraically closed valued fields, Γ will be o-minimal. Another application is to valued fields with algebraically closed residue field and different value groups, such as $\mathbb{Z}$; where still every definable subset of Γ is a Boolean combination of 0-definable sets and intervals.

Definition 1.2. *T is* metastable *(over Γ) if for any partial type P over a base C_0 there exists $C \supset C_0$ and a *-definable (over C) map $\gamma_c : P \to \Gamma$ with $tp(a/\gamma_c(a))$ stably dominated.*

In addition, we assume:

(E) Every type over an algebraically closed subset of a model of T^{eq} has an automorphism-invariant extension to the model.

We will say that C is a good base for P. A *good base* is a good base for all partial types over it.

The condition (E) on existence of automorphism-invariant types, is currently needed in the hypotheses of the descent lemma for stable domination.

Let $A \leq M \models T$. Let St_A be the family of all stable, stably embedded A-definable sets. This will be referred to as the *stable part of* T_A. We write $St_A(c)$ for $A(c) \cap St_A$, where $A(c) = \mathrm{dcl}(A \cup \{c\})$ is the smallest substructure of M containing $A \cup \{c\}$.

Similarly, let $\Gamma_A(c) = \mathrm{dcl}(A, c) \cap \mathrm{dcl}(A \cup \Gamma)$. If $A = \mathrm{dcl}(\emptyset)$ we omit it from the notation. By a Γ-set over A we mean an A- definable set D such that there exists an A-definable surjection $h : \Gamma^m \to D$ for some m. In practice we can take $D \subseteq \Gamma^m$ for some m. A Γ-group is a Γ-set with a definable group structure; since Γ is assumed to be stably embedded, the group structure is definable with parameters from Γ.

Remark 1.3. *(1) Instead of taking all stable, stably embedded A-definable sets, it is possible to take a proper subfamily $\mathcal{S}_A$ with reasonable closure properties. $\mathcal{S}$-domination is meaningful even for stable theories.*

(2) One can also replace the single sort Γ with a family of sorts Γ_n, or with a family of parametrized families of definable sets $\mathcal{G}_A$, with no loss for the results of the present paper.

As in stability theory, a range of finiteness assumptions is possible. We will use the following "finite rank" assumption (FD) below. Some terminology: We will refer to the Morley rank of a formula in the stable part as the (Morley) *dimension*.

A definable subset D is called *o-minimal* if it has a distinguished definable linear ordering $<$, such that every definable subset of D with parameters in a model of T is a finite union of intervals and points. There is a natural notion of dimension for definable subsets of D^m, with $\dim(D) = 1$. See [28].

The dimension $\dim(e/C)$ of a tuple of elements $e \in St_C$ (or $e \in \Gamma^n$) is defined to be the minimum dimension of a formula D over C with $e \in D$.

A structure B is acl-finitely generated over $A \subseteq B$ if $B \subseteq \mathrm{acl}(A(b))$ for some tuple b from B.

(FD) :

(1) Γ is o-minimal.

(2) Morley dimension is finite and definable in families: if D_t is a definable family of definable sets then $\{t : MR(D_t) = m\}$ is definable

(3) Let D be a definable set. The Morley dimension of $f(D)$, where f ranges over all definable functions (with parameters) such that $f(D)$ is stable, takes a maximum value $\dim_{st}(D) \in \mathbb{N}$.

Similarly, the o-minimal dimension of $g(D)$, where g ranges over all definable functions (with parameters) such that $g(D)$ is Γ-internal, takes a maximum value $\dim_o(D)$.

Recall that a definable set X is Γ-*internal* if $X \subseteq \mathrm{dcl}(\Gamma, F)$ for some finite set F; equivalently for any $M \prec M' \models T$, $X(M') \subseteq \mathrm{dcl}(M \cup \Gamma(M'))$. For the purposes of (FD) , we could equally well ask that $g(D) \subseteq \mathrm{dcl}(\Gamma)$, or simply, when Γ eliminates imaginaries, that $g(D) \subseteq \Gamma^n$ for some n.

Some statements will be simpler if we also assume:

(FD_ω) : In addition to (FD) , any set is contained in a good base M which is also a model. Moreover, for any acl- finitely generated $F \subseteq \Gamma$ and $F' \subseteq St_M$ over M, isolated types over $M \cup F \cup F'$ are dense.

Remarks

(1) Write $\dim_{st}^{def}(d/B) = min\{\dim_{st}(D) : d \in D, D \,\mathrm{B-}$ definable.$\}$. If $B' = B(d) := dcl(B \cup \{d\})$, let $\dim_{st}^{def}(B'/B) = \dim_{st}^{def}(d/B)$; this is well-defined. Note that we may have $\dim_{st}^{def}(B'/B) > \dim St_B(B')/B$.

(2) (FD) and (FD$_\omega$) are true for ACVF, with all imaginary sorts included (Lemma **??**). (FD) at least is valid for all C-minimal expansions of ACVF, in particular the Lipshitz-Robinson rigid analytic expansions. cf. [6], [15], [16].

(3) In practice, the main structural results will use finite weight hypotheses; this is a weaker consequence of (FD) . Recall that a definable type p in a stable theory is said to have *weight* $\leq n$ if for any M and independent $b_1, \ldots, b_{n+1}$ over M, and any $a \models p|M$, for some i, $a \models p|Mb_i$. In a definable set of Morley rank n, every definable type has weight $\leq n$.

(4) Many of our results remain valid in case $\Gamma \models Th(\mathbb{Z})$ instead of being o-minimal; the theory thus applies to Henselian valued fields with algebraically closed residue field of characteristic zero, and value group $\mathbb{Z}$ (see the last section of [8]).

A group is *generically metastable* if it has a generic type that is stably dominated (See Definitions 3.1, 4.1.) In this case, the stable domination is witnessed by a group homomorphism (Proposition 4.6). One cannot expect every group to be generically metastable. But one can hope to shed light on any definable group by studying the generically metastable groups inside it. We formulate the notion of a *limit metastable* group; it is a direct limit of connected metastable groups by a $*$-definable direct limit system.

Theorem 1.4. *Let T be a metastable theory with (FD$_\omega$) . Let A be a definable Abelian group. Then there exists a definable group $\Lambda \subset \Gamma^{eq}$, and a definable homomorphism $\lambda : A \to \Lambda$, with $K = \ker(\lambda)$ limit- metastable.*

In fact under these assumptions, K is the union of a definable directed family of definable groups, each of which is connected and generically metastable. Assuming only bounded weight (in place of (FD$_\omega$)), we obtain a similar result but with K ∞-definable.

In the non-Abelian case the question remains open. The optimal conjecture would be a positive answer to:

Problem 1.5. *(FD$_\omega$) Does any definable group G have a limit-metastable definable subgroup H with $H\backslash G/H$ internal to Γ?*

One goal of this paper is to relate definable groups in ACVF to group schemes over $\mathcal{O}_F$. We recall the analogous results for the algebraic and real semi-algebraic cases. First consider a field F of characteristic zero. Then the natural functor from the category of algebraic groups to the category of constructible groups is an equivalence of categories. This follows locally from Weil's group chunk theorem; nevertheless some additional technique is needed to complete the theorem. It was conjectured by Poizat and proved by [29] using definable topological manifolds, and by [11] using the stability theoretic notions: definable types, germs. These methods will be explained below. Let us only remark here that an irreducible algebraic variety has a unique generic behaviour, in that any definable subset has lower dimension or a complement of lower dimension; this is typical of stable theories. For definable groups over $\mathbb{Q}_p$,

Problem 1.6. *Let G be a definable group. Then there exist definable normal subgroups $(1) = G_0 \leq \cdots \leq G_n = G$ of G, such that G_{i+1}/G_i is a definable homomorphic image of a constructible group.*

Call a definable set D *purely imaginary* if there exists no definable map (with parameters) of D onto an infinite subset of the field K. Thus the elements of D are tuples of value group elements, residue field elements or elements of the sorts S_n, T_n of [7]; Note that over the algebraic closure L of a local valued field, D is purely imaginary iff $D(L)$ is countable. Call D *boundedly imaginary* if there exists no definable map on D onto an unbounded subset of Γ.

Proposition 1.7. *Let G be a generically metastable group definable in ACVF. Then there exists an algebraic group H over K and a definable homomorphism $f : G \to H(K)$ with boundedly imaginary kernel.*

If D is defined over a local field L, then D is boundedly imaginary iff $D(L')$ is finite for every finite extension of L. Hence

Corollary 1.8. *Let G be a generically metastable group definable in $ACVF_L$, with L a local field. Then there exists a definable homomorphism $f : G(L) \to H(L)$ with H an algebraic group over L, with finite kernel.*

Proposition 1.7 reduces the study of a generically metastable group definable in the field sort in ACVF to that of generically metastable subgroups of algebraic groups H. We proceed to describe these.

There exists an exact sequence $1 \to A \to H \to_f L \to 1$, with A an Abelian variety and L an affine algebraic group. We show (Lemma 4.5) that a definable subgroup G of H is generically metastable if and only if $G \cap A$ and $f(G)$ are. The Abelian variety case falls under the general Lemma 5.1; this will be treated separately. For linear groups, we have:

Theorem 1.9. *Let H be an affine algebraic group, and let G be a generically metastable definable subgroup of H. Then G is isomorphic to $H_1(\mathcal{O})$, H_1 an algebraic group scheme over $\mathcal{O}$.*

If G is Zariski dense in H, H_1 can be taken to be K-isomorphic to H.

Examples include $GL_n(\mathcal{O}); GL_n(\mathcal{O}/a\mathcal{O})$ where $\mathcal{O}$ is the valuation ring; and "congruence subgroups" such as the kernel of $GL_n(\mathcal{O}) \to GL_n(\mathcal{O}/a\mathcal{O})$.

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2. Preliminaries

We recall some material from stability and stable domination, in a form suitable for our purposes.

Let $\mathbb{U}$ be a universal domain (a saturated model.) $\models \phi$ is shorthand for $\mathbb{U} \models \phi$. By a *definable set* or function we mean one defined in T_A for some $A \leq \mathbb{U}$. If we wish to specify the base of definition, we say A-definable, or 0-definable if $A = \emptyset$.

By a *-definable function f, we mean an indexed sequence (f_i) of definable functions. Similarly a *-type.

Notation 2.1. *Given a type p over C with a unique $Aut(\mathbb{U}/C)$-invariant extension $\tilde{p}$ to $\mathbb{U}$, write:*

$a \downarrow_C b$ *if* $a \models \tilde{p}|acl(\{b, C\})$.

(This applies when p, $\tilde{p}$ are Δ-types; in this case, $\tilde{p}|E$ is defined to be the set of E-formulas, that follow from $\tilde{p}$.)

When the identity of C is clear, or when $C = acl(\emptyset)$, we write: $a \downarrow b$.

2.2. **Definable types.** Let Δ be a set of formulas $\phi(x; y)$. Here x is a fixed tuple of variables, while y may range over all variables. Let Δ_x be the Boolean algebra of formulas generated by those of the form $\phi(x; b)$, with $\phi \in \Delta$.

When not other specified, we assume Δ consists of all formulas in the appropriate variables. Though our applications will use this case, definability considerations will occasionally lead us to consider *finite* sets Δ.

By a Δ-definable type p over C we mean a Boolean homomorphism $\phi \mapsto (d_p)\phi$ carrying formulas $\phi(x;y) \in \Delta$ to formulas $(d_p)\phi(y)$ over C; so that if M is a model, then

$$p|C = \{\phi(x,b) : \phi \in \Delta, M \models (d_p)\phi(b)\}$$

Sometimes to specify the variable, when $p = p(x)$, we will write: $d_p x\phi(x,y)$.

We say p *concentrates on* D if $d_p D$ is true.

By a *definable function on* p we mean a definable function on some definable $D \in p$.

If r is a definable function on a definable type p, we define the pushforward r_*p by:

$$(d_{r^*p}u)\phi(u,v) = (d_p x)\phi(r(x),v)$$

So that if $a \models p|C$ then $r(a) \models r_*p|C$.

2.2.1. *Free products of definable types.* Let $p(x)$,$q(y)$ be definable types. Define $r(x,y) = p(x) \otimes r(y)$ by:

$$(d_r xy)\phi(xy,z) = (d_p x)(d_q y)\phi(xy,z)$$

for those formulas ϕ for which the right hand side is defined.

If p,q are complete (i.e. their domains are all formulas), then so is r.

Remark 2.3. *Let ϕ be a formula in variables x,y,z; the same formula may be viewed as $\phi(x;yz)$ or $\phi(y;xz)$ or $\phi(xy;z)$, defining three bipartite graphs. If $\phi(x;yz)$ lies in the domain of p, and $\phi(y;xz)$ lies in the domain of q and is stable, then $\phi(xy;z)$ lies in the domain of $p \otimes q$.*

Proof. $(d_q y)\phi(y;xz)$ is equivalent to a Boolean combination of formulas $\phi(b_i,xz)$; all these lie in $\mathrm{dom}(p)$ by assumption. □

We will occasionally use a more general construction. Assume $p(x)$ is a 0-definable type. Let $a \models p$ and let $q_a(y)$ be an definable type of the theory T_a.

Lemma 2.4. *There exists a unique definable type $r(x,y)$ such that for any C, if $(a,b) \models r|C$ then $a \models p|C$ and $b \models q_a|Ca$.*

Proof. Given a formula $\phi(xy,z)$, let $\phi^*(x,z)$ be a formula such that $\phi^*(a,z) = (d_{q_a}y)\phi(a,y,z)$. ϕ^* is not uniquely defined, but if ϕ',ϕ'' are two possibilities then $(d_p x)(\phi' \equiv \phi'')$. Therefore we can define:

$$(d_r xy)\phi(xy,z) = (d_p x)\phi^*(x,y,z)$$

It is easy to check that this definition scheme works. □

In fact over a model, it suffices that q_a be definable over $\mathrm{acl}(a)$. This follows from:

Lemma 2.5. *Let $M \leq N$ be models, and let $tp(a/N)$ be M-definable. Let $c \in \mathrm{acl}(Ma)$. Then $tp(ac/N)$ is definable over M. Indeed, $tp(a/N) \cup tp(ac/M) \models tp(ac/N)$.*

Proof. Let $\phi(x,y)$ be a formula over M such that $\phi(a,c)$ holds, and such that $\phi(a,y)$ has m solutions, with m least possible. If $\phi(a,y)$ does not imply a complete type over Na, there exists $\psi(u,x,y)$ over M and $d \in N$ such that $\psi(d,a,y)$ implies $\phi(a,y)$, and $\psi(d,a,y)$ has k solutions with $1 \leq k < m$. Since $tp(a/N)$ is M-definable, there exists $d' \in M$ satisfying the p-definition of the formulas below, and hence with

$$(\exists^k y)\psi(d',a,y), (\exists^{m-k}y)(\phi(a,y) \& \neg\psi(d',a,y))$$

But then either $\psi(d',a,c)$ or $\neg\psi(d',a,c)$, contradicting the minimality of m in either case. □

See Proposition 2.10 (4) for a stronger statement in the stably dominated case.

Let p be a definable type. Let $f = f_c(x)$ be a c-definable function.

When we consider definable functions $f(x, y), g(x, z)$, we will assume that the formula $f(x, y) = g(x, z)$ lies in Δ.

Definition 2.6 (Germs of definable functions)**.** *Two definable functions $f(x, b), g(x, b')$ are said to have the same p-germ if*

$$\models (d_p x) f(x, b) = g(x, b')$$

We say that *the p-germ of $f(x, b)$ is defined over C* if whenever $tp(b/C) = tp(b'/C)$, $f(x, b), f(x, b')$ have the same p-germ. Note that the equivalence relation:

"$b \sim b'$ iff $f(x, b), f(x, b')$ have the same p-germ"

is definable; the p-germ of $f(x, b)$ is defined over C iff $b/\sim \in dcl(C)$.

2.7. **Stably dominated types.**

Definition 2.8. *A partial type P is* stably dominated over C *if there exist C-definable maps $\alpha_i : P \to D_i$, D stable, $\alpha = (\alpha_i)_i$, such that $\alpha(a) \downarrow b$ implies*

$$tp(b/\alpha(a)) \models tp(b/a)$$

for any tuple b.

We call a definable set D *stable* if every formula $\phi(x; y)$ with $y = (y_1, \dots, y_m)$, such that ϕ implies $D(y_1) \& \cdots \& D(y_m)$, is stable. This is often referred to as *stable, stably embedded* in the literature. See e.g. [22] for a treatment of basic stability.

A type over C is said to be *stably dominated* if it is stably dominated over C via some α.

Proposition 2.9. *Let p be a complete type over $C = acl(C)$. If p is stably dominated, it has a C-definable extension to $\mathbb{U}$, and this extension is unique.*

Thus the $\downarrow$ - notation is applicable, as well as the notion of a p-germ.

Proposition 2.10. *Let $p = tp(a/C)$ be stably dominated.*

(1) (Symmetry) If $tp(b/C)$ is also stably dominated, $a \downarrow_C b$ iff $b \downarrow_C a$

(2) (Transitivity) $a \downarrow_C bd$ iff $a \downarrow_C b$ and $a \downarrow_{acl(Cb)} d$.

(3) (Base change) If $a \downarrow_C b$, then $tp(a/acl(Cb))$ is stably dominated.

(4) If $tp(d/C)$ and $tp(b/acl(Cd))$ are stably dominated, then so is $tp(bd/C)$. Conversely if $a \in dcl(Cb)$ and $tp(b/C)$ is stably dominated, then so is $tp(a/C)$.

(5) For any formula $\phi(x, y)$, $(d_p x \phi)$ is a positive Boolean combination of formulas $\phi(a_i, y)$, where the $a_1 \models p|C$, $a_2 \models p|Ca_1$, etc.

Proposition 2.11 (Descent)**.** *Let p, q be $\mathrm{Aut}(\mathbb{U}/C)$-invariant $\star$-types. Assume that whenever $b \models q|C$, the type $p|Cb$ is stably dominated. Then p is stably dominated.*

Question 2.12. *Can the descent lemma be proved without the additional hypothesis (E) on existence of invariant types? Does (E) follow from metastability over an o-minimal Γ?*

Proposition 2.13 (The strong germ lemma)**.** *Let p be stably dominated. Assume p as well as the p-germ of $f(x, b)$ are defined over $C = acl(C)$. Then there exists a C-definable function g with the same p-germ as $f(x, b)$.*

Proposition 2.14. *A definable type p is stably dominated iff for any definable function g on p into Γ, the p-germ of g is constant.*

Write $g(p)$ for the constant value of the p-germ. The property of p in the theorem is referred to as *orthogonality of p to Γ*. Note that this is strictly weaker than orthogonality of D to Γ for some definable $D \in p$.

See [8] for proofs of these propositions. Proposition 2.10 (4) is Lemma 5.10 there. Symmetry, transitivity and base change are easy consequences of the corresponding facts in stable theories; but descent (Theorem 9.3 there) is more difficult, and uses the additional hypothesis of extendibility of types to invariant types.

Lemma 2.15. *(FD) Let M be a good base, and let c be a (finite) tuple. Then $\Gamma_M(c)$ and $St_M(c)$ are are acl- finitely generated over M.*

Proof. Any tuple $d \in St_M(c)$ can be written $d = h(c)$ for some M-definable function. By the dimension bound in (FD), $\dim(d/M) \leq \dim_{st}^{def}(c/M)$. It follows that if $d_1, d_2, \ldots \in St_M(c)$ then $d_n \in \mathrm{acl}(M(d_1, \ldots, d_{n-1})$ for all sufficiently large n. So $St_M(c)$ is finitely generated in the sense of algebraic closure. □

Say $tp(a/C)$ is *strongly stably dominated* if there exists $\phi(x) \in tp(a/St_C(a))$ such that for any tuple b with $St_C(a) \downarrow b$, $tp(a/Cb)$ is isolated via ϕ. Equivalently, $tp(a/C)$ is stably dominated via some definable h, and $tp(a/C, h(a))$ is isolated. (We then say that $tp(a/C)$ is strongly stably dominated via h.)

Lemma 2.16. *(FD_ω) Let D be a formula over C_0, $h : D \to S$ a definable map to a stable definable set of maximal possible dimension $\dim(S) = \dim_{st}(D)$. Then there exists M containing C_0 and $a \in D$ such that with $\Gamma_M(a) = C$ we have $\dim_{st}(D) = \dim St_C(a)/C = \dim St_M(a)/M$, and $tp(a/C)$ is strongly stably dominated via h. (In particular, $tp(a/St_C(a))$ is isolated.)*

Proof. We may assume h is defined over C_0. Let $a_0 \in D$ be such that $\dim_{st}(D) = \dim h(a_0)/C_0$; extend C_0 to a good base M with $h(a_0)$ independent from M; then $\dim_{st}(D) = \dim St_M(a_0)/M = \dim h(a_0)/M$. Choose a_0 such that $\dim_o \Gamma_M(a_0)/M$ is as large as possible (given the other constraints.) Let $C = \Gamma_M(a_0)$; then C is acl- finitely generated over M, and $\Gamma_C(a_0) = C$. Let $C' = St_C(a_0)$. Let D' be a formula over C' implying D and such that for any a' with $D'(a')$, $C' \subseteq \mathrm{acl}(M(a'))$ (Lemma 2.15.) By (FD_ω) there exists $a \in D'$ such that $tp(a/C')$ is isolated. By choice of D' we have $C' \subseteq St_C(a)$ and hence $St_{C'}(a) \subseteq St_C(a)$. But then $\dim(St_{C'}(a)/C) \leq \dim_{st}(D) = \dim(C'/C)$ so $St_C(a) \subseteq \mathrm{acl}(C')$. Since $St_C(a)$ is acl-finitely generated over C, $tp(St_C(a)/C')$ is algebraic and in particular $St_C(a)/C'(a)$ is atomic, i.e. any tuple from $St_C(a)$ realizes an isolated type over $C'(a)$. But $tp(a/C')$ is isolated, so $tp(a, St_C(a)/C'(a))$ is atomic and hence $tp(a/St_C(a))$ is isolated. Similarly by maximality of $\dim_o \Gamma_M(a_0)/M$ and the fact that $\Gamma_M(a_0) \subseteq C' \subseteq \Gamma_M(a)$ we have $\Gamma_M(a_0) = C$. By metastability, $tp(a/C)$ is stably dominated; since $tp(a/C')$ is isolated and $C' = St_C(a)$, $tp(a/C)$ is strongly stably dominated. □

Lemma 2.17. *(FD) Let $f : P \to Q$ be a definable map between definable sets P, Q. Let $P_a = f^{-1}(a)$. Then there exists m such that if P_a is finite, then $|P_a| \leq m$.*

Proof. Say f, P, Q are 0-definable. By compactness, it suffices to show that if P_a is infinite then there exists a 0-definable set Q' with $a \in Q'$ and P_b infinite for all $b \in Q'$.
Claim. If P_a is infinite then either $\dim_{st}(P_a) > 0$ or $\dim_o(P_a) > 0$.

Proof. Let M be a good base, with $a \in M$. Let $c \in P_a \setminus M$. If $\Gamma_M(c) \neq M$ then $\dim_o(P_a) > 0$. Otherwise, by metastability, $tp(c/M)$ is stably dominated, say via f. If $f(c) \in M$, then $tp(b/M) \implies tp(b/Mc)$ for all b, and taking $b = c$ it follows that $c \in M$. Thus $f(c) \in St_M \setminus M$. It follows that $dim_{st}(P_a) > 0$. □

If $\dim_{st}(P_a) > 0$, then there exists a definable family of stable definable sets D_t with $\dim(D_t) = k > 0$, and a definable function $f(x,u)$ such that for some b and t, $f(P_a, b) = D_t$. Then the formula $(\exists u)(\exists t) f(P_y, u) = D_t$ is true of $y = a$, and implies that P_y is infinite. The case $\dim_o(P_a) > 0$ is similar. □

2.18. **Some o-minimal lemmas.** This subsection contains some lemmas on o-minimal partial orders and groups. The former will yield a definable generic type of limit metastable groups. The latter will be used to improve some statements from "almost internal" to "internal".

Let P be a partial ordering, and p a definable type. We say p is cofinal if for any $c \in P$, $\models (d_p x)(x \geq c)$. Equivalently, for every non-cofinal parametrically definable $Q \subseteq P$, $\models \neg(d_p x) Q(x)$.

Lemma 2.19. *Let P be a definable directed partial ordering in an o-minimal structure Γ. Then there exists a definable type p cofinal in P.*

Proof. We assume P is 0-definable, and work with 0-definable sets; we will find a 0-definable type with this property.

Note first that we may replace P with any 0-definable cofinal subset. Also if Q_1, Q_2 are non-cofinal subsets of P, there exist a_1, a_2 such that no element of Q_i lies above a_i; but by directedness there exists $a \geq a_1, a_2$; so no element of $Q_1 \cup Q_2$ lies above a, i.e. $Q_1 \cup Q_2$ is not cofinal. In particular if $P = P' \cup P''$, at least one of P', P'' is cofinal in P (hence also directed.)

If $\dim(P) = 0$ then P is finite, so according to the above remarks we may assume it has one point; in which case the lemma is trivial. We use here the fact that in an o-minimal theory, any point of a finite 0-definable set is definable.

If $\dim(P) = n > 0$, we can divide P into finitely many 0-definable sets P_i, each admitting a map $f_i : P_i \to \Gamma$ with fibers of dimension $< n$ (where Γ is o-minimal.) We may thus assume that there exists a 0-definable $f : P \to \Gamma$ with fibers of dimension $< n$. Let $P(\gamma) = f^{-1}(\gamma)$, and $P(a,b) = f^{-1}(a,b)$.

Claim 1. One of the following holds:

(1) For any $a \in \Gamma$, $P(a, \infty)$ is cofinal in P.
(2) For some 0-definable $a \in \Gamma$, for all $b > a$, $P(a,b)$ is cofinal.
(3) For some 0-definable $a \in \Gamma$, $P(a)$ is cofinal.
(4) For some 0-definable $a \in \Gamma$, for all $b < a$, $P(b,a)$ is cofinal.
(5) For all $a \in \Gamma$, $P(-\infty, a)$ is cofinal.

Proof. Suppose (1) and (5) fail. Then $P(a, \infty)$ is not cofinal in P for some a; so $P(-\infty, b)$ must be cofinal, for any $b > a$. Since (5) fails, $\{b : P(-\infty, b) \text{ is cofinal }\}$ is a nonempty proper definable subset of Γ, closed upwards, hence of the form $[A, \infty)$ or (A, ∞) for some 0-definable $A \in \Gamma$. In the former case, $P(-\infty, A)$ is cofinal, but $P(-\infty, b)$ is not cofinal for $b < A$, so $P(b, A)$ is cofinal for any $b < A$; thus (4) holds. In the latter case, $(-\infty, b)$ is cofinal for any $b > A$, while $(-\infty, A)$ is not; so $P([A, b))$ is cofinal for any $b > A$. Thus either (2) or (3) hold. □

Let p_1 be a 0-definable type of Γ, concentrating on sets X with $f^{-1}(X)$ cofinal. (For instance in case (1) p_1 concentrates on intervals (a, ∞).)

Claim 2. For any $c \in P$, if $a \models p_1 | \{c\}$ then there exists $d \in P(a)$ with $d \geq c$.

Proof. Let $Y(c) = \{x : (\exists y \in P(x))(y \geq c)\}$. Then the complement of $Y(c)$ is not cofinal in P, so it cannot be in the definable type p_1. Hence $Y(c) \in p_1 | \{c\}$. □

Now let $M \models T$. Let $a \models p_1 | M$. By induction, let q_a be an a-definable type, cofinal in $P(a)$, and let $b \models q_a | Ma$. Then $tp(ab/M)$ is definable (Lemma 2.4). If $c \in M$ then by Claim 2, there exists $d \in P(a)$ with $d \geq c$. So $\{y \in P(a) : \neg(y \geq c)\}$ is not cofinal in $P(a)$. Therefore this set is not in q_a. Since $b \models q_a|Ma$, we have $b \geq c$. This shows that $tp(ab/M)$ is cofinal in P. □

Discussion. If there exists a definable weakly order preserving map $j : \Gamma \to P$ with cofinal image, then we can use the definable type at ∞ of Γ, r_∞, to obtain a cofinal definable type of P, namely $j_* r_\infty$.

When Γ admits a field structure, perhaps such a map j always exists. In general, it is not always possible to find a one-dimensional cofinal subset of P. For instance, when Γ is a divisible ordered Abelian group, consider the product of two closed intervals of incommensurable sizes; or the subdiagonal part of a square.

At all events, one can define the limit of a function along a definable type, generalizing the limit along a one-dimensional curve.

Definition 2.20. *Given a definable function $g : D \to \Gamma$ and a definable type p of elements of D, define $\lim_p g = \gamma \in \{-\infty, \infty\} \cup \Gamma$ if for any neighborhood U of γ in the order topology, for generic $c \models p$ we have $g(c) \in U$.*

To see that the limit always exists, consider the definable type $g_* p$ on Γ itself. The definable types of elements of Γ are r_∞, $r_{-\infty}$, $r_a = (x = a)$, r_a^+ the type of elements infinitesimally bigger than a, and r_a^-. By definition $\lim_p g$ is $\infty, -\infty, a, a, a$ in the respective cases.

Given a definable space X over an o-minimal structure Γ, a definable set D, a definable type p on D, and a definable $g : D \to X$, we can define $\lim_p g = x \in X$ if for any neighborhood U of x, $g_* p$ concentrates on U. The following definition is equivalent to the one in [21].

Definition 2.21. *X is* definably compact *if for any definable type p on Γ and any definable function $f : \Gamma \to X$, $\lim_p f$ exists as a point in X.*

Recall that definable set X is Γ-*internal* if $X \subseteq \mathrm{dcl}(\Gamma, F)$ for some finite set F; equivalently for any $M \prec M' \models T$, $X(M') \subseteq \mathrm{dcl}(M \cup \Gamma(M'))$. The same condition with acl replacing dcl is called *almost internality*; thus X is almost Γ-internal if X is Γ'-internal for some Γ' defined over parameters F', and admitting a definable m-to-one function $f : \Gamma' \to Y$ into some definable $Y \subseteq \mathrm{dcl}(\Gamma^n)$, for some finite m.

Lemma 2.22. *Let G be a definable group. Assume G is almost internal to a stably embedded definable set Γ. Then there exists a finite normal subgroup N of G with G/N internal to Γ.*

Proof. (cf. [2], [9]). We work in a saturated model $\mathbb{U}$, possibly over parameters for a small elementary submodel. The assumption implies the existence of a definable finite-to-one function $f : G \to Y$, where $Y \subseteq \mathrm{dcl}(\Gamma)$. Given a definable $Y' \subseteq Y$, let $m(Y')$ be the least integer m such that over further parameters, there exists a definable m-to-one map $f^{-1}(Y') \to Z$, for some definable $Z \subseteq \mathrm{dcl}(\Gamma)$. Let I be the family of all definable subsets Y' of Y with ($Y = \emptyset$ or) $m(Y') < m(Y)$. This is clearly an ideal (closed under finite unions, and definable subsets). Let $F = \{Y \setminus Y' : Y' \in I\}$ be the dual filter. For $g \in G$, let $D(g)$ be the set of $y \in Y$ such that for some (necessarily unique) y', $gf^{-1}(y) = f^{-1}(y')$; and define $g_* y = y'$. The function $x \mapsto (f(y), f(gy))$ shows that $\{y : |f(gf^{-1}(y))| > 1\} \in I$; equivalently, $D(g) \in F$. Let $F|Z = \{W \cap Z : W \in F\}$. Let G_0 be the set of bijections $\phi : Y' \to Y''$ with $Y', Y'' \in F$, carrying the filter $F|Y'$ to $F|Y''$. Write $\phi \sim \phi'$ if ϕ, ϕ' agree on some common subset of their domains, lying in F; and let $G' = G_0/\sim$. Composition induces a group structure on G'. The function g_* on $D(g)$ lies in G_0, and we obtain a homomorphism $G \to G'$, $g \mapsto g_*/\sim$. Let N be the kernel of this homomorphism. Clearly $Aut(\mathbb{U}/\Gamma)$ fixes G' and hence G/N pointwise. It remains only to show that N is finite. In fact $|N| \leq m!$. For suppose $n_0, \ldots, n_{m!}$ are distinct elements of N. Then for some $y \in \cap D(n_i)$ we have $(n_i)_*(y) = y$. It follows that $n_i(f^{-1}(y)) = f^{-1}(y)$, and since $|f^{-1}(y)| = m$, for some $i \neq j$ we have $n_i g = n_j g$ for all $g \in f^{-1}(y)$. But then $n_i = n_j$, a contradiction. □

Lemma 2.23. *Let G be a definable group. Assume G is almost internal to an o-minimal definable set Γ. Then G is Γ-internal.*

Proof. By Lemma 2.22, there exists a definable surjective homomorphism $f : G \to B$ with B a group definable over Γ, and $N = \ker(f)$ a group of finite size n.

Let B^0 be the connected component of the identity in B; then B/B^0 is finite, and it suffices to prove the lemma for $f^{-1}(B^0)$. Assume therefore that B is connected.

If G has a proper definable subgroup G_1 of finite index, then $f(G_1) = B$ by connectedness of B. It follows that N is not contained in G_1, so $N_1 = N \cap G_1$ has smaller size than N. Hence using induction on the size of the kernel, G_1 is Γ-internal; hence so is G. Thus we may assume G has no proper definable subgroups of finite index.

Since the action of G on N has kernel of finite index, N must be central.

Let $Y = \{g^n : g \in B\}$. By [4], Theorem 7.2, there exists a definable function $\alpha : Y \to B$ with $n\alpha(b) = b$. Define $h : Y \to A$ by $h(b) = na$ where $f(a) = \alpha(b)$; this does not depend on the choice of a, and we have $fh(b) = nf(a) = ng(b) = b$. It follows that $f^{-1}(Y) = Nh(Y) \subseteq \mathrm{dcl}(N, \Gamma)$ is Γ-internal.

Similarly, let $[B, B] = \{[g, h] : g, h \in B\}$. Again there exists a definable $\alpha_1 : [B, B] \to B$ such that $(\exists y)[\alpha_1(b), y] = b$, and $\alpha_2 : [B, B] \to B$ such that $[\alpha_1(b), \alpha_2(b)] = b$. Define $h' : [B, B] \to A$ by $h'(b) = [a_1, a_2]$ where $f(a_i) = \alpha_i(b)$. Again h' is definable and well-defined, and shows that $h^{-1}[B, B]$ is Γ-internal.

Hence for any k, letting $X^{(k)} = \{x_1...x_k : x_1, ..., x_k \in X\}$, $(h^{-1}(Y \cup [B, B]))^{(k)} = h^{-1}((Y \cup [B, B])^{(k)})$ is Γ-internal. So we are done once we show:

Claim. Let B be any definably connected group definable in an o-minimal structure. Let $Y = Y_n(B) = \{g^n : g \in B\}$. Then for some $k \in \mathbb{N}$, $(Y \cup [B, B])^{(k)} = B$.

Proof If the Claim holds for a normal subgroup H of B with bound k', and also for B/H (with bound k''), then it is easily seen to hold for B (with bound $k' + k''$.)

We use induction on $\dim(B)$. If B has a nontrivial proper connected definable normal subgroup H, then the statement holds for H and for B/H. We may thus assume B has no such subgroups H

If B is Abelian then in fact $Y = B$, by [27]. Otherwise the center Z of B is finite. In this case by the same argument as above, the claim holds with $k = k_{B/Z} + |Z|$. By connectedness B has no nontrivial finite normal subgroups. Hence B is definably simple.

Now by [19], B is elementarily equivalent to a Lie group. So we may assume B is a simple Lie group. In this case it is known that every element is the product of a bounded number of commutators.

□

Let G be an Abelian group. A definable set is called *generic* if finitely many translates cover the group. Say G has the property (NGI) if the non-generic definable sets form an ideal.

It is shown in [18] that in any definably compact group in an o-minimal theory, (NG) holds; moreover any definable subsemigroup is a group. We include the deduction of the latter fact.

Lemma 2.24. *Let G be a group with (NG). Let Y be a definable semi-group of G, such that $Y - Y = G$. Then $Y = G$.*

Proof. Note first that $G \setminus Y$ is not generic.

Otherwise, some finite intersection $\cap_{i=1}^n (c_i + Y) = \emptyset$. By assumption, there exist $b_i \in Y$ with $b_i + c_i \in Y$; so if $b = \sum b_i$ then $b + c_i \in Y$; so by translating we may assume each $c_i \in Y$. But then $\sum c_i \in c_i + Y$ for each i, a contradiction.

Hence Y is generic. So $\cup_{i=1}^m d_i + Y = G$ for some $d_i \in G$. Again we find $e \in Y$ with $e + d_i \in Y$. So $\cup_{i=1}^n (e + d_i) + Y = G$. But $e + d_i + Y \subseteq Y$. So $Y = G$. □

2.25. **Valued fields: imaginaries and resolution.** Let K be an algebraically closed valued field, with valuation ring $\mathcal{O}_K$. The geometric language for valued fields has a sort for the valued field itself, and certain other sorts. In particular, there is a sort S_n such that $S_n(K)$ is the space of for the space of free $\mathcal{O}_K$-modules in K^n, or $GL_n(K)/GL_n(\mathcal{O}_K)$.

We let $M(K)$ be the set of K-points of all these sorts.

By a *substructure*, we mean a subset of $M(K)$, closed under the quantifier-free definable functions.

A substructure A of $M(K)$ is called *resolved* if any element of $S_n(A)$ - viewed as a lattice in K^n - has a basis in $K(A)$. When $\Gamma(A) \neq 0$ and A is algebraically closed this just amounts to saying that $A \prec M(K)$.

Recall in any first order theory that if A is a substructure of a model M, M is *prime* over A if any elementary map $A \to N$ into another model, extends to an elementary map $M \to N$; and *minimal* if there is no $M' \prec M$ with $A \subset M$. If a minimal model over A and a prime one exist, then any two minimal or prime models over A are isomorphic.

Proposition 2.26. *Let A be a substructure of $M(K)$, finitely generated over a subfield L of K, and assume $\Gamma(A) \neq 0$. Then there exists a minimal prime model $\widetilde{\mathbf{A}}$ over A.*

$\widetilde{\mathbf{A}}$ enjoys the following properties.

(1) *$\widetilde{\mathbf{A}}$ is a minimal resolution of A. Moreover it is the unique minimal resolution, up to isomorphism over A. It is atomic over A.*
(2) *$St_L(\widetilde{\mathbf{A}}) = St_L(A)$.*
(3) *Let $A \leq A'$, with A'/A finitely generated. Then $\widetilde{\mathbf{A}}$ embeds into $\widetilde{\mathbf{A}}'$ over A. If $A \leq A' \leq \widetilde{\mathbf{A}}$, then $\widetilde{\mathbf{A}}$ is the prime resolution of A'.*
(4) *Let $L'(A)$ be the structure generated by $L' \cup A$. Then $L'(\widetilde{\mathbf{A}})^{alg}$ is a prime resolution of $L'(A)$.*
(5) *If A/L is stably dominated, then $\widetilde{\mathbf{A}}/L$ is stably dominated.*
(6) *If A'/A is stably dominated, and $A' \downarrow_A \widetilde{\mathbf{A}}$, then $\widetilde{\mathbf{A}}'/\widetilde{\mathbf{A}}$ is stably dominated.*

Proof. The existence, uniqueness and minimality of $\widetilde{\mathbf{A}}$ are [8], Theorem 11.14.

It is also shown there that $k(\widetilde{\mathbf{A}}) = k(\mathrm{acl}(A))$ and $\Gamma(\widetilde{\mathbf{A}}) = \Gamma(\mathrm{acl}(A))$, where k is the residue field; and that $\widetilde{\mathbf{A}}/A$ is atomic, i.e. $tp(c/A)$ is isolated for any tuple c from A.

Uniqueness of the minimal resolution: Let B be a minimal resolution of A. Then the prime resolution $\widetilde{\mathbf{A}}$ embeds into B over A. As B is minimal, this embedding is an isomorphism.

Since L is a field, for any $B = acl(B)$ with $L \subset B$, $St_L(B) = dcl(B, k(B))$. This proves (2).

(3) is immediate from the definition of prime resolution: since $\widetilde{\mathbf{A}}'$ is a resolution of A, $\widetilde{\mathbf{A}}$ embeds into $\widetilde{\mathbf{A}}'$. If $A \leq A' \leq \widetilde{\mathbf{A}}$, then $\widetilde{\mathbf{A}}$ is clearly a minimal resolution of A'; hence by (1) it is the prime resolution.

(4) Let B be the prime resolution of $L'(A)$. Then $\widetilde{\mathbf{A}}$ embeds into B. Within B, $L'(\widetilde{\mathbf{A}})^{alg}$ is a resolution of $L'(A)$; by minimality of B, $B = L'(\widetilde{\mathbf{A}})^{alg}$.

(5) We may assume $L = L^{alg}$. Let $\bar{L}$ be a maximal immediate extension of L. Choose it in such a way that $tp(\widetilde{\mathbf{A}}/\bar{L})$ is a sequentially independent extension of $tp(\widetilde{\mathbf{A}}/L)$.

Since A/L is stably dominated, $\Gamma(\bar{L}(A)^{alg}) = \Gamma(\bar{L})$. According to (4), $B\bar{L}(\widetilde{\mathbf{A}})^{alg}$ is the prime resolution of $\bar{L}(A)^{alg}$, and so $\Gamma(\bar{L}(\widetilde{\mathbf{A}})^{alg}) = \Gamma(\bar{L})$. Thus by [8], Theorem 10.12, $tp(\widetilde{\mathbf{A}}/\bar{L})$ is stably dominated. Using descent, Proposition 2.11, $tp(\widetilde{\mathbf{A}}/L)$ is stably dominated.

(6) Let $\widetilde{\mathbf{A}}$ be a prime resolution of A, with $A' \downarrow_A \widetilde{\mathbf{A}}$. Then $tp(A'/\widetilde{\mathbf{A}})$ is stably dominated. By (5), $tp(B/\widetilde{\mathbf{A}})$ is stably dominated, where B is a resolution of $\widetilde{\mathbf{A}}(A')$. Since $\widetilde{\mathbf{A}}'$ embeds into B over A', $tp(\widetilde{\mathbf{A}}'/\widetilde{\mathbf{A}})$ is stably dominated.

□

Let C be a substructure of $M \models ACVF$, and let e be an (imaginary) element. Call e *purely imaginary* over C if $\mathrm{acl}(C(e))$ contains no field elements other than those in $\mathrm{acl}(C)$. If $\alpha \in \Gamma$, let $\alpha\mathcal{O} := \{x \in K : \mathrm{val}(x) \geq \alpha\}$, and $\alpha\mathcal{M} := \{x \in K : \mathrm{val}(x) > \alpha\}$.

Lemma 2.27. *The following are equivalent:*
(1) e is purely imaginary
(2) $C(e) \cap K \subseteq \mathrm{acl}(C)$
(3) For some $\beta_0 \leq 0 \leq \beta_1 \in \Gamma(C(e))$, $e \in \mathrm{dcl}(C, \beta_0, \beta_1, \beta_0\mathcal{O}/\beta_1\mathcal{M})$.

Proof. A nonempty ball in K cannot be a definable image of $(\beta R/\alpha R)^n$, since in some models of ACVF, $R/\alpha R$ is countable while every ball is countable. Hence any such definable image is finite. Thus (3) implies (2).

(2) implies (1): Let $d \in \mathrm{acl}(C(e))$ be a field element. The finite set of conjugates of d over C is coded by a tuple d' of field elements. By Galois theory, $d' \in C(e)$. By (2), $d' \in \mathrm{acl}(C)$. Since $d \in \mathrm{acl}(d')$ we have $d \in \mathrm{acl}(C)$.

(1) implies (2) trivially.

(2) implies (3): By [7], and using (2), there exists an e-definable $\mathcal{O}$-submodule Λ of K^m (for some m), such that e is a canonical parameter for Λ (over $\mathrm{acl}(C)$.) The K-space $V = K \otimes_{\mathcal{O}} \Lambda$ is coded by an element w of some Grassmanian $G_{m,l}$; by (2) we have $w \in K_0 := K \cap \mathrm{acl}(C)$; since K_0 is an algebraically closed field, V is K_0-isomorphic to K^l, so we may assume $V = K^l$.

Dually, let $V' = \{v \in V : Kv \subseteq \Lambda\}$. Then V' is a $K \cap \mathrm{acl}(C(e)) = K_0$-definable K-vector subspace of V. Replacing Λ by the image in V/V', we may assume $V' = (0)$. It follows ([7]) that $V' \subseteq \beta_0\mathcal{O}^l$ for some $\beta_0 < 0$.

Let $v_1, \dots, v_l$ be a standard basis for $V = K^l$. Since $v_i \in K \otimes_{\mathcal{O}} \Lambda$, we have $c_i v_i \in \Lambda$ for some $c_i \in \mathcal{O}$. So $\sum r_i c_i v_i \in \Lambda$ for all $r_1, \dots, r_l \in \mathcal{O}$. Let $\Lambda'(\beta) = \{\sum r_i v_i : \mathrm{val}(r_i) > \beta\}$. Then $\Lambda'(\beta) \subseteq \Lambda$ for sufficiently large β. (Namely for $\beta \geq \max \mathrm{val}(c_i)$). $\{\beta : \Lambda'(\beta) \subseteq \Lambda\}$ is definable, hence has the form $\{\beta : \beta \geq \beta_1\}$ for some $C(e)$-definable β_1.

Thus Λ is determined by its image in $\beta_0\mathcal{O}/\beta_1\mathcal{M}$. Pick $c \in \beta_0\mathcal{O}/\beta_1\mathcal{M}$; then $r \mapsto rc$ is an isomorphism $\mathcal{O}/(\beta_1 - \beta_0)\mathcal{M} \to b_0\mathcal{O}/b_1\mathcal{M}$. By [14], $\mathcal{O}/(\beta_1 - \beta_0)\mathcal{M}$ is stably embedded; hence so is $\beta_0\mathcal{O}/\beta_1\mathcal{M}$. Thus $e \in \mathrm{dcl}(C, \beta_0\mathcal{O}/\beta_1\mathcal{M})$.

□

Lemma 2.28. *Any definable function $f : (\alpha\mathcal{O}/\beta\mathcal{M})^n \to \Gamma$ is bounded.*

Proof. Suppose first $n = 1$. Since parameters are allowed, we may assume $\alpha = 0$, and consider $f : \mathcal{O}/\beta\mathcal{M} \to \Gamma$, defined over C. Let q be the type of elements of Γ greater than any element of $\Gamma(C)$. For $\gamma \models q$, let $X(\gamma)$ be the pullback to $\mathcal{O}$ of $f^{-1}(\gamma)$. This is a finite Boolean combination of balls of valuative radii $\delta_1(\gamma), \dots, \delta_m(\gamma)$, with $0 \leq \delta_i(\gamma) \leq \beta$. But any C-definable function into a bounded interval in Γ is constant on q. Thus $X(\gamma)$ is a finite Boolean combination of balls of constant valuative radii $\delta_1, \dots, \delta_m$. However it is shown in [7] that any definable function on a finite cover of Γ into balls of constant radius has finite image. Hence $X(\gamma)$ is constant on q. But if if $X(\gamma) = X(\gamma') \neq \emptyset$ then for any $x \in X(\gamma)$, $\gamma = f(x + \beta\mathcal{M}) = \gamma'$. It follows that $X(\gamma) = \emptyset$ for $\gamma \models q$, i.e. f is bounded.

Now given $f : (\alpha\mathcal{O}/\beta\mathcal{M})^2 \to \Gamma$, let $F(x) = \sup\{f(x,y) : y \in \alpha\mathcal{O}/\beta\mathcal{M}\}$. Then F is bounded, so f is bounded. This shows the case $n = 2$, and the general case is similar. $\square$

Corollary 2.29. *Let D be a C- definable set in ACVF. Then the following are equivalent:*

(1) There exists a definable surjective map $g : (\mathcal{O}/\beta\mathcal{O})^n \to D$.

(2) There is no definable function $f : D \to \Gamma$ with unbounded image.

(3) For some $\beta_0 \leq 0 \leq \beta_1 \in \Gamma(C)$, for any $e \in D$, $e \in \mathrm{dcl}(C, \beta_0\mathcal{O}/\beta_1\mathcal{M})$.

An ∞-definable set D satisfying (3) will be called *boundedly imaginary*. By compactness, D is then contained in a definable set D' satisfying (1-3).

Proposition 2.30. *ACVF is metastable, with (FD) and (FD$_\omega$).*

Proof. Let P be a partial type, and let M be a maximally complete algebraically closed valued field, with P defined over M. Let $\gamma(a)$ enumerate $\Gamma(M,a) = dcl(M,a) \cap \Gamma$. Then $tp(a/\gamma(a))$ is stably dominated by Theorem 10.12 of [8].

To see that (FD$_\omega$) holds, let $M \models ACVF$, C finitely generated over M (imaginary sorts allowed.) The resolution N of C is a model of ACVF, and is atomic over M (Lemma 2.26). Hence over C are dense. $\square$

Finally, a characterization of independence of stably dominated types of field elements in ACVF in terms of a "maximum modulus" principle.

Proposition 2.31. *(Maximum modulus) Let p be a stably dominated C-definable type on an affine variety V defined over $K \cap C$, $P = p|C$. Let F be a regular function on V over $L \supset C$. Then* $\mathrm{val}F$ *has an infimum $\gamma^F_{\min} \in \Gamma(C)$ on P. Moreover for $a \models P$, $a \models p|L$ iff* $\mathrm{val}F(a) = \gamma^F_{\min}$ *for all such F.*

Proof. [8] Theorem 14.12. $\square$

Corollary 2.32. *Let U, V be varieties over the algebraically closed valued field C. Let p, q be a stably dominated types over C of elements of U, V respectively. Let F be a regular function on $U \times V$. Then there exists $\gamma = \gamma_F \in \Gamma$ such that*

(1) If $(a,b) \models p \otimes q$ then $\mathrm{val}F(a,b) = \gamma$.

(2) For any $a \models p$, $b \models q$, we have $\mathrm{val}F(a,b) \geq \gamma$.

(3) Assume U, V are affine. If $a \models p, b \models q$ and $\mathrm{val}F(a,b) = \gamma_F$ *for all regular F on $U \times V$, then $(a,b) \models p \otimes q$.*

Proof. U admits a finite cover by open affines, and p concentrates on one of these affines; so we may assume U is affine. Let $b \models q$. Then the statement follows from Theorem 2.31. (See [8] Theorem 14.13.) $\square$

3. Groups with definable generics

Let us clarify the notion of an ∞-definable group G. The required data is a sequence $G_1 \supset G_2 \supset \ldots$ of definable sets and maps, and a definable map $m : (G_2)^2 \to G_1$, such that $m(G_{n+1})^2 \subset G_n$. Composition gives two maps $G_3 \to G_1$, $m(m(x,y),z)$ and $m(x, m(y,z))$; we assume they are equal. Further, there exists a unit element $1 \in \cap_n G_n$, $m(1,x) = m(x,1) = 1$; and an inverse map $x \mapsto x^{-1}$ $(G_2 \to G_2)$, such that $m(x, x^{-1}) = m(x^{-1}, x) = 1$. For $x_1, \ldots, x_n \in G_n$, we can then write unambiguously $x_1 \cdot \ldots \cdot x_n$ to denote their product (but not for more than n elements.)

Let $G = \cap_n G_n$: in the sense that in any model M, we set $G(M) = \cap_n G_n(M)$; so that $G(M)$ is a group (with multiplication m.) Two ∞-definable groups G, H are considered equal if $H(M) = G(M)$ for all M; equivalently, the $\{H_n\}, \{G_n\}$ are isomorphic as directed limit systems.

Essentially everything we will say goes through for $\star$-definable groups; this notion differs only in that the inclusion maps $G_{n+1} \to G_n$ are replaced by arbitrary definable maps. For simplicity, we will deal explicitly with ∞-definable groups. Thus a 'group' is ∞-definable unless otherwise stated.

Let G be a group, and let Δ be a set of formulas $\phi(x; y)$. Here x refers to elements of G, while y may range over G^n or elsewhere. Let Δ_x be the Boolean algebra of formulas generated by those of the form $\phi(x; b)$, with $\phi \in \Delta$. *We assume Δ_x is closed under left translations and inversion* .

Note that if $\Delta' = \{\phi(y_1 x y_2, y_3) : \phi(x, y) \in \Delta\}$, then Δ'_x is left- and right- translation invariant.

If p is a definable type over C, we let ${}^a p$ (the left translate of p by a) be the definable type such that for any $C' \supset C \cup \{a\}$,

$$d \models p|C' \text{ iff } ad \models {}^a p|C'$$

Similarly the right translate p^b.

Note that ${}^b p$ is Cb-definable.

Definition 3.1. *Let G be a group, p a definable Δ- type of elements of G . p is* left-generic *in G if for any $C = acl(C)$ with p defined over C, and $b \in G$, p^b is defined over C. p is* right-generic *if for any such C, b, ${}^b p$ is defined over C.*

Lemma 3.2. *Right generics have boundedly many left translates. If Δ is finite, right Δ-generics have finitely many left-translates.*

Proof. the first statement is immediate, since only boundedly many types are C-definable. The second statement follows by compactness, since when Δ is finite, the equivalence relation: ${}^b p ={}^{b'} p$ is definable: it holds iff for each $\phi(x, y) \in \Delta$, $\models (\forall y)(d_p x)(\phi(b'x, y) \equiv \phi(bx, y))$. ☐

Call a definable type p *symmetric* if whenever q is a definable type, p, q definable over C, $b \models q|C$, $a \models p|acl(C \cup \{b\})$, then $b \models q|acl(C \cup \{a\})$. (We will see that stably dominated types are symmetric in this sense.)

Lemma 3.3. *Any symmetric left generic is right generic. Any two symmetric generics differ by a left translation. If Δ is finite, and a symmetric generic Δ-type exists, there exists a definable group G^0_Δ of finite index stabilizing all generics. This group has no Δ-definable subgroups of finite index.*

Proof. Let p be a left generic. Using the inverse map $x \mapsto x^{-1}$, we see that G also has right generics. Let q be any right generic of G. Let C be an elementary submodel with p, q defined over C. Let $a \models p|C$, $b \models q|acl(C \cup \{a\})$. Then $ab \models^a q|(C \cup \{a\})$, by definition of ${}^a q$. By right genericity of q, ${}^a q$ is C-definable. By symmetry, $a \models p|acl(C \cup \{b\})$. So $ab \models p^b|(C \cup \{b\})$. As p is left generic, p^b is C-definable. Thus ${}^a q|C = tp(ab/C) = p^b|C$; since C is an elementary submodel, these definable types are equal.

Now if p_1, p_2 are left-generic, then for any right-generic q, for some a_1, a_2, b we have ${}^{a_1} q = {p_1}^b$, ${}^{a_2} q = {p_2}^b$; so ${}^{{a_1}^{-1} a_2} p_1 = p_2$.

This shows that any two left generics differ by a left translation. In particular by 3.2 there are boundedly many left generics.

Let G^0 be the right stabilizer of all left generics. So G^0 is of bounded index in G. By translating into G^0 we see that it has left and right generics; the left generic is unique. G^0 can have no subgroups of finite index, since the left generic is unique. If p is the left generic of G^0

and q is a right generic, $a \models p$, $b \models q$, $a \downarrow b$, then $tp(ab)$ is left generic so equals p. This shows q is also unique. Thus $p^b = p$ and $a^q = q$; since we saw above that $p^b = a^q$, we have $p = q$. When Δ is finite, G^0 is definable and hence of finite index. □

It follows that if G has a symmetric left generic (for $\Delta = \{\text{all formulas}\}$), then G has at most $|L|$ definable subgroups of finite index, over any set of parameters. In general, in this situation, the intersection of all definable subgroups of finite index is denoted G^0. For any sufficiently saturated M, $G(M)/G^0(M)$ is a profinite group, denoted G/G^0 since (up to a canonical isomorphism) it does not depend on the choice of M.

Any generic type has a translate lying in G^0. A *principal* generic is one lying in G^0. In case a symmetric left generic exists, the principal generic is unique. If $G = G^0$ we say that G is *connected*.

Remark 3.4. *Let p be a symmetric definable generic type of elements of a definable group G. Then the following are equivalent:*

(1) *p is the unique definable generic type of G.*
(2) *For all $g \in G$, ${}^g p = p$.*
(3) *$G = G^0$.*

Proof. Assume (1). By definition of genericity, and associativity, for $g \in G$, ${}^g p$ is right generic. Hence by uniqueness, ${}^g p = p$.

Conversely given (2), let q be generic. Then by Lemma 3.3, ${}^g p = q$ for some g. By (1), $p = q$.

The equivalence with (3) is immediate. □

Lemma 3.5. *Let T be a theory in a language L; let H be a group ∞-definable over C, with a symmetric definable generic type. Then H^0 is ∞-definable over some $C_0 \subset C$, with $|C_0| \leq |L| + \aleph_0$. (We will say that C_0 is* small.*)*

Proof. A definable type $q(x)$ is determined by the function $\phi \to (d_q x)\phi$, where $\phi = \phi(x, y) \in L$. Thus every definable type is C_0-definable for some small $|C_0|$, in this sense. Apply this to the principal generic of H □

3.6. **Stabilizers.** The stabilizer of a definable type can be viewed as an adjoint notion for the generic of a definable group. If p is a generic of G, then G^0 will be the stabilizer of p. If q is a definable type and is in some sense no bigger than $Stab(q)$, then it is a translate of a generic of $Stab(q)$.

Definition 3.7. *Let p be a definable Δ-type of elements of a definable group G.*

$$Stab(p) = \{c : {}^c p = p\}$$

The definition makes it clear that $Stab(p)$ is a *subgroup* of G. When Δ is generated by a formula ϕ, $q =^c p$, we have: $d_q\phi(x, y) = d_p\phi(cx, y)$; so : $c \in Stab(p)$ iff $q = p$ iff $d_q\phi = d_p\phi$ iff

$$\models (\forall y)(d_p x)(\phi(x, y) \equiv \phi(cx, y))$$

Thus when Δ is finite, $Stab(p)$ is a definable group. In general, therefore, it is an intersection of definable groups.

Lemma 3.8. *Let p be a definable type of elements of a definable group G. If ${}^a p = p$ for $a \in Q$, and Q generates G, then p is a generic of G. In particular if p is B-definable and $p|B \subset Stab(p)$, then p is a definable generic type of $Stab(p)$.*

Proof. $Stab(p)$ is a subgroup, so if $Q \subset Stab(p)$ then $G \subset Stab(p)$; it follows that p is generic of G.

Let $g \in Stab(p)$. Let $a \models p|B$¿ Then $ga \models p|Ba$. In particular $ga \models p|B$, and $g = (ga)a^{-1}$. Thus $p|B$ generates $Stab(p)$. The second statement thus follows from the first. □

3.8.1. Let p, q be two definable Δ-types, and let

$$Stab(p, q) = \{c : {}^c p = q\}$$

Then $Stab(p, q)$ may be empty. If nonempty, $Stab(p, q)$ is a regular torsor for both $Stab(p)$ and $Stab(q)$. In particular these two groups are then conjugate (by any element of $Stab(p, q)$.)

3.8.2. *A more general variant.* Let $p(x)$ be a complete definable type, and let h be a definable function, $h(a, x) = y$, with parameters a in $\mathbb{U}$. Then we have a definable type $q(y) = {}^h p(x)$:

$$(d_q y)\phi(y, u) \equiv (d_p x)\phi(h(a, x), u)$$

Let $\mathfrak{G}_0(p)$ be the family of $\mathbb{U}$-definable functions h such that ${}^h p = p$. $\mathfrak{G}_0(p)$ forms a semi-group under composition; one has the quotient semi-group of *p-germs* of elements of $\mathfrak{G}_0(p)$. The invertible germs form a group, denoted $\mathfrak{G}(p)$.

If p is a type of elements of a group G, then the stabilizer $Stab(p)$ defined above embeds naturally into $\mathfrak{G}(p)$: c maps to the germ of left translation by c.

Remark 3.9. *If p is left-generic, let G^0_p be the intersection of $Stab(q)$ over all left-translates q of p; they are bounded in number. So G^0_p has bounded index in G; and p is a translate of a left-generic type of G^0_p*

3.10. **A general remark on interpreting groups.** A structure M interprets a structure N if there exists a 0-definable set D over M, and a surjective function $f : D \to N$, such that for any 0-definable (in N) $R \subset N^k$, $f^{-1}R \subset D^k$ is 0-definable in M.

It suffices to have this for any basic R (in the language of N), if this is understood to include equality.

In order to interpret a *group* N, it suffices to find a function $f : D \to G$ and $m \in \mathbb{N}$ such that:

(1) $f(D)$ generates G in m steps. I.e. any element of G is a product of $\leq m$ elements of $f(D)$ and their inverses.
(2) $\{(d_0, \ldots, d_{2m}) \in D^{2m+1} : f(d_0) \cdot \ldots \cdot f(d_{2m}) = 1\}$ is a definable subset of D^{2m+1}.

Indeed in this case we obtain a surjective function $\mu : D^{\leq m} \to G$ defined by $(\mu(d_1, \ldots, d_l) \mapsto f(d_1) \cdot \ldots \cdot f(d_m)$. To see that the pullback multiplication is definable, show by induction on l that $\{(d, e, e') \in D^l \times D^{\leq m} \times D^{\leq m} : \mu(d)\mu(e) = \mu(e')\}$ is definable. For $l = 1$, this follows from assumption (ii). For $l + 1$, we have:

$$\mu(d_0 \frown d)\mu(e) = \mu(e') \equiv (\exists e'' \in D^{\leq m})(\mu(d)\mu(e) = \mu(e'')\&\mu(d_0)\mu(e'') = \mu(e')))$$

.

The pullback of equality can be viewed as a special case.

3.11. **Group chunks.** This idea, in the context of algebraic groups, is due to Weil.

Let G be a definable group, with principal generic type p. The $p{\otimes}p$-germ of multiplication is called the *group chunk* corresponding to (G, p).

An *abstract group chunk* is a C-definable type p and a C-definable function F (or a $p{\otimes}p$-germ of such a function), such that:

(1) If $a \models p|C$, and $b \models p|acl(Ca)$, then $F(a, b) \models p|acl(Ca)$, and $b \in dcl(a, F(a, b))$, and $a \in dcl(b, F(a, b))$.

(2) $\models (d_p x)(d_p y)(d_p z) F(x, F(y, z)) = F(F(x, y), z)$

Proposition 3.12. *Let p, F be an abstract group chunk. Then (p, F) is (definably isomorphic to) the chunk of an ∞-definable group G (in $\mathbb{U}^{eq}$.) In other words p can be identified with a generic type of G, in such a way that $(d_p x)(d_p y) F(x, y) = x \cdot y$.*

Proof. Let $P = p|C$, and let $Q = P \times P$. For $a \in P$, The p-germ of $F(a, x)$ is invertible. The germ g_a of $F(a, x)$ is thus an element of the group $\mathfrak{G}(p)$ of § 3.6. Let $\mathbf{G}$ be the subgroup of $\mathfrak{G}(p)$ generated by the elements g_a and their inverses. By § 3.10, it suffices to see:

(1) Any element of $\mathbf{G}$ is a product $g_a {g_b}^{-1}$.
(2) $\{(a_0, \ldots, a_m) : g_{a_0}(g_{a_1})^{-1} \ldots g_{a_m} = 1\}$ is definable.

The second point is immediate from the definability of p, and the fact that the map $\mathfrak{G}_0(p) \to \mathfrak{G}(p)$ is a well-defined homomorphism. As for the first, it suffices to show that a product $g_a {g_b}^{-1} g_c {g_d}^{-1}$ has the required form. Note that by (iii), when $(a, b) \models p \otimes p$, $g_a g_b = g_{F(a,b)}$; and thus also ${g_a}^{-1} g_b = g_c$ where $F(a, c) = b$. Applying this, Thus given any $a, b, c, d \in P$, let $e \models p|acl(Cabcd)$; then ${g_d}^{-1} g_e = g_{e'}$ for some e', and $e' \models p|acl(Cabc)$; so $g_c g_{e'} = g_{e''}$ for some $e'' \models p|acl(Cab)$; continuing this way, we obtain $g_a {g_b}^{-1} g_c {g_d}^{-1} g_e = g_{e''''}$ and so $g_a {g_b}^{-1} g_c {g_d}^{-1} = g_{e''''} {g_e}^{-1}$ as required. □

The uniqueness of G in 3.12 is guaranteed by the following:

Proposition 3.13. *Let G_1, G_2 be ∞-definable groups, and let p be the unique generic type of G_1. Let $F : G_1 \to G_2$ be a partial definable map, such that $(d_p x)(d_p y) F(xy) = F(x) F(y)$. Then there exists a unique homomorphism $H : G_1 \to G_2$ such that $\models (d_p x) F(x) = H(x)$.*

Proof. Uniqueness of H is clear, since the solution set P to p generates G: if $g \in G$, $a \models p|acl(C, b)$, then $ag \in P$.

Existence is also clear, provided we show that $F(a)F(b) = F(c)F(d)$ when $a, b, c, d \in P$ and $ab = cd$. Let $e \models p|acl(Cabcd)$. It suffices to show that $F(a)F(b)F(e) = F(c)F(d)F(e)$. But $F(b)F(e) = F(be), F(d)F(e) = F(de)$ by the property of F. Moreover, $be \models p|acl(Ca)$, and so $F(a)F(be) = F(abe)$ Similarly the right hand side equals $F(cde)$. Since $abe = cde$, equality holds.

H is definable by compactness. We can also write: $H(a) = b$ iff $(d_p x) F(ax) = bF(x)$. □

The analogous statements and proofs work for *group actions.*

3.14. **$\star$-definable groups with definable generics are projective limits of definable groups.** For simplicity, we will deal with ∞-definable groups.Thus we show that ∞-definable groups with definable generics are intersections of definable groups.

Actually we will prove something more general, by weakening the assumption of a (complete) definable generic to what we will call a "largeness notion".

Fix a saturated model $\mathbb{U}$. By a *largeness notion* with support G, we mean a filter μ on the Boolean algebra of $\mathbb{U}$-definable subsets of G_1, such that:

(1) If $Q_b = \{a : (a, b) \in R\}$ where R is a definable set, then for some definable set $d_\mu R$, $Q_b \in \mu$ iff $b \in d_\mu(R)(\mathbb{U})$.
(2) Each $G_n \in \mu$.
(3) (translation invariance) If $Q_b \in \mu$, $g \in G$, then $gQ_b \in \mu$.

Note using (2) that if $Q_b, Q'_{b'}$ are two definable sets, and $Q_b \cap G = Q'_{b'} \cap G$, then by compactness $Q_b \cap G_n = Q'_{b'} \cap G_n$ for some n, and so $Q_b \in \mu$ iff $Q_b \cap G_n \in \mu$ iff $Q'_{b'} \cap G_n \in \mu$ iff $Q'_{b'} \in \mu$. Thus μ can be viewed as a filter on relatively definable subsets of G; this explains (3).

Note that if G has a definable generic type, then the principal generic gives a largeness notion.

Proposition 3.15. *Let $G = \cap_{n \geq 1} P_n$ be an ∞-definable group, supporting a largeness notion μ. Then G is the intersection of a sequence of definable subgroups of a definable group.*

Proof. Let P_n be as above. Let $Q_n = P_3 \cap d_\mu(xy^{-1} \in P_n)$. Thus for $a \in P_3(\mathbb{U})$, $a \in Q_n$ iff $aP_n \in \mu$.
Claim. $G = \cap_{n \geq 2} Q_n$

Proof. Let $a \in G$. Then $a^{-1} \in P_{n+1}$, so $a^{-1}P_{n+1} \subset P_n$, $P_{n+1} \subset aP_n$, and thus as $P_{n+1} \in \mu$ we have $aP_n \in \mu$. So $a \in Q_n$.

Conversely, let $a \in \cap_n Q_n$. Then each $aP_n \in \mu$. Since μ is closed under finite intesections, and $\mathbb{U}$ is saturated, there exists $b \in \cap_n aP_n \cap \cap_n P_n$. So $b \in G$, and $b = ac$, $c \in G$. It follows that $bc^{-1} = (ac)c^{-1} = a(cc^{-1}) = a \cdot 1 = a$ so $a \in G$. □

Thus for some $n_0 \geq 3$, for all $n \geq n_0$, $Q_n \subset P_4$. Let $n > n_0$.
Claim. $GQ_n \subset Q_n$.

Proof. Let $g \in G, b \in Q_n$. As $b \in P_3$, $gb \in P_2$. Now $g(bP_n) = (gb)P_n$. Since $bP_n \in \mu$, also $g(bP_n) \in \mu$. Thus $(gb)P_n \in \mu$ and so $gb \in Q_n$. □

Let $S_n = \{x \in P_3 : (\forall y \in Q_n) xy \in Q_n\}$. If $a, b \in S_n$, then for any $x \in Q_n$, $bx \in Q_n$, so $a(bx) \in Q_n$. In particular, taking $x = 1$, $ab \in Q_n$, so $ab \in P_3$. Thus by definition $ab \in S_n$. So S_n forms a semigroup under m. We have $S_n \subset Q_n$, and by the last claim, $G \subset S_n$.

Finallly, let $H_n = \{x : x, x^{-1} \in S_n\}$. Then clearly H_n is a group under m, and $G = \cap_n H_n$. □

Corollary 3.16. *Let $R = \cap P_n$ be an ∞-definable ring, supporting a largeness notion μ invariant both for both additive translations and for multiplication by units of R. Assume every element of R is a sum of units. Then R is the intersection of a sequence of definable subrings of a definable ring.*

Proof. By Proposition 3.15 applied to the additive group, we may assume that the P_n have the structure of additive groups, with P_{n+1} a subgroup of P_n. By compactness, we can refine the sequence so that the multiplication map is defined and gives a bilinear map $m : P_n \to P_n \to P_{n-1}$. Now follow the proof of Proposition 3.15 with respect to the multiplicative group of units R^*. Let $Q_n = P_3 \cap d_\mu(xy^{-1} \in P_n$. By the additive invariance of μ and the distributive law, Q_n is a subgroup of P_3. As in the Claim, we have $R = \cap_n Q_n$. Let $R_n = \{x \in Q_n : m(x, Q_n) \subseteq Q_n\}$. Then R_n is a ring. Again as in the first Claim, but now multiplicatively, $R^* \subseteq R_n$; thus $R \subseteq R_n$. Conversely we have $\cap_n R_n \subseteq \cap_n Q_n = R$. □

3.17. **Products of types in definable groups.** By a *piecewise definable* set, we mean a sequence H_n of definable sets, with injective definable maps $H_n \to H_{n+1}$, viewed as inclusion maps. The direct limit is thus identified with the union, and denoted H. A *piecewise definable group* is a piecewise definable set together with maps $m_n : H_n \times H_n \to H_{n+1}$, compatible with the inclusions, and inducing a group structure $m : H \times H \to H$.

By a definable (or ∞-definable) subgroup G of H we mean a (∞-) -definable subset $G \subset H_k$ for some definable piece H_k of H; such that $m(G^2) \subset G$ and (G, m) is a subgroup of (H, m).

Let $p_1, \ldots, p_n$ be a definable type of elements of H, and let w be an element of the free group F on generators $\{1, \ldots, n\}$. We construct a definable type $p_w = w_*(p_1, \ldots, p_n)$. If w is the product of the generators $1, 2$, we also write $p_1 * p_2$ for p_w. Let

$a_i \models p_i | acl(Ca_1 \ldots a_{i-1})$. Let $a_w = w(a_1, \ldots, a_n)$ be the image of w under the homomorphism $F \to H$ with $i \mapsto a_i$. Let $p_w | C = tp(a_w/C)$.

These are definable types.

The words of the free group will be denoted by expressions such as $1\bar{2}3$; $\bar{2}$ is the inverse of the generator 2.

If a single type p is given, rather than a sequnce p_w will refer to the sequence $(p, p, \ldots, p)$. Let p^{*n} denote $p_{123\cdots n}$, $p^{\pm 2n[+1]} = p_{1\bar{2}3\bar{4}\cdots(\bar{2n})[(2n+1)]}$.

4. Generically metastable groups

Definition 4.1. *A* generically metastable group *is a group with a stably dominated left-generic.*

Lemma 4.2. *If A, B are generically metastable groups, so is $A \times B$. If A is generically metastable and $f : A \to C$ is a definable surjective homomorphism, then C is generically metastable.*

Proof. Let p, q be the principal generic types of A, B. Then $p \otimes q$, $f_* p$ are the generics of $A \times B, C$ respectively; the verifications are easy. □

Lemma 4.3. *Assume N is a generically metastable definable or ∞-definable subgroup of G, and G/N has a stably-dominated type invariant under G-translations. Then G is generically metastable.*

In particular, if $N \trianglelefteq G$ and $N, G/N$ are generically metastable, then so is G.

Proof. Let p_N be the principal generic type of N, the unique generic of the connected component N^0 of N, and let $p_{G/N}$ be a G-invariant stably dominated type of G/N.

Assume first that N is connected, $N = N^0$.

If $n \in N$ then ${}^n p = p$. Thus if $cN = dN$, so that $c = dn$ for some $n \in N$, then

$$ {}^c p_N = {}^{dn} p_N = {}^d ({}^n p_N) = {}^d p_N $$

Thus the definable type ${}^c p_N$ depends only on the coset $S = cN$; denote it p_S.

We obtain a definable type p_G of G: a realization of p_G over C is a realization of p_S over C, s where $s \models p_{G/N}|C$, and S is the coset of N corresponding to s. Here $p_{G/N}$ is the given G-invariant definable type of G/N.

Then p_S is stably dominated. (Descent: Proposition 2.11.) So is $p_{G/N}$. Hence by transitivity (Proposition 2.10 (4)) p_G is stably dominated. By construction it is translation invariant.

In general, the natural map $r : G/N^0 \to G/N$ has profinite fibers. Thus $p_{G/N}$ lifts to G/N^0; i.e. the set Q of definable types q of G/N^0 with $r_* q = p_{G/N}$ is nonempty. Moreover it is profinite. Now G acts on Q by translation. So a co-profinite ∞-definable subgroup G_1 of G must fix some element $q \in Q$. The above proof now applies to G_1, using q. So G_1 is generically metastable, and hence so is G. □

Lemma 4.4. *Let G be a generically metastable group, N a definable subgroup, $X = N \backslash G$ the right coset space, $\eta : G \to X$ the map: $\eta(g) = Ng$. Assume there exists a definable $Y \subset G$ with $\eta|Y$ finite-to-one. Then N is generically metastable.*

Proof. Let p be the generic type of G^0, and let $g \models p|C$ (where N, G, X, Y, η, p are C-definable.) Let $h = \eta(g)$. Let $g_1 \in Y$, $\eta(g_1) = h$. Then $g_1 \in acl(C, h) \subset acl(C, g)$. Thus $tp((g, g_1)/C)$ is stably dominated (Proposition 2.10 (4))). Let $n = g g_1^{-1} \in N$. Then $n \in dcl(C, g, g_1)$, so by Lemma 2.10, $q := tp(n/C)$ is stably dominated. Because of the finite ambiguity in the choice of g_1, the above may not pick out a unique type q, but at worst a finite set Q of types meet the description.

We now show that Q is $N \cap G^0$-invariant. Let $n' \in N \cap G^0$; take $g \models p|Cn'$. Let $h = \eta(g)$, $g_1 \in Y$, $\eta(g_1) = h$. Let $n = g g_1^{-1}$. By definition, $n \models q|C$,q a typical element of Q, and $n'n \models^{n'} q|Cn'$. Note that $n'n = (n'g) g_1^{-1}$, and $g_1 \in Y$, $\eta(g_1) = h = \eta(n'g)$. Thus $tp(n'n/C) \in Q$

also. So Q is a finite, $N \cap G^0$-invariant collection of definable types; hence all types in Q are generic. □

Corollary 4.5. *Let G be an algebraic group, N an algebraic subgroup. Let H be a definable subgroup of G in $ACVF$, with H generically metastable. Then $H \cap N$ is generically metastable.*

Proof. Let X be the coset space $N \backslash G$; it is an algebraic variety. Let $\eta : G \to X$. Let $X_H = \eta(H) = N_H \backslash H$ (where $N_H = H \cap N$.) This is all defined over some $M \models ACVF$. Let $g \in H$ be generic, $h = \eta(g)$. Then $N = M(h)^{alg} \models ACVF$; so there exists $d \in H(N)$ with $\eta(d) = h$. Let Y' be a formula over M, true of (d, h), with a finite-to-one projection to X_H; let Y be the projection of Y to G. Then the hypotheses of Lemma 4.4 hold true of $(H, N_H, X_H, \eta|H, Y)$. Thus N_H is generically metastable. □

Proposition 4.6. *Let G be a generically metastable group. Then there exists a *-definable stable group $\mathfrak{g}$, and a *-definable homomorphism $g : G \to \mathfrak{g}$, such that the generics of G are stably dominated via g.*

Assuming G is definable, the base is algebraically closed, and (FD) holds. Then $\mathfrak{g}$ and $g : G \to \mathfrak{g}$ can be taken to be definable.

Proof. Let $\theta(a)$ enumerate $St_C(a)$. Consider the map f_a, $f_a(b) = \theta(ab)$. The p-germ is strong, and is in $St_C(a)$, so it factors through $\theta(a)$: $f_a = f'_{\theta a}$. By stable embeddedness, it factors through $\theta(b)$ too: let $c = f_a(b) = f'_{\theta a}(b)$. Since $c \in St_C$, $tp(\theta(a), c/C, \theta(b)) \implies tp(\theta(a), c/C, b)$. Thus $c \in dcl(\theta(a), \theta(b))$; i.e. $\theta(ab) = c = F(\theta(a), \theta(b))$ for $a \models p$, $b \models p|Ca$. Now the hypotheses of the group chunk theorem 3.12 are satisfied, so F is a restriction of the multiplication map on a *-definable group $\mathfrak{g}$. θ is generically a homomorphism, and by Proposition 3.13, it extends to a group homomorphism.

If $g' : G \to \mathfrak{g}'$ is another such homomorphism, then $g(a)$ is part of $St_C(a)$, when a realizes the generic type of G. Thus for some definable function r, $g'(a) = r(g(a))$. r extends uniquely to a group homomorphism $R : \mathfrak{g} \to \mathfrak{g}'$, and $g' = R \circ g$.

By Proposition 3.15, $\mathfrak{g}$ is a projective limit $\varprojlim \mathfrak{g}_i$ of definable groups over some directed partially ordered index set I; for $i > j$ we have a homomorphism $\pi_i : \mathfrak{g}_i \to \mathfrak{g}_j$. Let $g_i : G \to \mathfrak{g}_i$ be the natural homomorphism. Since there are no descending chains of definable subgroups of $\mathfrak{g}_i$, every ∞-definable subgroup is definable; in particular the image of g_i is definable. Replacing $\mathfrak{g}_i$ by this image, we may assume g_i and hence all maps π_i are surjective.

Assume (FD). Let M be a good base and let a realize the generic type of G^0 over M. Then $St_M(a)$ is acl- finitely generated over M (Lemma **??**); say $St_M(a) = M(d)$. Then $\dim(d/M(g_i(a))$ decreases as i increases; so it stabilizes at some i_0. Since all $g_i(a)$ are in $M(d)$, it follows that $g_i(a) \in \mathrm{acl}(M(g_{i_1}(a)))$ for all $i \geq i_1$. It follows that $tp(d/M)$ is dominated via g_i.

□

The homomorphism g is not uniquely determined by G; even if G is stable, there may be a nontrivial homomorphism of this kind. We do however have a maximal *-definable homomorphism into stable pro-definable groups.

Remark 4.7. *There exists a *-definable stable group $\mathfrak{g}$, and a *-definable homomorphism $g : G \to \mathfrak{g}$, maximal in the sense that any homomorphism $g' : G \to \mathfrak{g}'$ into a stable group factors through g. The kernel of this maximal g is uniquely determined. If G is stably dominated it will be stably dominated via this maximal homomorphism.*

Lemma 4.8. *Let G be a generically metastable group, with generics dominated by a surjective group homomorphism $g : G \to \mathfrak{g}$. Then $tp(a/C)$ is generic in G (i.e. $a \models r|C$ for some definable generic r of G) iff $tp(g(a)/C)$ is generic in $\mathfrak{g}$.*

Proof. Assume $tp(g(a)/C)$ is generic in $\mathfrak{g}$. Let p be a generic type of G, stably dominated via $g : G \to \mathfrak{g}$. Let $b \models p|acl(Ca)$. Then $q =^a p$ is a definable generic. Hence q is stably dominated via g. Note that $ab \models q|C$. Since $g(a)$ is generic in $\mathfrak{g}$, we have $g(a)g(b)\downarrow_C g(b)$; since g is a homomorphism, $g(ab)\downarrow_C g(b)$. Since p, q are both stably dominated via g we have $ab\downarrow_C b$. Thus $ab \models q|acl(Cb)$. So $a \models r = q^{b^{-1}}|Cb$, and in particular $a \models r|C$. Since r is generic, the lemma is proved. □

Corollary 4.9. *Let G be a generically metastable group, with generics dominated by a group homomorphism $g : G \to \mathfrak{g}$. let H be a definable subgroup of G. If $g(H) = \mathfrak{g}$, then H has finite index in G.*

Proof. Work over $C = acl(C)$. Let p be the principal generic type of G. Let $b \in H \cap G^0$ be such that $g(b)$ is generic in $\mathfrak{g}$ over C. By Lemma 4.8, $tp(b/C)$ is generic in G. Thus a generic of G lies in H, so H has finite index in G. □

Corollary 4.10. *(FD_ω) Let G be a generically metastable definable group, with generics dominated by a group homomorphism $g : G \to \mathfrak{g}$. Then G^0 is definable, i.e. $[G : G^0]$ is finite.*

Proof. G/G^0 is profinite. Extend the base C so as to have a representative of each class of G/G^0. As in Lemma 2.16 we may further extend the base, and then find $a \in D$, $C' = \Gamma_C(a)$, such that $g(a)$ has dimension n over C', and $tp(a/C', g(a))$ is isolated. From the first fact it follows (Lemma 4.8) that $tp(a/C')$ is a generic of G. So it is a translate (by some element in $G(C) \subseteq G(C')$) of the generic p of G^0. It follows that $p|C'$ is isolated.

On the other hand, $G = \cap_i G_i$ where $\{G_i\}$ is a bounded family of definable subgroups of finite index, closed under intersections. $p|C'$ is generated by all formulas $\neg g(x) \in E$ (with E a C'-definable non-generic subset of $\mathfrak{g}$) together with all formulas $G_i(x)$. So $tp(a/C', g(a))$ is generated by the formulas $G_i(x)$. Being isolated, it is generated by a single formula $G_i(x)$. It follows that every generic of G_i lies in G^0. Go $G_i = G^0$, and G^0 is definable. □

Here is a characterization of generics that does not explicitly mention g and $\mathfrak{g}$.

Corollary 4.11. *Let G be a generically metastable group, $\dim_{st}(G) < \infty$. Then the generic types of G over C are precisely the types $tp(c/C)$ maximizing, for generic h, $\dim(St(hc)/C(h))$.*

Proof. By Proposition 4.6 and Lemma 4.8: we have $\operatorname{acl} St(hc) = \operatorname{acl}(g(h)g(c))$ so $\dim(St(hc)/C(h)) = \dim(g(h)g(c)/C(h)) = \dim(g(c)/C(h)) \leq \dim(g(c))$; if this is maximal, then $g(c)$ is generic in $\mathfrak{g}$ over $C(h)$, and hence over C. □

(We could have used the Lascar rank of the stable part, in place of Morley dimension.)

Note that the proof uses rather than proves the existence of a generic.

If G is piecewise-definable, in a superstable theory, and p is a type of maximal rank in G, then p is a translate of a generic of a definable subgroup of G, $Stab(p)$. For stably dominated types, the analog need not hold:

Example 4.12. *(ACVF).* Let $G = (G_a)^2$. Let $P = \{(x, y) : \operatorname{val}(x - y^2) \geq v\}$ where v is a fixed positive element of the value group. Then the maximal dimension of the stable part of any element of G over any base equals 2. P also extends to a type p whose stable part has rank 2. But though p has maximal rank, it is not a generic type. (We do have: p^{*2} is generic.)

A certain converse to Proposition 4.6:

Proposition 4.13. *Let G be a definable group in a metastable theory, with $\dim_{st}(G) = n < \infty$. Assume G admits a surjective definable homomorphism $g : G \to \mathfrak{g}$, with $\mathfrak{g}$ stable of dimension n. Then there exists an ∞- definable normal subgroup T of G with:*

(1) G/T almost internal to Γ

(2) T connected and generically metastable via $g|T$.

(3) $g(T) = \mathfrak{g}^0$, and

T is uniquely determined by (1,2) or by (2,3).

Proof. Let Z be the collection of definable types q (over some B) of elements of G with g_*q generic in $\mathfrak{g}$, and q stably dominated.

Claim 1. $Z \neq \emptyset$.

Proof. Let C be a good base for G. Let $q_1 = tp(a_1/C)$ be any type over C avoiding $g^{-1}(Y)$ for all non-generic C-definable $Y \subset \mathfrak{g}$. Then g_*q_1 is generic in $\mathfrak{g}$. Let $\gamma = \operatorname{dcl}(C(a_1)) \cap \operatorname{dcl}(\Gamma)$. Let $q = tp(a/\operatorname{acl}(C, \gamma))$. Any $\operatorname{acl}(C, \gamma)$-definable subset of $\mathfrak{g}$ is already $\operatorname{acl}(C)$-definable, so q also avoids all pullbacks of non-generic sets, and g_*q is generic in $\mathfrak{g}$. By metastability, q is stably dominated, and extends to a unique definable type; thus $Z \neq \emptyset$. □

Claim 2. Any two elements of Z are (left or right) translates of each other.

Proof. Let $q, r \in Z$; say both are B-definable. Let $a \models q|B$, $b \models r|B(a)$. Then $g(a), g(b)$ are B-independent generic elements of $\mathfrak{g}$; hence also $g(a), g(a)g(b)^{-1}$ and $g(b), g(a)g(b)^{-1}$ are independent pairs of elements of $\mathfrak{g}$ over B. Let $c = ab^{-1}$. Then $g(c) = g(a)g(b)^{-1}$. Since $\dim_{st}(G) = n = \dim(St_B(c)/B)$, we have $St_B(c) \subseteq \operatorname{acl}(g(c))$. Hence $g(a) \downarrow_B c$. By stable domination, $a \models q|B(c)$. Similarly $b \models r|B(c)$. It follows that $q =^c r$. □

It follows that $T := Stab(q)$ is an ∞-definable group, not depending on $q \in Z$. Since Z is invariant under conjugation, T is normal. Moreover if a, b are independent realizations of q over B, then the proof of the second claim shows that $a \downarrow_B ab^{-1}$, hence $ab^{-1} \in S$. It follows that $r = q^{b^{-1}}$ is a type of elements of S, and must be a generic type of S. Thus S is generically metastable. Also $g(ab^{-1})$ is a B-generic element of $\mathfrak{g}$, so $g_*(r)$ is a generic of $\mathfrak{g}$.

By Lemma 3.15, or over a larger base directly from the definition of a stabilizer, T is the intersection of definable groups S_i.

Claim 3. G/S_i is almost Γ-internal.

Proof. Otherwise, fix S_i, a good base C, and a type $\bar{q} = tp(\bar{a}/C)$ of elements of G/S_i, with $\bar{q}$ not almost Γ-internal. The element $\bar{a}$ corresponds to a coset $U_{\bar{a}}$ of S_i. Pick $a \in U_{\bar{a}}$; if $c \in S$ is generic then $ac \in U_{\bar{a}}$ while $g(ac) = g(a)g(c)$, so replacing a by ac we may assume $g(a)$ is generic in $\mathfrak{g}$ over C. By the proof of Claims 1 and 2, for some γ from Γ, $q = tp(a/C(\gamma))$ is stably dominated, and $q \subseteq Sa$, i.e. q is contained in a single coset of S. It follows that each aS_j is algebraic over $C(\gamma)$, and in particular $\bar{a} = aS_i \in \operatorname{acl}(C(\gamma))$; contradicting the choice of $\bar{q}$. □

Let T' be another connected generically metastable group. Then $T'/(T \cap T')$ is almost Γ-internal but also connected generically metastable, hence is trivial; so $T' \subseteq T$. By Corollary 4.9 and Lemma 3.15, T/T' is profinite; but T is connected, so $T = T'$. □

4.14. **Generically metastable subgroups of maximal rank.** While not all generically metastable subgroups are definable, we will show that subgroups whose residual rank is $\dim_{st}(D)$ are.

Lemma 4.15. *Let G be a definable group in a metastable theory. Let T be an ∞- definable normal subgroup of G with G/T almost internal to Γ, and T connected and generically metastable.*

(1) (FD) There exists a definable subgroup S of G with $S^0 = T$.

(2) (FD$_\omega$) T itself is definable.

Proof. By Lemma 3.15, T is the intersection of definable groups S_i; and G/S_i is almost Γ-internal. Assume (FD) . By Proposition 4.6, there exists a definable homomorphism $g : T \to \mathfrak{g}$, with $\mathfrak{g}$ stable, and T generically metastable via $g|T$. By compactness, g extends to a definable homomorphism on some S_i; replacing G by this S_i, we may assume $g : G \to \mathfrak{g}$.

(1) By Lemma 2.22, some quotient G/S by a finite normal subgroup is Γ-internal; define $\dim(G/S)$ to be the o-minimal dimension of any such quotient. By (FD) , for any $a \in A$, the rank of $\Gamma_B(a)$ over B is finite, so the o-minimal dimension of G/S_i is bounded independently of i. Thus for some i, for all $j \geq i$, $\dim(G/S_j) = \dim(G/S_i)$. It follows that the natural map $G/S_j \to G/S_i$ has zero-dimensional fibers, hence it has finite kernel. This shows that for some i, for all $j \geq i$, S_i/S_j is finite.

Thus all S_j (with $j \geq i$) are generically metastable via g, and (1) holds.

(2) Assume (FD$_\omega$) . Then by Corollary 4.10 , T is definable. □

Corollary 4.16. *(FD$_\omega$) Let H be a definable group, G be a connected ∞-definable generically metastable subgroup of H, with a stable homomorphic image of dimension $n = \dim_{st}(H)$. Then G is definable.*

Proof. By Proposition 4.6, there exists a definable homomorphism $g : G \to \mathfrak{g}$ with $\mathfrak{g}$ ω-stable, such that G is stably dominated via g. So $\dim(g(a)) = n$ for $a \models p$, p the generic of G. By Proposition 3.15, G is an intersection of definable groups G_i; by compactness, g extends to a homomorphism $G_i \to \mathfrak{g}_i$, with $\mathfrak{g}_i$ a stable definable group (for some i.) Then $g(G_m)$ is a descending sequence of definable subgroups of $\mathfrak{g}_i$; by ω-stability of the stable part, it must stabilize; i.e. $\mathfrak{g}_m = \mathfrak{g}$ for large enough m. By Proposition 4.13 there exists an ∞-definable T of H with H/T almost internal to Γ, T connected and generically metastable via $g|T$, $g(T) = \mathfrak{g}^0$. By Lemma 4.15, T is definable. By the uniqueness in Proposition 4.13, $T = G$. □

Corollary 4.17. *Let A be a definable Abelian group in a metastable theory.*

(FD) Any connected generically metastable ∞-definable subgroup of A is contained in a definable generically metastable subgroup.

(FD$_\omega$) Any connected generically metastable ∞-definable subgroup of A is contained in a definable connected generically metastable subgroup.

Proof. By Proposition 3.15, H is an intersection of definable groups H_i; for simplicity assume $H = \cap_{i\in\mathbb{N}} H_i$, $H_0 \supset H_1 \supset \ldots$. Then $dim_{st}(A/H_i)$ is non-decreasing with i, and eventually stabilizes; we may assume it is constant, $dim_{st}(A/H_i) = n$. Clearly $\dim_{st}(A/H) \leq n$; but also since A/H_0 is a homomorphic image of A/H, $\dim_{st}(A/H) \geq \dim_{st}(A/H_0) = n$. By Proposition 5.4, A/H contains a generically metastable ∞-definable subgroup $\mathbf{S}$ with stable homomorphic image of dimension n. Let $\mathbf{S}^i$ be the image of G in A/H_i. So $\mathbf{S}^i$ is generically metastable, $\mathbf{S}$ is the inverse limit of the $\mathbf{S}_i$, and for large enough i (say for $i = 1$), $\mathbf{S}^1$ has a stable homomorphic image of dimension n. By Corollary 4.16, $\mathbf{S}^1$ is definable. Let S_1 be the pullback of $\mathbf{S}^1$ to A. So S_1 is definable, and for any $j \geq 1$, S_1/H_j has a stable homomorphic image of dimension $n = \dim_{st}(S_1/H_j)$. By Proposition 4.13 there exists a unique ∞-definable group T_j with $H_j \leq T_j \leq S_1$, S_1/T_j almost Γ-internal, and T_j/H_j generically metastable and connected. By Lemma 4.15 T_j/H_j is definable, hence T_j is definable. If $k > j$ then $H_k \leq H_j$; since T_k/H_k is generically metastable connected, and S_1/T_j is almost Γ-internal, we have $T_k/H_k \leq T_j/H_k$, i.e. $T_k \leq T_j$. Since S_1/T_j is (almost) Γ-internal, the argument of Lemma 4.15 (1) shows that for large enough j (say for $j = 1$), for all $k \geq j$, T_k has finite index in T_1; i.e. the T_k all have the same connected component of 1, an ∞-definable gruop T. Then $T/H_k \cap T$ is generically metastable for all k, so T/H is generically metastable. By Proposition 4.3, T is generically metastable. Hence so are the T_i. By Lemma 4.15, if (FD$_\omega$) holds, T is definable. □

A connected generically metastable definable group H is called *certifiable* (over C) if it is an element of a uniformly definable family of definable groups $H_c : c \in Q$, with Q definable over C, such that each H_c is connected and generically metastable.

Lemma 4.18. *(FD$_\omega$) Let A be a definable Abelian group. There exists a base C and C-definable families $\mathcal{S}^\nu$ of definable subgroups S^ν_t of A such that*

(1) Any S^ν_t is connected, generically metastable.

(2) Any connected, generically metastable ∞-definable subgroup of A (over any set of parameters) is contained in some S^ν_t.

An equivalent statement is that any connected, generically metastable ∞- definable subgroup of A is contained in a C-certifiable one. By Corollary 4.17, it suffices to show that any connected, generically metastable definable subgroup of A is contained in a C-certifiable one.

Proof. For a definable Abelian group B, define invariants n, k, l as follows: $n = \dim_{st}(B)$. Let $Z(B)$ be the collection of definable subgroups S of B with stable homomorphic images of dimension n; by Lemma 5.4, $Z(B) \neq \emptyset$. Let $Z_2(B) = \{(T, K) : T \in Z(B), \dim_{st}(T) = n, K \leq T, T/K\ \Gamma\text{- internal}\}$. Let $k = \max\{\dim(T/K) : (T, K) \in Z_2(B)\}$. (By (FD), such a maximum exists.) If $(T, K) \in Z_2(B)$ then by Proposition 4.13 and Lemma 4.15, K is generically metastable, and by $[K : K^0]$ is finite. Since $Z_2(B)$ is uniformly definable, by Lemma 2.17 there exists a bound l on $[K : K^0]$, valid for all such K.

The set of triples (n, k, l) is ordered lexicographically.

Pick a definable generically metastable definable $B_0 \leq A$ such that $(n, k, l)(A/B_0)$ is as small as possible. (If $(T/B_0, K/B_0)$ attains the maximum for A/B_0, then the pullbacks to A show that $(n, k, l)(A) \geq (n, k, l)(A/B_0)$. Thus increasing B_0 has the effect of decreasing $(n, k, l)(A/B_0)$.) For any generically metastable $H \leq A$, $H + B_0$ is also generically metastable; so it suffices to find a family $\{S_t\}$ for A/B_0. Thus (after augmenting the base so that B_0 is definable) we may assume $(n, k, l)(A/B) = (n, k, l)(A)$ for any connected generically metastable definable $B \leq A$. Let $(n, k, l) = (n, k, l)(A)$. S

Claim. Let S be a definable subgroup of A admitting a surjective homomorphism $h : S \to \mathfrak{g}$ to a stable group of dimension n. Let W be a Γ-group, $\dim_o(W) = k$, and let $g : S \to W$ be a homomorphism with kernel L. Assume $[L : L^0] = l$. Then L^0 is connected, generically metastable, and certifiably so (over C.)

Proof. Say $S, h, \mathfrak{g}, g, W, L, L^0$ are all d-definable. Clearly they lie in a family $(S_t, h_t, g_t, \mathfrak{g}_t, W_t, L_t, L'_t)$. such that $\mathfrak{g}$ is stable, $\dim \mathfrak{g}_t = n(A)$, W_t is a Γ-group, $h_t : S_t \to \mathfrak{g}$ is a surjective homomorphism, $g_t : S_t \to W_t$ is a surjective homomorphism with kernel L_t, and L'_t is a definable subgroup of L_t of index l. By Proposition 4.13 and Lemma 4.15, L_t is generically metastable. By definition of l we have $[L_t : L^0_t] \leq l$; so $L^0_t = L'_t$. Thus L'_t is connected and generically metastable, for any element of the family. Let $\{L'_t\}$ be the projection to a family of subgroups of A. This is a definable family (using definability of Morley dimension and o-minimal dimenstion, (FD) .) Any L'_t is connected and generically metastable, showing certifiability. □

Let B be a generically metastable definable subgroup of A. Let $(T, K) \in Z_2(A/B)$ with $\dim_{st}(T) = n, \dim_0(T/K) = k, [K : K^0] = l$. Let (S, L, L') be the pullbacks to A, so that

$B \leq L' \leq L \leq S$ and $T = S/B, K = L/B, K^0 = L'/B$. Clearly S admits a homomorphism onto a stable group of dimension n, and $\dim_o(S/L) = \dim_o(T/K) = k$, $[L : L'] = l$.

Since $(n,k,l)(A) = (n,k,l)$ it follows that L is generically metastable, and (since $[L : L^0] \leq l$) that $L' = L^0$. By the Claim, L' is certifiable connected generically metastable subgroup. Since $B \subseteq L'$, the Proposition is proved. □

5. Abelian groups

We will use the notation of § 3.17.

Lemma 5.1. *Let H be a piecewise definable, or even piecewise *-definable, Abelian group, p a symmetric definable type of elements of H. Assume H has p-weight $< 2n$, in the sense that:*

Whenever $b \in H$, $(a_1, \dots, a_{2n}) \models p^{\otimes 2n}$, $a_i \models p|b$ for some i.

Then there exists an ∞-definable subgroup G of H with generic type $p^{\pm 2n}$. p is contained in a coset of G.

Proof. Let $(a_1, a_2, \dots, a_{2n}) \models p^{\otimes 2n}$, and let $b = a_1^{-1} a_2 \cdot \ldots \cdot a_{2n}$.

By the weight assumption, $a_i \models p|b$ for some i. Say i is odd (the even case is similar.) Since the group is commutative, $tp(a_1, a_2, \dots, a_{2n}/b)$ is $Sym(n)$-invariant,so $a_1 \models p|b$.

Let G be the stabilizer of $p^{\pm 2n}$, and $C = Stab(p^{\mp 2n-1}, p^{\pm 2n})$. (Recall § 3.8.1.) Then $a_1^{-1} \in C$, so $p^{\pm 1}$ is a type of elements of C. It follows that $p^{\pm 2}$ and hence also $p^{\pm 2n}$ is a type of elements of G. By Lemma 3.8, it is the generic type.

□

Remark 5.2. *It follows that p generates a coset of the ∞-definable group G, in finitely many iterations of the function: $(x, y, z) \mapsto xy^{-1}$. If p is stably dominated, it follows that G is connected and generically metastable.*

For the rest of this section we work with a metastable theory.

Let us say that a set D has bounded weight if for some n, every stably dominated type of elements of D has weight $\leq n$. In ACVF, every definable set has bounded weight (if it is a subset of an algebraic variety V, the weight is bounded in fact by the dimension of V.) In EVDF, the same is true of a definable set of finite differential order.

Proposition 5.3. *Let G be a definable Abelian group of bounded weight. Let A_i $(i \in I)$ be connected generically metastable ∞-definable subgroups of G (defined over a set C.) Then there exists a generically metastable ∞-definable subgroup B containing all A_i.*

Proof. The sum of two generically metastable groups is again generically metastable, by Lemma 4.2: $A + B$ is a homomorphic image of $A \times B$.

Let $\mathfrak{F}$ be the family of all C-definable functions into Γ. By metastability, enlarging C if necessary, we have for any $c \in G$: $tp(c/C, (f(c) : f \in \mathfrak{F}))$ is stably dominated.

Let p_i be the principal generic of A_i. Consider the partial type:

$$q_0 = \{(d_{p_i} y) f(x) = f(yx) : i \in I, f \in \mathfrak{F}\}$$

Claim 1. q_0 is consistent.

It suffices to show that any finite number of formulas, concerning say $p_1, \dots, p_n$, can be satisfied. As above, let $A = A_1 + \ldots + A_n$, and let p be the principal generic of A.

If $(b, c) \models p_i \otimes p$, then by genericity of p, $bc \models p|b$. By symmetry for stably dominated types, $b \models p_i|c$. Now if $f \in \mathfrak{F}$, as p is stably dominated, and thus orthogonal to Γ, there exists γ_f such that for any $c' \models p|C$, $f(c') = \gamma_f$. In particular, $f(c) = \gamma_f = f(bc)$. Thus $\models (d_{p_i} y) f(c) = f(yc)$. Since this is true for each $i = 1, \dots, n$ and $f \in \mathfrak{F}$, q_0 is consistent.

Let $c \models q_0$, $C' = C \cup (f(c) : f \in \mathfrak{F})$, $C'' = \mathrm{acl}(C')$.

Claim 2. $tp(c/C'')$ is generically p_i-invariant for each i.

Indeed if $a \models p_i|Cc$, then $tp(ac/C) = tp(c/C)$ (consider two-valued functions in $\mathfrak{F}$). Thus $tp(ac, (f(ac) : f \in \mathfrak{F})/C) = tp(c, (f(c) : f \in \mathfrak{F})/C)$. But $f(ac) = f(c)$; so $tp(ac, (f(c) : f \in \mathfrak{F})/C) = tp(c, (f(c) : f \in \mathfrak{F})/C)$. This shows $tp(c/C') = tp(ac/C')$. In fact

As noted, $tp(c/C')$ is stably dominated. By 5.2, $tp(c/C'')$ generates a coset B' of a generically metastable group B. B' depends on the extension of $tp(c/C')$ to a type over C'', but B does not. It follows that B is generically p_i-invariant, and hence $A_i \subseteq B$. □

Lemma 5.4. *(FD). Let G be a $\star$-definable Abelian group. Then G contains a generically metastable ∞- definable subgroup S with stable homomorphic image of dimension* $\dim_{st}(G)$*.*

Proof. Say $\dim_{st}(G) = n$. Let $C = acl(C)$ be a base substructure, $g \in G$ with $\dim(St_C(g)/C) = n$. Let $\gamma = \mathrm{dcl}(C(g)) \cap \mathrm{dcl}(\Gamma)$, $p = tp(g/C, \gamma)$. So p is stably dominated, with stable part p_{st} of dimension n.

A real valued function r on types is *subadditive* if $r(tp(ab/C)) \leq r(tp(b/C)) + r(tp(a/Cb))$.

Claim 1. $\dim(St_C(x)/C)$ is subadditive.

Proof.

$$\dim(St_C(a,b)/C) \leq \dim(St_C(a,b)/C, St_C(a)) + \dim(St_C(a)/C) = \dim(St_C(a,b)/Ca) + \dim(St_C(a)/C)$$

Claim 2. $p^{\pm m}$ is also stably dominated, with stable part of dimension n. More generally, if q is stably dominated, $\dim(p_{st}) = \dim(q_{st}) = n$, $a \models p|C$, $b \models q|Ca$, $c = ab$, then $\dim(St_C(c)/C) = n$, and $tp(c/C)$ is stably dominated.

Proof. $2n \leq \dim(St_C(a,b)/C) = \dim(St_C(a,c)/C) \leq \dim(St_{Cc}(a)/Cc) + \dim(St_C(c)/C) \leq n + \dim(St_C(c)/C)$. So $\dim(St_C(c)/C) \geq n$. But $\dim_{st}(G) = n$, so $\dim(St_C(c)/C) = n$. Since $tp(ab/C)$ is stably dominated, so is $tp(c/C)$.

By Lemma 5.1, G contains a generically metastable group S, with generic type $p^{\pm 2n}$. By the claims, the stable part of the generic of S has dimension n.

□

5.5. **Limit - metastable groups.**

Definition 5.6. *Let q be a $\star$-type of Γ, over a small set C_0. For $t \models q$, let S_t be an ∞-definable subgroup of G: $S_t =_{def} \cap_n S_t^n$, where S_t^n is a Γ- definable family of subgroups of G. We call (S_t) a limit-metastable family for G if:*

1. *S_t is a connected generically metastable subgroup of G.*
2. *q carries a C_0-invariant partial ordering $\leq$. Any small set has an upper bound in this ordering. If $t \leq t' \models q$, then $S_t \subset S_{t'}$.*
3. *If $W \subset G$ is connected and generically metastable, ∞-definable over C, then $W \subseteq S_t$ for some $t \models q$.*

$\cup_{t' \models q} S_{t'} = H$ is ∞-definable (with no parameters.) We say that H is the limit-metastable group for G. If $G = H$ we say that G is limit-metastable.

Lemma 5.7. *Assume S_t is the connected component of a definable group R_t. Then the partial ordering on q is (relatively) definable.*

Proof. If $S_t \subseteq R_t$, then necessarily $S_t \subseteq R_t^0 = S_{t'}$. Hence $S_t \subseteq S_{t'}$ iff $(d q_t x)(x \in R_t)$, where q_t is the generic type of S_t. (Another argument, assumign (FD): $S_t \subseteq S_{t'}$ iff $R_t/(R_t \cap R_{t'})$ is finite.) □

In particular this is the case when the S_t are definable groups. When this is so we will say that the system is *definable.*

We thus view H as the limit of a Γ-internal direct limit system of generically metastable groups. It is clearly independent of the particular limit-metastable family.

Two different behaviors are possible, according to whether or not the direct limit system has a maximal element. The latter is equivalent to the existence of a maximal connected generically metastable subgroup of G.

Lemma 5.8. *Let G be a definable Abelian group of bounded weight. There exists a limit-metastable ∞-definable subgroup for G.*

Proof. Let C_1 be an $\aleph_1$-saturated elementary submodel of a large saturated model for the theory. Let $C_1 \subseteq C$ be such that metastability for G is witnessed over C.

Let $\mathfrak{F}$ be the family of all C-definable functions into Γ. Let $A_i (i \in I)$ be the family of all connected generically metastable ∞-definable subgroups of G, defined over C. Let q_0 be the partial $\star$-type found in the proof of Proposition 5.3. Let $c \models q_0$, $t = \mathfrak{F}(c)$, and let S_t be the generically metastable group found there.

By Lemma 3.5, we can take t to be a small tuple. Moreover we can find a small $C_0 \subset C$ such that S_t is $C_0(t)$-definable. Let $q = tp(t/C_0)$.

If W is a connected generically metastable group, we must show that for *some* $t' \models q$, $W \subset S_{t'}$. For this purpose we can replace W by a conjugate, under the group of automorphisms of the universal domain over C_0. Thus we may assume W is defined over C. In this case, $W \subset S_t$. This gives (3).

Define the partial ordering by: $S_t \subset S_{t'}$. The directedness follows from Proposition 5.3, together with (3). □

Theorem 5.9. *Let A be a definable Abelian group in a metastable theory of bounded weight.*

There exists an ∞-definable limit- metastable subgroup K of A, with A/K almost internal to Γ.

If K is itself generically metastable, then A/K is a pro-definable Γ-group.

If (FD_ω) holds, then K is definable, has a definable generic type, and equals the kernel of a definable $\lambda : A \to \Lambda$ into a Γ-group.

Remark 5.10. *For the definability of Λ it suffices to assume, in place of (FD_ω), that every ∞-definable connected generically metastable subgroup of A is contained in a certifiable one.*

Proof. First suppose A contains no nontrivial generically metastable ∞-definable subgroups. Then by Lemma 5.1, no type of elements of A is stably dominated. As the theory is metastable, for some base set C, there exist C-definable maps $f_j : A \to \Gamma$ such that the fibers of $f = (f_j)_j$ are stably dominated. Thus the common kernel of the maps f_j is finite. By compactness finitely many of the f_j have finite kernel. By Lemma 2.23, one can take λ to be an isomorphism.

This argument works for $\star$-definable groups as well. Specifically if $B = \cap_i B_i$ is an intersection of definable subgroups of A, and $A/B = \underleftarrow{lim}_i A/B_i$ has no nontrivial generically metastable ∞-definable subgroups, then each A/B_i is isomorphic to a Γ-group; by stable embeddedness of Γ the group structures and the homomorphisms $A/B_i \to A/B_j$ for $j \geq i$ are also definable over Γ.

Next suppose A has a maximal ∞-definable generically metastable subgroup B. By Proposition 3.15, we have $B = \cap_i B_i$ for certain definable subgroups B_i of A. We can view A/B as a *-definable group (a projective limit of the groups A/B_i.) By Lemma 4.3, A/B can have no nontrivial generically metastable ∞-definable subgroups; otherwise the pullback to A would be a generically metastable ∞-definable group. By the previous case, A/B is definably isomorphic to a pro-definable group Λ of Γ^{eq}. The isomorphism $A/B \to \Lambda$ gives an homomorphism $\lambda : A \to \Lambda \subset \Gamma^{eq}$ with kernel B.

In general, let B be the limit-metastable subgroup of A. B is ∞-definable:

$$b \in B \iff (\exists t \models q)(b \in S_t) \iff (\exists t \models q) \bigwedge_n (b \in S_t^n)$$

If ψ ranges over all finite sets of formulas in q, we find

$$b \in B \iff \bigwedge_n \bigwedge_\psi (\exists t)(\psi(t) \& b \in S_t^n)$$

Let C be a good base for A (Definition 1.2).

Let $\mathfrak{F}$ be the family of all C-definable functions $f : A \to E$, where E is C-definable and $E \subseteq \mathrm{acl}(C, \Gamma)$.

Claim. If $f(c) = f(d)$ for $f \in \mathfrak{F}$, then $cd^{-1} \in B$.

In fact every fiber of $\mathfrak{F}$ lies in a coset of some S_t. This is proved as in Proposition 5.3 Claim 2, and implies the claim. Thus A/B is definably isomorphic to $\mathfrak{F}(A)/E$ for some ∞-definable equivalence relation E.

Now assume that among the connected generically metastable ∞-definable subgroups of A, the certifiable ones are cofinal. Replacing S_t by a larger certfiable group, we can take $(S_t : t \models q)$ to be a uniformly definable family of connected generically metastable definable groups, such that every connected generically metastable definable group is contained in some S_t. Let Q be a definable set containing q, such that S_t is connected and generically metastable for $t \in Q$. Then $\cup_{t \in Q} S_t = \cup_{t \models q} S_t$. (Any S_t with $t \in Q$ is connected generically metastable, and thus by cofinality of the family contained in some S'_t, $t \models q$.) Define a partial ordering on Q by $S_t \subseteq S_{t'}$. This is directed, since again if $t, t' \in Q$ there exists $t'' \models q$ with $t, t' \leq t''$. Thus the limit-metastable group is definable, and hence Λ is definable in Γ^{eq}.

By Lemma 2.19, Q has a cofinal definable type $\tilde{q}$. For $t \models \tilde{q}$ let p_t be the unique generic type of S_t. Let r be the definable type as in Lemma 2.4. If $c \in \Lambda$, and $t \models \tilde{q}|c$, then $c \in S_t$ (since $c \in S_{t'}$ for some $t' \in Q$, and $\tilde{q}$ is cofinal.) Hence if $a \models p_t|\{c, t\}$ then $ca \models p_t|\{c, t\}$. It follows that $a \models r|\{c\}$ and $ca \models r|\{c\}$; showing that r is Λ-translation invariant. □

Corollary 5.11. *(FD_ω) Let A be a definable Abelian group. There exists a universal pair (f, B) with B a Γ-internal definable group, and $f : A \to B$ a definable homomorphism. In other words for any (f', B') of this kind there exists a unique definable homomorphism $h : B \to B'$ with $f' = hg$.*

Proof. Let L be the limit-metastable subgroup of A. Then L is definable. For any pair (f', B') as above, f vanishes on any connected generically metastable group. Since L is a union of such groups, f vanishes on L. But A/L is Γ-internal, so the canonical homomorphism $A \to A/L$ clearly solves the universal problem. □

In particular, in ACVF, for $0 < a \in \Gamma$, there cannot be a compatible sequence of homomorphisms $\psi_n : A \to C(n!a)$ for each n, where $C(b)$ is the definable subquotient $C(b) = \{c \in \Gamma : (\exists m \in \mathbb{N})(-mb < C < mb)\}/\mathbb{Z}b$ of $(\Gamma, +)$.

Corollary 5.12. *(FD_ω) Let A be a definable Abelian group. There exists a smallest definable subgroup A^0 of A of finite index.*

Proof. As in Corollary 5.11, any subgroup A' of finite index must contain each connected generically metastable group, and hence the limit group L. The question thus reduces to the o-minimal group A/L, where it is known, [28]. □

Lemma 5.13. *(FD_ω) Let A be a definable Abelian group. There exists a universal pair (f, B) with B a stable definable group, and $f : A \to B$ a definable homomorphism. Equivalently, there exists a smallest definable subgroup K with A/K stable.*

Proof. Clearly if A/K and A/K' are stable, so is $A/(K \cap K')$. it suffices to show that there are no strictly descending chains $A \supset K_1 \supset K_2 \ldots$of such subgroups. By (FD) $\dim(A/K_n)$ is bounded; so we may assume it is constant, $\dim(A/K_n) = d$. Then K_n/K_{n+1} is finite. By Corollary 5.12, the chain stabilizes. □

Corollary 5.14. *(FD_ω) Let A be a connected definable Abelian group. Then for almost all primes p, $pA = A$.*

Proof. Let L be the limit metastable group of A. We have $L = \cup_{t \models q} B_t$ where q is a complete type and B_t is connected generically metastable. Let A_t be the maximal stable quotient of B_t. Then A_t has finite Morley rank m. Let $A_t(p) = \{x \in A_t : px = 0\}$. Then $A_t(p)$ can be infinite for at most m values of p. For any other p, $A_t(p)$ is finite and hence $pA_t = A_t$. It follows that pB_t has finite index in B_t (Lemma 4.9); so $pB_t = B_t$. Thus $pL = L$. On the other hand $p(A/L) = A/L$ since A/L is a connected o-minimally definable group. So $pA = A$. □

5.15. **Metastable fields.** By a rng we mean an Abelian group with a commutative, associative, distributive multiplication (but possibly without a multiplicative unit element.) If a rng has no zero-divisors, the usual construction of a field of fractions makes sense.

Proposition 5.16. *Let F be a metastable field of bounded weight. There exists an ∞- definable subrng D of F, with $(D, +)$ connected generically metastable.*

Moreover F is the field of fractions of D.

(FD_ω) If D is a ring, then the generic type of $(D, +)$ is also generic for $(D^*, \cdot)$.

Proof. Let n be the maximal weight of a stably dominated type of elements of F. Using Lemma 5.1, find a definable subgroup M of F^* with a stably dominated generic type p of weight n. Let D be a subgroup of $(F, +)$ with generic type $p^{\pm 2n}$, generated by $p^{\pm 2}$ (additively). Then p clearly stabilizes D multiplicatively. Since D is generated by $p^{\pm 2n}$, so does D; so D is a subrng of F.

Let F' be the field of fractions of D. Suppose $a \in F$, $a \notin F'$. Then the map $(x, y) \mapsto x + ay$ takes $D^2 \to F$ injectively. (For if $x + ay = x' + ay'$ and $(x, y) \neq (x', y')$ then $y \neq y'$ and $a = (x' - x)/(y - y')$.) Thus $D + aD$ is definably isomorphic to D^2, hence is metastable of weight $2n$; but this contradicts the weight bound n on stably dominated types of F. So $F = F'$ is the field of fractions of D.

Note also that a generic element of M is a unit, and every element is a sum of generics, so the units D^* generate D additively.

By Corollary 3.16, D is the intersection of definable subrings D_n.

Let N be the unique normal ∞-definable subgroup of $(D, +)$ such that D/N is stable (cf. Proposition 4.6.) Then clearly N is multiplicatively invariant under the units of D. Thus N is actually an ideal of D, so D/N is a rng.

Now assume that $1 \in D$, and (for simplicity) (FD_ω) . Then D is definable. D/N is a definable connected ω-stable ring. It follows that modulo the Jacobson radical, D/N is a finite union of fields. Hence the additive generic type of D/N is also a generic type of $(D/N)^*$.

Let $c \in R$ be an additive generic. Then the ideal cR has image in R/J containing the generic $c + J$, which is invertible, so the image equals R/J. By Lemma 4.9, $cR = R$. So c is invertible. □

Additional remarks

(1) In place of $1 \in D$ it suffices to assume that R/J is not nil, i.e. there is no n with $(\forall x_1, \ldots, x_n)(x_1 \cdot \ldots \cdot x_n = 0)$ in R/J. This holds for instance if R is closed under roots.

(2) $(F, +)$ is properly limit metastable, unless the limit metastable group is 0 (in this case F is Γ-internal) or F is stable.

(3) Let $M = \cup M_t (t \models q)$ be the limit metastable system of F^*. As above one can construct a generically metastable rng R_t, stabilized by M_t. If $t < s$ we have $R_t \subseteq R_s$ and $N_t \subseteq N_s$ (since $R_t/(R_t \cap N_s)$ is stable.) The limit $R = \cup R_t$ cannot equal F (unless F is stable), since if $a \in R_t \setminus N_t$ then $a^{-1} \notin R_s$ for $s > t$, so $a^{-1} \notin R$.

(4) The maximal stable quotient B of R_t does not depend on t. We have $h_t : R_t \to B$. If $s \models q$ and $t \models q|s$, the inclusion $R_s \to R_t$ induces a homomorphism $f : B \to B$, which also does not depend on s, t. Hence $f^2 = f$. So $B = I \oplus A$ where $A = f(B)$ and $I = \ker(f)$. We have $f|A = Id$. The direct limit of the B under these maps is A. If $I = (0)$ then the system degenerates to a single element, by stable domination of R_s via A.

(5) Let $S_t = \{r \in R : rR_t \subseteq R_t\}$. This ring contains M_t. Hence $S = \cup S_t$ contains M. It follows that $V = F^*/S^*$ is Γ-internal. Note that V admits a natural subsemigroup $V^+ = (R \setminus (0))/S^*$, and that since any element of F is a quotient of two elements of R (in fact of any R_t), we have $V^+ - V^+ = V$.

(6) Since by (3) $R \neq F$ we have $S \neq F$, so $V^+ \neq V$. By Lemma 2.24, V does not have fsg, so is not definably compact. By [21] V contains a definable one-dimensional group without torsion.

6. Valued fields: Generically metastable subgroups of algebraic groups

In this section, we work with definable groups in ACVF. Let K be an algebraically closed valued field; $\mathcal{O}$ denotes the valuation ring. Occasionally we will assume K to be saturated.

We repeat the statement of Proposition 1.7. Recall Lemma 2.29

Proposition 6.1. *Let H be a connected, generically metastable definable or ∞-definable group. Then there exists an algebraic group G and a definable homomorphism $f : H \to G$, with boundedly imaginary kernel.*

If H is defined over $C = \mathrm{acl}(C)$, then G, f can be found over C.

Proof. Let $F = C \cap K$ Let p be the definable generic type of H; let $(a_1, a_2) \models p^{\otimes 2}|C$, $a_3 = a_1 a_2$.

Let $\tau = \{a_1, a_2, a_3, a_{12} = a_1 a_2, a_{23} = a_2 a_3, a_{123} = a_1 a_2 a_3\}$. We view this six-element set as a *matroid*, given by specifying the collection of algebraically dependent subsets of τ. This data is called a group configuration.

For $c \in \tau$, let $A(c) = \mathrm{acl}(C(c)) \cap K$; this is the set of field elements in the algebraic closure of c over C. Pick a tuple $\alpha(c) \in A(c)$ such that $A(c) = F(\alpha(c))^{alg}$.

We view $c \mapsto \alpha(c)$ as a map on τ into the matroid of algebraic dependence in the algebraically closed field K over F. Then α preserves both dependence and independence. For independence this is clear. For dependence we need:

Claim 1. Let E_1, E_2 be two algebraically closed substructures of a model of ACVF, all sorts allowed. Let L_i be the set of field elements of E_i. If $c \in \mathrm{acl}(E_1 \cup E_2)$ and $c \in K$, then $c \in (L_1 L_2)^{alg}$.

Hence $\alpha(\tau)$ is isomorphic to τ. According to the group configuration theorem for stable theories, applied to the theory ACF over the model F, there exists an ACF- definable group G with parameters in F, such that $a(\tau)$ is a group configuration for G; in particular there exist $b_1, b_2 \in G$, $b_{12} = b_1 b_2$ such that $A(a_i) = F(b_i)^{alg}$. Since ACF-definable groups are (definably isomorphic to) algebraic groups, we can take G to be an algebraic group over F. Compare [10] (Proposition 3.1).

We work in the group $H \times G$. Let $c_i = (a_i, b_i)$. By Proposition 2.10 (4), $tp(c_i/C)$ is stably dominated, and so has a unique C- definable extension q_i. Let $Z = Stab(q_2, q_3)$. This is a coset of $S = Stab(q_2, q_2)$; and q_1 is a generic type of Z. (I.e. $q = (q_1)^{{c_1}^{-1}}$ is a generic type of S.) Let $J = \{h \in H : (h, 1) \in S\}$.

Claim 2. J is boundedly imaginary.

Proof. If $h \in J$ then for $(a, b) \models q_2|C(h)$, $(ha, b) \models q_2|C$. Now $tp(a/bC)$ is purely imaginary, and so is $tp(ha/bC)$. By Lemma 2.27, $\beta_0 \leq 0 \leq \beta_1 \in \Gamma(C(a, b))$, $a \in \mathrm{dcl}(C, \beta_0, \beta_1, \beta_0\mathcal{O}/\beta_1\mathcal{M})$. Since $tp(ab/C)$ is stably dominated, we have $\beta_0, \beta_1 \in \Gamma(C)$, and $a \in \mathrm{dcl}(C, \beta_0\mathcal{O}/\beta_1\mathcal{M})$. Similarly there exist $\beta_0', \beta_1' \in \Gamma(C)$ with $ha \in \mathrm{dcl}(C, \beta_0'\mathcal{O}/\beta_1'\mathcal{M})$. Thus $h \in \mathrm{dcl}(a, ha, C) \subseteq \mathrm{dcl}(C, \beta_0\mathcal{O}/\beta_1\mathcal{M}, \beta_0'\mathcal{O}/\beta_1'\mathcal{M})$. By Corollary 2.29, some C-definable $tp(h/C)$ is boundedly imaginary. This holds for any $h \in J$; so by compactness J is boundedly imaginary. $\square$

Let $J' = \{g \in G : (1, g) \in S\}$. By a dual argument, using that $b \in \mathrm{acl}(a)$ when $(a, b) \models q_2$, we see that J' is a finite normal subgroup of G.

Note that S projects onto H; this is because the projection contains $a^{-1}a'$ for $(a, a') \models p_1$; in other words it contains a realization of $p_1|M$; but p_1 generates H.

Thus J is a boundedly imaginary normal ∞-definable subgroup of H.

Now S can be viewed as the graph of a homomorphism $f : H \to G/J'$ with kernel J. Now G/J' is isomorphic to an algebraic group defined over F; so replacing G by G/J' we may assume $f : H \to G$. S is ∞-definable (an intersection of definable groups); but being the graph of a function, it must be definable relative to H. (We have $(a, b) \notin S$ iff $(a, b') \in S$ for some $b' \neq b$.) In particular J is definable relative to H.

$\square$

The proof of Lemma 6.1 goes through, at least over a model, assuming only that the generic type of H is symmetric (rather than stably dominated.) Lemma 2.5 replaces Proposition 2.10 (4). However in our context this gains no generality, since in ACVF every symmetric definable type is stably dominated.

Additional Remarks

(1) Let $f' : H \to G'$ be another homomorphism into an algebraic group. Since J is purely imaginary, $f'(J)$ is finite. Thus $\ker(f')$ contains a finite index subgroup of $\ker(f)$.

(2) Assume H is Abelian, or more generally that H is a subgroup of a definable group H' such that Question 6.16 has a positive answer for definable subgroups of H'. Then one can demand that $\ker(f)$ be be as small as possible, i.e. that $\ker(f)/\ker(f)^0$ be least possible. It follows that

(*) Let G' be an F-algebraic group, and let $f' : H \to G'$ be a definable homomorphism (over any base). Then there exists a unique definable homomorphism $g : f(H) \to G'$ with $f' = g \circ f$.

With a small additional argument, one can show that (*) is valid for any algebraic group G', not necessarily defined over F.

Remark 6.2. *Let H be a purely imaginary definable group, and let B be a generically metastable subgroup. Then B is boundedly imaginary.*

Proof. Let p be the principal generic type of B; with H, B, p defined over C. Then $\Gamma(C(a)) = \Gamma(C)$ for $a \models p|C$. Since $p|C$ is purely imaginary, it follows that it is boundedly imaginary. Any element of B^0 is a product of two realizations of $p|C$, so B^0 is boundedly imaginary. It follows that some definable set B' containing B^0 is boundedly imaginary. Finitely many translates of B' cover B, so B too is boundedly imaginary. $\square$

Proposition 6.3. *Let H_0 be a definable group, such that 6.16 has a positive answer for subgroups of H_0.*

Let H be an ∞- definable limit metastable subgroup of H. Assume this data is defined over $C = \mathrm{acl}(C)$, and let $F = K \cap C$.

Then there exists an algebraic group G and a definable homomorphism $f : H \to G$, with purely imaginary kernel.

For any definable homomorphism $f' : H \to G'$ into an F-algebraic group over C, there exists a unique $g : f(H) \to G'$ with $f' = g \circ f$.

Proof. Let $H_t (t \models q_0)$ be a limit metastable system for H; so q_0 is a Γ-internal type with a directed definable ordering, $H_t \leq H_{t'}$ for $t \leq t'$, and $\cup_t H_t = H$. We may extend q_0 to a definable type q, compatible with the directed ordering (Lemma 2.19).

For $t \models q_0$, by Proposition 6.1 there exists an algebraic group G and a definable homomorphism $f_t : H \to G$, with boundedly imaginary kernel J_t. G is $\mathrm{acl}(C(t))$-definable; but as $t \in \mathrm{dcl}(C, \Gamma)$ we have $K \cap \mathrm{acl}(C(t)) = K \cap C = F$. So G is C-definable and does not depend on t. As in Remark 2, J_t^0 is definable, and any boundedly imaginary subgroup of H_t contains J_t^0. As in Remark (2) may replace J_t by the smallest subgroup (containing J_t^0) such that the quotient embeds into an algebraic group. Then $J_t \subseteq (J_{t'} \cap H_t)$ if $t' \geq t$.

If $t \models q_0$ and $t' \models q|C(t)$, then $f_t(J_{t'} \cap H_t)$ is finite since $J_{t'}$ is purely imaginary. This is a finite set of elements of G, and so (as $\mathrm{acl}(C(t)) \cap K = \mathrm{acl}(C) \cap K$) it cannot depend on t'; so $f_t(J_{t'} \cap H_t)$ is a fixed finite group F_0. Thus $J_t^* = J_{t'} \cap H_t = f_t^{-1}(F_0)$ is a fixed normal subgroup of H_t, not depending on t'. Note that if $t'' \models q|C(t,t')$ then $J_{t'}^* \cap H_t = (J_{t''} \cap H_{t'}) \cap H_t = J_{t''} \cap H_t = J_t^*$. It follows that one can replace each J_t by J_t^* and obtain: $J_{t'} \cap H_t = J_t$.

We obtain induced embeddings $\phi_{t,t'} : f_t(H_t) \to f_{t'}(H_{t'})$, such that $\phi_{t,t'} \circ f_t = f_{t'}|H_t$. Clearly $\phi_{t',t''} \circ \phi_{t,t'} = \phi_{t,t''}$.

Let $\Phi_{t,t'}$ be the Zariski closure of the graph of $\phi_{t,t'}$. This is a Zariski closed, connected subgroup of $G \times G$ of dimension equal to $\dim(G)$, and projecting onto G in both directions (since $\dim(f_t(H_t)) = \dim(G)$.) So $\Phi_{t,t'}$ is the graph of an isogeny. At all events it is coded by elements of K, and so cannot depend on t or t'; $\Phi_{t,t'} = \Phi$.

The relation $\phi_{t',t''} \circ \phi_{t,t'} = \phi_{t,t''}$ shows that $\{(a,b,c) : \phi_{t,t'}(a) = b, \phi_{t',t''}(b) = c\}$ is contained in $\{(a,b,c) : (a,c) \in \Phi\}$. But the former set is Zariski dense in the connected component of $\Phi \circ \Phi$. Thus $\Phi \circ (\Phi \sim \Phi)$ where $\sim$ denotes commensurability. Let Φ' be the Zariski closure of $\phi_{t,t'}^{-1}$. We have

$$\Phi \subseteq \Phi \circ (\Phi \circ \Phi') = (\Phi \circ \Phi) \circ \Phi') \sim \Phi \circ \Phi' \sim Id_G$$

As Φ is a connected, $\Phi = Id_G$. So $\phi_{t,t'}(x) = x$.

Define $f(x) = f_t(x)$ where $x \in H_t$. This is then well-defined. Given a map $f' : H \to G'$, there exists a unique $g_t : f_t(H_t) = f(H_t) \to G'$ with $f' = g_t f_t$, so we must have $g_t = g_{t'}\ f_t(H_t)$; let $g = \cup_t g_t$.

□

Remark 6.4. *Assume in Proposition 6.1 or Proposition 6.3 that H has no nontrivial Abelian normal subgroups. Then one can find f with the following universal property:*

(**) for any definable homomorphism $f' : H \to G'$ into an algebraic group, there exists a unique definable homomorphism of algebraic groups $g : G \to G'$ with $f' = g \circ f$.

Indeed by factoring out any normal Abelian subgroups of G, we may assume G is semi-simple. In this case G as an algebraic group has only finitely many connected finite central extensions. Let $\mathcal{F}$ be the family of those finite central extensions G' of G that admit a homomorphism $f' : H \to G'$, lying over f. If (f', G') and (f'', G'') are two such, let $f'''(h) = (f'(h), f''(h)) \in G' \times G''$, and let G''' be the Zariski closure of $f'''(H)$; then G''' dominates both G' and G''. It follows that $\mathcal{F}$ has a maximal element. By a similar construction one shows that (**) holds. Uniqueness of g follows from Zariski density of $f(H)$.

Remark 6.5. *The examples of § 6.17 show that the assumption of limit metastability cannot be eliminated. Without it, it seems likely that one can find an isogeny of H into a quotient of an Ind-algebraic group.*

6.6. **Stably dominated subgroups of algebraic gruops.** By an $\mathcal{O}$-variety (or variety over $\mathcal{O}$), we mean a flat, reduced scheme over Spec$\mathcal{O}$, admitting a finite open covering by schemes isomorphic to $\mathcal{O}[X_1, \ldots, X_n]/I$. Since $\mathcal{O}$ is a valuation ring, Spec$R$ is flat over Spec$\mathcal{O}$ iff no nonzero element of $\mathcal{O}$ is a 0-divisor in R. So $\mathcal{O}[X_1, \ldots, X_n]/I$ is flat iff $I = IK[X_1, \ldots, X_n] \cap \mathcal{O}[X_1, \ldots, X_n]$. Hence there are no infinite descending chains of $\mathcal{O}$-subvarieties.

If V is an $\mathcal{O}$-variety, we write $V(K)$ for $(V \otimes Spec(K))(K)$. Let $V_k = V \otimes_{\mathcal{O}} k$, and let $r : V(\mathcal{O}) \to V_k(k)$ be the natural map. By flatness, $\dim V_k = \dim V_K$.

For any set $Z \subseteq \mathcal{O}_K^n$, if $I = \{f \in \mathcal{O}[X_1, \ldots, X_n] : f|Z = 0\}$ and $R = \mathcal{O}[X_1, \ldots, X_n]/I$,then then Spec$R$ is flat over Spec$\mathcal{O}$. If $Z = V(K) \cap \mathcal{O}_K^n$ for some K-variety V, then Spec$R(\mathcal{O}) = Z$.

If W is a scheme over $\mathcal{O}$, and V' a subvariety of $W_K = W \otimes_{\mathcal{O}} K$, there exists a unique $\mathcal{O}$-subvariety of W such that $V_K = V'$, and $V(\mathcal{O}) = V'(K) \cap W(\mathcal{O})$.

For example, if $V' \subset \mathbb{A}^n$ is an affine variety over K, defined by a radical ideal $P \subset \mathcal{O}[X]$, we let $V = \mathrm{Spec}\mathcal{O}[X]/P \cap \mathcal{O}[X]$. Let $V_k \subset k^n$ denote the zero set of the image of $P \cap \mathcal{O}[X]$ in $k[X]$. In this case we denote the affine coordinate ring $K[X]/P$ by $K[V]$, and $\mathcal{O}[V] = \mathcal{O}[X]/(P \cap \mathcal{O}[X])$.

Lemma 6.7. *Let V be a scheme over $\mathcal{O}$, with $\dim V_K = n$. Let q be a K-definable type of elements of $V(\mathcal{O})$ with r_*q of Morley rank $\geq n$. Assume V is defined over $B = acl(B)$; then so is q. In fact there are only finitely many q with this property.*

Proof. Say q is defined over $B' \supset B$. Let $a \models q|B'$. Then $r(a) \downarrow_B B'$. $B(a)$ is a field extension of B of transcendence degree n, and also of residual transcendence degree n. (Indeed $tr.deg._{B'} B'(r(a)) = n$, so $tr.deg._B B(r(a)) \geq n$.) It follows that $a \downarrow_B B'$.

Since $\dim V_k \leq n$, the set of types in question is the set of q with r_*q a generic type of V_k. This is an elementary class. (If predicates are added to designate $(d_q x)\phi$ for all ϕ, then the set of q corresponds to the set of expansions to a certain partial theory.) Since it has boundedly many elements, it can only have finitely many. □

Let G be a group scheme over $\mathcal{O}$, with generic fiber G_K and special fiber G_k. Then $G(\mathcal{O}) \subset G_K(K)$, and we have a definable group homomorphism $r : G(\mathcal{O}) \to G_k$.

Proposition 6.8. *Let p be a definable type of elements of $G(\mathcal{O})$. Assume $\dim G_K = \dim G_k = d$, and r_*p is a generic type of G_k. Then p is a generic type of $G(\mathcal{O})$.*

When G is of finite type, $G(\mathcal{O})$ has finitely many generic types.

Proof. Consider translates $q =^g p$ of p, $g \in G(\mathcal{O})$. Clearly $r_*q =^{r(g)} r_*p$. By Lemma 6.7, q is defined over $acl(B)$, where B is a base of definition of G, p. So p is generic.

Since $G(\mathcal{O})$ has a stably dominated generic, all generics are translates of each other. Thus all generics q have r_*q generic. By Lemma 6.7 again, there are only finitely many generics. □

We will see that all generically metastable ∞-definable groups, with their generics, may be obtained this way; but Example 6.14 shows that we cannot take G to be of finite type.

6.9. **Linear groups.**

Proposition 6.10. *Let G be an affine algebraic group over K, H a Zariski dense definable subgroup of $G(K)$. Let p be a definable type of elements of H. Then the following are equivalent:*

(1) *p is the unique definable generic of H, and p is stably dominated.*

(2) *For any regular function f on G, p attains the highest modulus of f on H; i.e for some γ_f,*
$$\models (d_p x)(\mathrm{val} f(x) = \gamma_f)$$
and for any $x \in H$, $\mathrm{val} f(x) \geq \gamma_f$.

Proof. (1) $\Longrightarrow$ (2) :
Since p is stably dominated, for any regular f there exists γ_f with $(d_p x)(\mathrm{val} f(x) = \gamma_f)$. If $(a,b) \models p \otimes p$ then $ab \models p$; so $\mathrm{val} f(ab) = \gamma_f$. By Proposition 2.32, for any $a, b \models p$ we have $\mathrm{val} f(ab) \geq \gamma_f$. But since p is the unique generic, any element c of H is a product of two realizations of p. Thus $\mathrm{val} f(c) \geq \gamma_f$.

(2) $\Longrightarrow$ (1): Note using quantifier elimination, that (2) characterizes p uniquely. Also, since γ_f does not depend on x, p is orthogonal to Γ, hence stably dominated by Proposition 2.14.

On the other hand, (2) is invariant under H-translations. Thus if (2) holds of p, it holds of every translate, so every translate equals p. By Remark 3.4, p is the unique generic type of H. □

Note that connectedness of G follows from the assumptions (any function constant on connected components of G is regular.)

Let G be an algebraic group scheme over $\mathcal{O}$. $\mathrm{res} G(\mathcal{O})$ is a definable, hence algebraic, subgroup of $(\mathrm{res} G)(k)$. It may be a proper subgroup. Let $n = \dim(G_k)$. This in turn, when G is not of finite type, may be smaller than $\dim(G_K)$.

Proposition 6.11. *Let G be an affine algebraic group scheme over $\mathcal{O}$. Assume:*
(*) if $f \in K[G]$ and $|f(x)| \leq 1$ for $x \models p$, then $f \in \mathcal{O}[G]$.
Let p be a definable type of elements of $G(\mathcal{O})$. Then p is the unique generic of $H = G(\mathcal{O})$ iff $r_* p$ is the unique generic type of $\mathrm{res}\, H$.

Proof. In an algebraic group chunk, to show that a type is the unique generic is to show that any regular function f vanishing on the type vanishes on the whole chunk. If f over k vanishes on $r_* p$, lifting to $\mathcal{O}$ we obtain a regular function $F \in \mathcal{O}[G]$ with $\mathrm{val}(F(a)) > 0$ for $a \models p$. By Proposition 6.10 (2), $\mathrm{val}(F(a')) > 0$ for all $a' \in G(\mathcal{O})$. So F vanishes on $\mathrm{res}(G(\mathcal{O}))$.

The converse uses (*). Let $F \in K[G]$. Since p is stably dominated, for some γ, for any $a \models p$, $|F(a)| = \gamma$. If $\gamma = 0$, then by assumption any K- multiple of F lies in $\mathcal{O}[G]$, so $F = 0 \in K[G]$. Otherwise, we may assume $\gamma = 1$. By assumption we may take $F \in \mathcal{O}[G]$. Suppose $|F(a')| = |c| > \gamma$, $a' \in G(\mathcal{O})$; (we may take $a' \in G(\mathcal{O}_0)$, a fixed submodel.) Then $c^{-1} \in \mathcal{O}$, and $|c^{-1} F(a)| < 1$ for $a \models p$, i.e. $\mathrm{res} c^{-1} F(a) = 0$. By Zariski density in G_k, $\mathrm{res} c^{-1} F$ vanishes on $\mathrm{res} G(\mathcal{O})$; so $|c^{-1} F(a')| < 1$ for all $a' \in G(\mathcal{O})$; a contradiction. □

Proposition 6.12. *genchar3 Let G be an affine pro-algebraic group over K. Let H be a Zariski dense ∞- definable subgroup of G, with unique stably dominated generic type p. Then there exists a group scheme $\mathbb{H}$ over $\mathcal{O}$ and an isomorphism $\phi : G \to \mathbb{H}_K$, such that $\phi(H) = \mathbb{H}(\mathcal{O})$.*
Moreover, $\mathbb{H}$ is a pro-group variety over $\mathcal{O}$.
If H is definable, there exists an affine group variety H' over $\mathcal{O}$ such that $H \simeq H'(\mathcal{O})$.

Proof. p is Zariski dense in G (and G is connected) by Proposition 6.10.

G, H are defined over some subfield $K_0 = (K_0)^a$ of K. Let $R_0 := K_0[G]$ be the affine coordinate ring of G.

Define $R = \{f \in K_0[G] : (d_p x) \mathrm{val} f(x) \geq 0\}$. This is an $\mathcal{O}$-subalgebra of R_0.

Claim 1. $R \otimes_{\mathcal{O}} K = R_0$.

Proof. Let $0 \neq r \in R_0$. Since p is stably dominated, it is orthogonal to Γ. Thus for some $c \in K_0$, for $a \models p | K_0$, $|r(a)| = |c|$. If $c = 0$, then r vanishes on p. Since p is Zariski dense in

G, r vanishes on G, i.e. $r = 0 \in R_0$, contradicting the choice of r. So $c \neq 0$, and $c^{-1}r \in R$. This shows that the natural map $R \otimes_O oK \to R_0$ is surjective. Injectivity is clear since R has no $\mathcal{O}$-torsion. □(Claim.)

Let $\mathbb{H} = \mathrm{Spec}R$. So $\mathbb{H}_K := \mathbb{H} \times_{\mathrm{Spec}\mathcal{O}} \mathrm{Spec}K = spec(R \otimes_{\mathcal{O}} K) = G$. We identify $\mathbb{H}_K$ with G. So p is a type of elements of $G(K)$ and in fact, by definition of R, of $G(\mathcal{O})$.

The morphisms $x \mapsto x^{-1} : G \to G$ and $(x, y) \mapsto xy : G^2 \to G$ correspond to two operations:

$$i : R_0 \to R_0,\ i(r)(g) = r(g^{-1})$$

$$c : R_0 \to R_0 \otimes_{K_0} R_0,\ c(r) = \sum_{i=1}^{n} r_i \otimes s_j,\ r(gh) = \sum_{i=1}^{n} r_i(g) s_i(h)$$

Note that any $\mathcal{O}$-subalgebra R' of R_0 is $\mathcal{O}$-torsion-free, hence a flat $\mathcal{O}$-module. Thus the maps $R' \otimes_{\mathcal{O}} R' \to R' \otimes_{\mathcal{O}} R \to R \otimes_{\mathcal{O}} R$ are injective. We identify $R' \otimes_{\mathcal{O}} R'$ with its image in $R \otimes_{\mathcal{O}} R$.

Let us say that an $\mathcal{O}$-subalgebra R' of R_0 is *Hopf* if $i(R') \subseteq R'$ and $c(R') \subseteq R' \otimes_{\mathcal{O}} R'$.

Claim 2. R is Hopf.

Proof. : Co-inversion: Let $g = i(r), r \in R$. Clearly $\mathrm{val}(g(x)) = \mathrm{val} f(x^{-1}) \geq 0$ for $x \models p$. Hence $g \in R$.

Co-multiplication: Let $(a, b) \models p^2$, and $r \in R$. Write $r(ab) = \sum_j g_j(a) h_j(b)$ (a finite sum, with $g_j, h_j \in R_0$.) By [8] (Lemma 12.4; see Claim there), we may assume $|r(ab)| = \max_j |g_j(a)||h_j(b)|$. Note no g_j is zero; and then by renormalizing g_j and h_j, we may assume $|g_j(a)| = 1$ (as in the first Claim.) But then

$$\max_j |h_j(b)| = |r(ab)| \leq 1$$

Since both a and b realize $p|K_0$, for generic $a' \models p|K_0$ we have $|g_j(a')| = 1$, $|h_j(a')| = |c_j|$ with $|c_j| \leq 1$. As in Claim 1, we can take $c_j \in K_0$. Thus $g_j, h_j \in R$. □(Claim).

Claim 3. p is the unique generic type of $\mathbb{H}(\mathcal{O})$.

Proof. Proposition 6.10 (2) and the definition of R. □

It follows that $H = \mathbb{H}(\mathcal{O})$. Let $\mathcal{F}$ be the family of finitely generated $\mathcal{O}$-subalgebras of R that are Hopf. If $R' \in \mathcal{F}$ then $\mathrm{Spec}R'$ is a group $\mathcal{O}$-variety. To show that $\mathbb{H}$ is a pro-group variety over $\mathcal{O}$, it suffices to show that R is the direct limit (i.e. the union) of $\mathcal{F}$.

Note that if S is generated by R', R'' as an $\mathcal{O}$-algebra, then then S is closed under c if R', R'' are. Indeed $c : R \to R \otimes_{\mathcal{O}} R$ is an $\mathcal{O}$-algebra homomorphism, so $\{r : c(r) \in S \otimes_{\mathcal{O}} S\}$ is an $\mathcal{O}$-subalgebra of S, hence equal to S since it contains R', R''. Moreover i is an automorphism of R of order 2; if $c(R') \subseteq R' \otimes R'$, then the same holds for the $\mathcal{O}$-algebra $i(R')$; since the $\mathcal{O}$-algebra generated by $R' \cup i(R')$ is closed under i, it is Hopf. Hence it suffices, given $r \in R$, to find a finitely generated $\mathcal{O}$-subalgebra R' of R with $r \in R'$ and $c(R') \subseteq R' \otimes R'$.

Claim 4. Let $r, a_i, b_i \in R$. If $(g, h) \models p^2$ and $r(gh) = \sum_{i=1}^{n} a_i(g) b_i(h)$, then $c(r) = \sum a_i \otimes b_i$.

Proof. Since p is Zariski dense in G, $p^{(2)}$ is Zariski dense in G^2. So if $r(gh) = \sum_{i=1}^{n} a_i(g) b_i(h)$ for $(g, h) \models p^2$ then this holds for all $(g, h) \in G^2$. □

By virtue of this claim, all elements x, y, g_j, h considered below can be taken to be independent realizations of p.

Fix $r \in R$.

Write $c(r) = \sum_{i=1}^{n} a_i \otimes b_i$, with n least possible, and $a_1, \ldots, a_n, b_1, \ldots, b_n \in R$. (Proof of Claim 2.)

The expression $r(xy) = \sum_{i=1}^{n} a_i(x) b_i(y)$ shows that $\{r(gy) : g \in G\}$ spans a finite-dimensional K-space. Similarly $\{r(gyh) : g, h \in G\}$ spans a finite-dimensional K-subspace V

of $K[G]$. Note that $V \cap R$ is a lattice in V. Let R' be the $\mathcal{O}$-algebra generated by $V \cap R$. We will show that $a_i, b_i \in R'$, so that $r \in A = \{x : c(x) \in R' \otimes_{\mathcal{O}} R'\}$. By construction, R' and hence A are left,right G-invariant. It follows that A contains V and hence R'. So R' is Hopf.

We saw in Claim 2 that the a_i, b_i lie in R. Thus it suffices to show that they lie in V. By symmetry, it suffices to show that $a_i \in V$.

Let $(g_1, \ldots, g_n) \models p^{(n)}|K_0$. Then $r(xg_j) = \sum_i a_i(x) b_i(g_j)$.

Claim 5. The matrix $b = (b_i(g_j))_{1 \le i,j \le n}$ is invertible over K.

Proof. Suppose otherwise. Then there exists a nonzero vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha \cdot b = 0$. We may assume $a_n = 1$. Since $g_1, \ldots, g_n$ are independent, some g_j must be independent from α (which has weight $n-1$). Now $\sum \alpha_i b_i(g) = 0$ for generic $g \models p|K_0(\alpha)$. By Claim (4), $\sum \alpha_i b_i = 0$; thus $b_1, \ldots, b_n$ are K-linearly dependent, contradicting the minimality of n. □

Now the $r(xg_j)$ are in V by definition. By Claim 5, $a_i(x) \in V$. □

This gives Theorem 1.9.

Lemma 6.13. *Let $V \subset \mathbb{A}^n$ be an irreducible affine K-variety, and assume $V(\mathcal{O}) := V(K) \cap \mathcal{O}^n \neq \emptyset$. Then $V(\mathcal{O}) \cap \mathcal{O}^n$ is Zariski dense in V.*

Proof. (A similar argument was given by Scanlon, in a slightly different context.) Let L be a large algebraically closed field extending K. Let $a \in V(L)$ be a K-generic point. Thus for any $b \in V(K)$ there exists a K-algebra homomorphism $h : K[a] \to K$ with $h(a) = b$. In particular, there exists such an h with $h(a) \in V(\mathcal{O})$. It follows that $\mathcal{M}_K$ generates a proper ideal of $\mathcal{O}_K[a]$. (Otherwise, for some $m \in \mathcal{M}_K$ and $f \in \mathcal{O}_K[X]$, $mf(a) = 1$; but then applying h we would have $mf(h(a)) = 1$.) Thus $\mathcal{M}_K$ extends to a maximal ideal M' of $\mathcal{O}_K[a]$, and thence to a maximal ideal M'' of some valuation ring $\mathcal{O}_L$ of L, with $\mathcal{O}_K[a] \subset \mathcal{O}_L$. Thus the valuation on K can be extended to L in such a way that $a \in \mathcal{O}(L)$. By model completeness of ACVF, there exists $a' \in V(K)$ outside any given proper K- subvariety of V, with coordinates in $\mathcal{O}$. □

Example 6.14. *An ∞-definable generically metastable group that is not definable, not purely inertial, and not the connected component of a definable group.*

Let $K_0 = \mathbb{C}(t)^a$, with $\mathrm{val}(t) > 0$. Let $H_n = \{(x,y) \in G_a \times G_m : \mathrm{val}(y - \sum_{i=1}^n 1/n!(tx)^n) \ge \mathrm{val}(t^{n+1})\}$. H_n is a definable subgroup of $(G_a \times G_m)(\mathcal{O})$, isomorphic to $(G_m \times G_a)(\mathcal{O})$. $H = \cap H_n$ is also generically metastable; but it is dominated by the map $(x,y) \mapsto \mathrm{res}(x)$. The Zariski closure of H has dimension 2, but the generic type of H has a residual part of transcendence degree one. □

Lemma 6.15. *Let G be a group scheme over $\mathcal{O}$. For each n, let $\phi_n(g) = g^n$, and assume $\phi_n : \phi_n^{-1}(V) \to V$ is a finite morphism. Then $G(\mathcal{O})$ is connected.*

Proof. By properness, $\phi_n : G(\mathcal{O}) \to G(\mathcal{O})$ is surjective. So any finite quotient has order prime to n. This holds for all n, so $G(\mathcal{O})$ has no finite quotients. □

Question 6.16. *in ACVF, is G/G^0 always finite?*

The answer is presumably yes (cf. Problem 1.5 and the proof of Lemma 5.12). This would imply that Lemma 5.11 holds for non-Abelian groups too.

Example 6.17 (Versions of tori)**.** The value group of an algebraically closed valued field is an ordered $\mathbb{Q}$-vector space, admitting quantifier elimination. Every group definable over the value field has a chunk in common with the group $V = \Gamma^n$. Interesting variants of Γ^n are known. If Δ is a finitely generated subgroup of Γ^n, the convex hull $C(\Delta)$ of Δ (with respect to the product partial ordering of V) is an Ind-definable subgroup (a non-definable direct limit of definable sets). Δ is of course itself not-definable. But the quotient $C(\Delta)/\Delta$ is canonically isomorphic to a group intereppretable over Γ. See [20].

These examples lift to the tori $T = {G_m}^n$ over algebraically closed valued fields. Let $r : T \to V$ be the homomorphism induced by the residue map. Let Δ be a finitely generated subgroup of T. Then $C(\Delta) = r^{-1}(C(r\Delta)$ is Ind-definable, and $C(\Delta)/\Delta$ is definable in ACVF. For example, fix $t \in K$ with $\mathrm{val}(t) = \tau > 0$, and define a group structure on $A(t) = \{x : \mathrm{val}(x)0 \leq \mathrm{val}(x) \leq \tau\}$ by: $x * y = xy$ if $xy \in A(t)$, $x * y = xy/t$ otherwise. This group admits a homomorphism into a Γ-definable group, with generically metastable kernel.

6.18. **Abelian varieties.**

Definition 6.19. *Let V be an affine variety over K. A definable subset W is* bounded *if for any regular function f on V, $\mathrm{val} f$ is bounded below on W.*

For a general K-variety V, a definable subset W is bounded *if there exists an open affine covering $V = \cup_{i=1}^m U_i$, and a bounded $W_i \subseteq U_i$, with $W = \cup_i W_i$.*

This definition is due to [?], §6.1; the assumption there that the valuation is discrete is inessential. If V' is a closed subvariety of V and W is bounded on V, then clearly $W \cap V'$ is bounded on V'. In the affine case, if V has coordinate ring $K[f_1, \ldots, f_n]$, it suffices that the f_i be bounded. In the case of projective space $\mathbb{P}^n$, a standard covering by bounded sets is given in projective coordinates by: $U_i = \{(x_0 : \cdots : x_n) : (\forall j \leq n)(\mathrm{val}(x_i) \leq \mathrm{val}(x_j))\}$. Complete varieties are bounded as subsets of themselves.

Assume the base C is a model, so $C = \mathrm{dcl}(F)$ where $F = C \cap K$; F is an algebraically closed valued field. Let $\mathfrak{F}$ be the family of all C-definable functions on V into Γ. Recall (Proposition 2.14) that a C-definable type r is stably dominated iff for any $f \in \mathfrak{F}$ there exists $\gamma = f(r) \in \Gamma(M)$ such that if $c \models r|C$ then $f(c) = \gamma$.

Let q be a definable type extending a type q_0 over C, and let $(p_t : t \models q_0)$ be a family of stably dominated $C(t)$-definable types on V.

We say that the family $\{p_t\}$ is *uniformly bounded at q* if there exists open affine U such that for any regular function f on U, there exists $\alpha \in \Gamma$ such that if if $t \models q|C(\alpha, f))$ and $c \models p_t|C(\alpha, f, t)$ then $c \in U$ and $\mathrm{val} f(c) \geq \alpha$.

Lemma 6.20. *Let p_t be a family of stably dominated types. Assume p_t concentrates on a bounded $W \subseteq V$ (i.e. $W \in p_t$). Then $\{p_t\}$ is uniformly bounded at q.*

Proof. The types p_t (when $t \models q_0$) concentrate on one of the bounded affine sets W_i in Definition 6.19. Let U be the corresponding affine U_i. Any regular function on U_i is bounded on all of W_i, hence in particular on generic realizations of the p_t. □

Recall Definition 2.20.

Lemma 6.21. *Assume $\{p_t\}$ is* uniformly bounded at q. *Then there exists a unique C-definable type $p_\infty = \lim_q p_t$ such that for any $f \in \mathcal{O}_V(U)$, if $a \models p_\infty$ then* $\mathrm{val} f(a) = \lim_q f(q_t)$. *p_∞ is stably dominated. If $h : V \to W$ is an isomorphism of varieties, then* $\lim_q h_* p_t = h_* \lim_q p_t$.

Proof. By assumption we cannot have $\lim_q q_t(f) = -\infty$. The set of f such that $\lim_q q_t(f) = +\infty$ is a prime ideal I; part of the condition on p_∞ is that $(f = 0) \in p_\infty$ iff $f \in I$. Let V' be the zero set of I. Let $U' = V' \cap U$. The affine coordinate ring of U' is $\mathcal{O}_V(U)/I$; hence an element of $K(V') = K(U')$ can be written g/h with $g, h \in \mathcal{O}_V(U)$ and $h \notin I$; and $\lim_q q_t(h) \neq \pm\infty$. Hence we can define a valuation $\mathbf{v}$ on $F(V')$ (extending the given valuation on F) by $\mathbf{v}(g/h) = \lim_q q_t(g) - \lim_q q_t(h)$. This determines a valued field extension F^+ and hence gives a complete type p_∞ of elements of V'. Note that $p_\infty(f) = \lim_q q_t(f) \in C$ for any

$f \in F(V')$. In particular the choice of U is immaterial. It is clear that p_∞ is definable; and that that $\Gamma(F^+) = \Gamma(F)$, so p_∞ is orthogonal to Γ. By Lemma 2.14 it is stably dominated. The functoriality is evident. □

Lemma 6.22. *Let G be a bounded definable subgroup of an algebraic group $\tilde{G}$ over K. Let H_t be a Γ-family of connected stably dominated definable subgroup, forming a directed system under inclusion, $H = \cup_t H_t$. Assume G/H is internal to Γ. Then H is generically metastable. Moreover G/H is definably compact.*

Proof. Let $q(t)$ be a definable type cofinal in the partial ordering: $H_t \subseteq H_{t'}$. Let p_t be the generic type of H_t. By Lemma 6.20 the family is uniformly bounded at q, so $p_\infty = \lim_q p_t$ exists.

Claim. Let H, H' be connected generically metastable definable subgroups of G. Let p, p' be their generic types. Then $H \subseteq H'$ iff $p * p' = p'$.

Proof. If $H \subseteq H'$, then $p * p' = p'$ by definition of genericity for p. Conversely if $p * p' = p'$ then a generic of H is a product of two realizations of p'; in particular it lies in H'; but any element of H is a product of two generics, hence any element of H lies in H'. □

By Lemma 6.21, we have $\lim_q^a p_t =^a \lim_q p_t$ for any $a \in G$. Let $s \models q|C, t \models q|C(s)$. If $a \models p_s|C(s)$, then ${}^a p_t = p_t$, and hence ${}^a p_\infty = p_\infty$. Thus $H_s \subseteq Stab(p_\infty)$. So $H \subseteq Stab(p_\infty)$. On the other hand, as p_∞ is generically metastable, and G/H is internal to Γ, the function $x \mapsto Hx$ is constant on p_∞, so p_∞ lies in a single double coset Hb. It follows that p_∞ is a generic type of a coset of H, so H is generically metastable.

Now let q' be a definable type on Γ, and let $h :\to G/H$ be a definable function. For $t \in \Gamma$, let p'_t be the generic type of $h(t)$, viewed as a coset of H; this is a translate of the generic type of H. Let $p'_\infty = \lim_r p'_t$. Then p'_∞ is stably dominated, so it concentrates on a unique coset of H, corresponding to an element $e \in G/H$. Tracing through the definitions we see that $e = \lim_r h$. Since r, h are arbitrary, G/H is definably compact (Definition 2.21). □

Corollary 6.23. *Let A be an Abelian variety over K. Then A has a unique maximal stably dominated definable subgroup B. There exists a definably compact group C defined over Γ, and a definable isomorphism $\phi : A/B \cong C$.*

If A is defined over a field $F \leq K$, then B and ϕ are are also defined over F.

□

If $A \cong G(K)$ where G is an Abelian scheme over $\mathcal{O}_K$ with good reduction, then by Lemma 6.7, Lemma 6.8, $G(K) = G(\mathcal{O}_K)$ is generically metastable. Since $G(K)$ is divisible, it has a unique definable generic. Thus in this case the definably compact quotient B is trivial.

If F is locally compact, then the set $B(F)$ of points of B lifting to F-points will be a *finite* subgroup of the definably connected group C. On the other hand if $F = \mathbb{Q}_p((t))$, $B(F)$ can be a finite extension of $\mathbb{Z}$.

After appropriate base change, A becomes isomorphic to a group over the residue field. It is interesting to compare this with the classical theory of semi-stable reduction.

6.24. **Residually Abelian groups.**

Example 6.25. *In ACVF, there exists a stably dominated non-Abelian group, with Abelian stable part.*

Proof. Let $A = \{G_a\}^2$, and let $\beta : A^2 \to G_a$ be a non-symmetric bilinear map; defined over the prime field. E.g. $\beta((a_1, a_2), (a_1', a_2')) = a_1 a_2' - a_2 a_1'$. Let t be an element with $\mathrm{val}(t) > 0$. Define on A^2:

$$(a, b) \star (a', b') = (a + a', b + b' + t\beta(a, a'))$$

□

In fact, Simonetta found a C-minimal example:

Example 6.26. *If we work in* $G_a(\mathcal{O}/t^2\mathcal{O})$, *where* $\mathrm{val}(t) > 0$, *we can also just take* $a \star b = a + b + t\beta(a, b)$. *cf.* [26].

The group in the above example does lift to an algebraic group structure on $\mathcal{O}$.

We may still ask if a G be connected metastable of weight 1 is nilpotent.

We believe that a definably simple group definable in ACVF is either finite or an algebraic group over the valued field or over the residue field. The existing metastable technology yields a proof of the Abelian case.

Proposition 6.27. *Let A be a nonzero Abelian group definable in ACVF. Then there exist definable subgroups $B \leq C \leq A$, with $B \neq C$, and C/B definably isomorphic (with parameters) to an algebraic group over the residue field or a definable group over the value group.*

Proof. Let $\lambda : A \to \Lambda$ be as in Theorem 5.9. If $\lambda(A) \neq 0$ we can take $C = A, B = \ker(\lambda)$. If $\lambda(A) = 0$, then A is limit-metastable. In particular A contains a nonzero generically metastable definable Abelian group C. C has a k-algebraic group as a nontrivial quotient.

□

6.28. **Interpretable fields.**

Proposition 6.29. *Let F be an infinite field definable in ACVF. Then F is definably isomorphic to the residue field or the the valued field.*

Proof. If F is stable, then by [7] it is definable over the residue field (after base change), hence by [29] or [11] it is definably isomorphic to the residue field. There are no infinite fields definable over Γ, since Γ is a model of the theory DOAG of divisible ordered Abelian groups, without additional structure. We will also use below the fact that any one-dimensional torsion-free group defined in DOAG is definably isomorphic to $(\Gamma, +)$. (See [5] for a more general statement.)

By Remark (6) to Proposition 5.16, F^* has a definable homomorphic image V definable over Γ, containing a torsion-free one-dimensional definable group. Since this group is isomorphic to $(\Gamma, +)$, F admits a definable unbounded map into Γ. Hence F is not boundedly imaginary (Lemma 2.29).

Let D be a subng of F, with $(D, +)$ connected generically metastable, and such that F is the field of fractions of D (Proposition 5.16). There exists a surjective definable map $D \times D \to F$ (namely $(x, y) \mapsto x/y$ for nonzero y, $(x, 0) \mapsto 0$.) If D were boundedly imaginary, then F would be too; so D is not boundedly imaginary. If I' is any nonzero ideal of D, say $c \in I'$; then $D \cong Dc \subseteq I'$; so I' is not boundedly imaginary either.

Let f be the homomorphism of Proposition 6.3 on $(F, +)$ into an algebraic group. Let I be the kernel. If $c \in D$, then $d \mapsto f(cd)$ is another such homomorphism, so it must factor through f, $f(cd) = g(f(d))$; thus if $f(d) = 0$ then $f(cd) = 0$, i.e. if $d \in I$ then $cd \in I$. So I is an ideal of F, and is purely imaginary; moreover (Remark 6.2) for any stably dominated subgroup A of $(F, +)$, $I \cap A$ is boundedly imaginary. In particular $I \cap D$ is boundedly imaginary, so by the above discussion $I \cap D = (0)$. It follows that $I \neq F$, so $I = (0)$. Hence f is an isomorphism onto a subgroup of an algebraic group G. It follows that any purely imaginary definable subset of F^n is finite.

Let L be the limit metastable subgroup of $(F^*, \cdot)$. Then L also is definable over the field sort, and hence so is he semi-direct product $H = L \ltimes (F, +) \subseteq F^2$. Let $\phi : H \to G$ be the homomorphism of Lemma 6.3. As above the kernel is a normal subgroup, and being imaginary it is finite; but H has no finite nontrivial normal subgroups, so ϕ is an embedding.

We proceed as as in [23], with some changes of local reasoning. The Zariski closure of $\phi(F)$ is commutative. Let A_0 be the maximal semi-Abelian subgroup of A. View L as a subgroup of H. Then the Zariski closure $\bar{L}$ of $\phi(L)$, being connected, acts trivially on A_0, and it follows that $A_0 \cap \phi(F) = (0)$. Factoring out A_0 we obtain an embedding ψ of F in a vector group W. Identify F with $\psi(F)$; we may assume F is Zariski dense in W. Now $\bar{L}$ is connected, hence acts linearly on W, and the action of L on F factors through the action of $\bar{L}$. For $b \in L$ we write $b \cdot w$ for $\phi(b)w$; this extends the action of L on F to a linear action of L on W. If $r_i \in L$ and $\sum n_i r_i = 0$, then $\sum n_i r_i(y) = 0$ for $y \in F$, and by Zariski density of F we have $\sum n_i r_i(y) = 0$ for $y \in W$. Thus the action may be extended to an action of the ring R' generated by L on W, again extending the given action of R' on F. Finally if $0 \neq r \in R'$ then r acts on F as an invertible linear transformation, since the image contains $r(F) = F$ and hence by Zariski density W. Thus we can extend the action to the field of fractions of R', namely to F. We have thus extended the action of F on F by multiplication, to an action of F on W.

Let Z be some nonzero orbit of L on F. Then Z is definably isomorphic to L, and hence has the same dimension as F. (Recall F^*/L^* is Γ-internal.) So Z contains a nonempty open subset U of W. Let $c \in U$; then any sufficiently nearby c' is L-conjugate to c. In particular for any $\alpha \in \mathcal{O}$ with $\mathrm{val}(\alpha - 1)$ sufficiently large, there exists $h(\alpha) \in L$ with $h(\alpha) \cdot c = \alpha c$. Since F is a field, $h(\alpha)$ is uniquely defined. For any $b \in L$ we have

$$\alpha(b \cdot c) = b \cdot (\alpha c) = b \cdot (h(\alpha) \cdot c) = h(\alpha) \cdot (b \cdot c)$$

so $h(\alpha), \alpha$ agree on Z. The linear span of Z is W, so$h(\alpha)$, α agree on W.

Let $E = \{\alpha \in K : (\exists b \in F)(\forall x \in W)(\alpha x = bx)\}$. This is clearly a subfield of K, and it contains a neighborhood of 1. Hence it contains a neighborhood N of 0. If $0 \neq x \in K$ then $x^{-1}N \cap N$ includes some nonzero $u \in N$, so $xu \in N$ and $x = (xu)/u \in E$. So $E = K$. We have thus defined an embedding of rings $K \to F$. Since F has bounded dimension, for some m no definable subset of F admits a definable map onto K^m; so $\dim_K F < m$. Since K is algebraically closed we have $K \cong F$. □

We naturally expect that any infinite non-Abelian simple group definable in ACVF is isomorphic to an algebraic group defined over the residue field or valued field. A proof along the above lines may be possible assuming a positive solution to Problem 1.5, at least for ACVF, along with an interpretation of an ordered proper semi-group structure on $H \backslash G / H$

Example 6.30. *[Definable generic types of algebraic groups]*

Let K be an algebraically closed valued field; given an algebraic group G, look at the definable generic types of $G(K)$.

1. $SL_n(\mathcal{O}_K)$ is generically metastable, with a unique definable generic.
2. $G_m(K)$ has two definable generics: of elements of large absolute value, and of infinitesimal absolute value. Attributable to Γ.
3. $G_a(K)$ has a unique definable generic: of elements of large absolute value. This is a limit-metastable example; here definable generic types correspond to cofinal definable types in the partially ordered set $(\Gamma^{<0}, <)$ indexing the generically metastable subgroups $\alpha\mathcal{O}$. The fact that the poset has a unique cofinal definable type is however a phenomenon of dimension one:

(4) $G_a^2(K)$ has a large family of generics. For any definable curve E in $(\Gamma^{<0})^2$, cofinal in the sense that for any (a, b) there exists $(d, e) \in E$ with $d < a, e < b$, there exists a definable generic type of G_a^2 whose projection to $(\Gamma^{<0})^2$ concentrates on E.

(5) The groups described in Example 6.17 admit a homomorphism to a definably compact group over Γ, and hence have no invariant definable types at all.

(6) Let G be the solvable group of upper diagonal matrices. Then G has one-sided definable left-generics, right-generics, as well as two-sided generics. One of the latter is given as follows:

Let (x_{ij}) be the matrix coefficients of an element of $x \in GL_n(K)$. A two-sided generic is determined by: $|x_{ij}| >> |x_{i'j'}|$ when $(i, j) < (i', j')$ lexicographically (and $i < j$.)

(7) For $n > 1$, $SL_n(K)$ (or $GL_n(K)$) has no definable left generic type. For suppose p is a definable type. Then $(d_p x)(x_{11}| \geq |x_{21}|)$ or $(d_p x)(x_{11}| < |x_{21}|)$. Multiplying on the left by the elementary matrix with 1's on the diagonal, and t (with $|t| > 1$) in the $(2, 1)$-entry (respectively the $(1, 2)$-entry), this situation becomes reversed. Thus p is not invariant under left multiplication. Since $GL_n(K)$ has no proper subgroups of finite index (definable or not), p cannot be a left generic.

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