

VALUED FIELDS, METASTABLE GROUPS

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ABSTRACT. We introduce a class of theories called *metastable*, including the theory of algebraically closed valued fields (ACVF) as a motivating example. The key local notion is of definable types dominated by their stable part. A theory is metastable (over a sort Γ) if every type over a sufficiently rich base structure can be viewed as part of a Γ -parametrized family of stably dominated types. We initiate a study of interpretable groups in metastable theories of finite rank. Groups with a stably dominated generic type are shown to have a canonical stable quotient. Abelian groups are shown to be decomposable into a part coming from Γ , and a definable direct limit system of groups with stably dominated generic. In the case of ACVF, among affine definable groups we characterize the groups with stably dominated generics.

1. INTRODUCTION

The paper [1], continuing earlier work, studied the category of quantifier-free definable sets over valued fields, especially with respect to imaginaries. As usual, the direct study of a concrete structure of any depth is all but impossible, if it is not aided by a general theory. We first tried to find an generalization of stability (or simplicity) in a similar format, capable of dealing with valued fields as stability does with differential fields, or simplicity with difference fields. To this we encountered resistance; what we found instead ([2]) was not a new analogue of stability, but a new method of utilizing classical stability in certain unstable structures.

Even a very small stable part can have a decisive effect on the behavior a quite “large,” unstable type. This is sometimes analogous to the way that the (infinitesimal) linear approximation to a variety can explain much about the variety; and indeed in some cases casts tangent spaces and Lie algebras in an unexpected model theoretic role.

Two main principles encapsulate the understanding gained:

(1) Certain types are dominated by their stable parts. They behave “generically” as stable types do.

(2) Uniformly definable families of types make an appearance; they are indexed by the linear ordering Γ of the value group, or by other, piecewise-linear structures interpretable in Γ . An arbitrary type can be viewed as a definable limit of stably dominated types (from (1)).

A general study of stably dominated types was initiated in [2]; it is summarized in §2. (2) was only implicit in the proofs there. We state a precise version of the principle, and call a theory satisfying (2) *metastable*. We concentrate here on finite rank metastability.

(1) is given a general group-theoretic rendering in Proposition 4.6. Generically stably dominated groups are defined, and it is shown that the a group homomorphism into a stable group controls them (generically.) Theorem 5.9 clarifies the second principle in the context of Abelian groups. A metastable Abelian group of finite rank is shown to contain nontrivial generically stably dominated groups T_α , unless it is internal to Γ . Moreover, the groups T_α are shown to form a definable direct limit system, so that the group is described by three ingredients: Γ -groups, generically stably dominated groups, and definable direct limits of groups.

We then apply the theory to groups interpretable over algebraically closed valued fields. Already the case of Abelian varieties is of considerable interest; all three ingredients above occur, and the description beginning with a definable map into a piecewise linear definable group L takes a different aspect than the classical one. (The points of L over a local field will form a finite group, related to the group of connected components in the Neron model.) Non-commutative interpretable groups include the congruence subgroups of algebraic groups, and quotients by them.

Previous work in this direction used other theories, inspired by topology; see [7] regarding the p -adics.

We now describe the results in more detail. Our main notion will be that of a generically metastable group. In a metastable theory, we will try to analyze arbitrary groups, and to some extent types, using them. In the case of valued fields, this notion is related to but distinct from compactness (over those fields where topological notions make sense, i.e. local fields.) For groups defined over local fields, generic metastability implies compactness of the group of points over every finite extension. Abelian varieties with bad reduction show that the converse fails; this failure is explained by another aspect of the theory, definable homomorphic quotients defined over Γ .

Let T be a first order theory.

Definition 1.1. *stdomdef* A partial type P over C is stably dominated if there exists over C a $*$ -definable map $\alpha : P \rightarrow D$, D stable, such that for any tuple b , $\alpha(a) \downarrow b$ implies

$$tp(b/C, \alpha(a)) \models tp(b/Ca)$$

Let Γ be a sort of T . By a *substructure* of a models of T we mean a subset closed under definable functions. We write $A \leq M \models T$ for short. A *parameterically definable* set is a definable set in T_A for $A \leq M \models T$.

Assume:

1) Γ is stably embedded: every subset of Γ^n defined with parameters in a model M of T is definable from parameters in $\Gamma(M)$. Equivalently, when M is saturated, the natural $Aut(M) \rightarrow Aut(\Gamma(M))$ is surjective. (cf. [4], appendix.)

2) Γ is orthogonal to the stable part: no infinite definable subset of Γ^{eq} is stable.

In the case of algebraically closed valued fields, Γ will be O-minimal. Another application is to valued fields with algebraically closed residue field and different value groups, such as $\mathbb{Z}$; where still every definable subset of Γ is a Boolean combination of 0-definable sets and intervals.

Definition 1.2. *meta* T is metastable (over Γ) if for any partial type P over a base C_0 there exists $C \supset C_0$ and a $*$ -definable (over C) map $\gamma_c : P \rightarrow \Gamma$ with $tp(a/\gamma_c(a))$ stably dominated.

In addition, we assume every type over an algebraically closed subset of a model of T^{eq} has an automorphism-invariant extension to the model.

We will say that C is a good base for P . A *good base* is a good base for all partial types over it.

The condition (E) on existence of automorphism-invariant types, currently needed in the hypotheses of the descent lemma for stable domination, is presently needed for the proof of the descent lemma ?? (and only there.)

Question 1.3. (1) Can the descent lemma be proved without the additional hypothesis (E)?
(2) Does (E) follow from metastability over an o-minimal Γ ?

Let $A \leq M \models T$. Let St_A be the family of all stable, stably embedded A -definable sets. This will be referred to as the *stable part* of T_A . We write $St_A(c)$ for $A(c) \cap St_A$, where $A(c) = dcl(A \cup \{c\})$ is the smallest substructure of M containing $A \cup \{c\}$.

Similarly, let $\Gamma_A(c) = \text{dcl}(A, c) \cap \text{dcl}(A \cup \Gamma)$. If $A = \text{dcl}(\emptyset)$ we omit it from the notation. By a Γ -set over A we mean an A -definable set D such that there exists an A -definable surjection $h : \Gamma^m \rightarrow D$ for some m . In practice we can take $D \subseteq \Gamma^m$ for some m . A Γ -group is a Γ -set with a definable group structure; since Γ is assumed to be stably embedded, the group structure is definable with parameters from Γ .

Remark 1.4. (1) *Instead of taking all stable, stably embedded A -definable sets, it is possible to take a proper subfamily $\text{mathcal{S}}_A$ with reasonable closure properties. $\text{mathcal{S}}$ -domination that is meaningful even for stable theories.*

(2) *One can also replace the single sort Γ with a family of sorts Γ_n , or with a family of parametrized families definable sets $\mathcal{G}_A$, with no loss for the results of the present paper.*

As in stability theory, a range of finiteness assumptions is possible. We will use the following “finite rank” assumption (FD) below. Some terminology: We will refer to the Morley rank of a formula in the stable part as the (Morley) *dimension*.

A definable subset D is called *o-minimal* if it has a distinguished definable linear ordering $<$, such that every definable subset of D with parameters in a model of T is a finite union of intervals and points. There is a natural notion of dimension for definable subsets of D^m , with $\dim(D) = 1$. See [15].

The dimension $\dim(e/C)$ of a tuple of elements $e \in St_C$ (or $e \in \Gamma^n$) is defined to be the minimum dimension of a formula D over C with $e \in D$.

A structure B is *acl-finitely generated* over $A \subseteq B$ if $B \subseteq \text{acl}(A(b))$ for some tuple b from B . (FD):

- 1) Γ is o-minimal.
 - 2) Morley dimension is definable in families: if D_t is a definable family of definable sets then $\{t : MR(D_t) = m\}$ is definable
 - 3) Let D be a definable set. The Morley dimension of $f(D)$, where f ranges over all definable functions (with parameters) such that $f(D)$ is stable, takes a maximum value $\dim_{st}(D)$.
- Similarly, the o-minimal dimension of $g(D)$, where g ranges over all definable functions (with parameters) such that $g(D)$ is Γ -internal, takes a maximum value $\dim_o(D)$.

Some statements will be simpler if we also assume:

(FD $_\omega$): In addition, any set is contained in a good base M which is also a model. Moreover, for any acl- finitely generated $F \subseteq \Gamma$ and $F' \subseteq St_M$ over M , isolated types over $M \cup F \cup F'$ are dense.

Remarks

Write $\dim_{st}^{def}(d/B) = \min\{\dim_{st}(D) : d \in D, D \text{ B-definable.}\}$; and if $B' = B(d)$. Let $\dim_{st}^{def}(B'/B) = \dim_{st}^{def}(d/B)$; this is well-defined. Note that we may have $\dim_{st}^{def}(B'/B) > \dim St_B(B')/B$.

(FD $_\omega$) is true for ACVF, with all imaginary sorts included. (FD) at least is valid for all C -minimal expansions of ACVF, in particular the Lipshitz-Robinson rigid analytic expansions. cf. [5], [8], [9]. (FD) is also valid for definable sets of finite differential order in Scanlon’s model completion of valued differential fields, (Proposition 2.21).

In practice, the main structural results will use finite weight hypotheses; this is a weaker consequence of (FD). Recall that a definable type p in a stable theory is said to have *weight* $\leq n$ if for any M and independent $b_1, \dots, b_{n+1}$ over M , and any $a \models p|_M$, for some i , $a \models p|Mb_i$. In a definable set of Morley rank n , every definable type has weight $\leq n$.

A group is *generically metastable* if it has a generic type that is stably dominated (See Definitions 3.1, 4.1.) In this case, the stable domination is witnessed by a group homomorphism (Proposition 4.6). One cannot expect every group to be generically metastable. But one can hope to shed light on any definable group by studying the generically metastable groups inside it.

We formulate the notion of a *limit metastable* group; it is a direct limit of connected metastable groups by a $*$ -definable direct limit system.

Theorem 1.5. *ab0* Let T be a metastable theory with (FD_ω) . Let A be a definable Abelian group. Then there exists a definable group $\Lambda \subset \Gamma^{eq}$, and a definable homomorphism $\lambda : A \rightarrow \Lambda$, with $K = \ker(\lambda)$ limit- metastable.

In fact under these assumptions, K is the union of a definable directed family of definable groups, of which is connected and generically metastable. Assuming bounded weight (in place of (FD_ω)), we obtain a similar result but with K ∞ -definable.

In the non-Abelian case the question remains open. The optimal conjecture would be a positive answer to:

Problem 1.6. (FD_ω) Does any definable group G have a limit-metastable definable subgroup H with $H \backslash G/H$ internal to Γ ?

Turning to valued fields, we obtain some structure theory for generically metastable definable groups. It would be important to clarify the general nature of category of *interpretable* groups, even the generically metastable ones, and their relation with points of groups schemes over the natural quotient rings. We believe that if G is generically metastable, there exist definable normal subgroups $(1) = G_0 \leq \dots \leq G_n = G$, such that G_{i+1}/G_i is a definable homomorphic image of a definable group. We plan to return to this in a sequel.

Call a definable set D *purely imaginary* if there exists no definable map (with parameters) of D onto an infinite subset of the field K . Thus the elements of D are tuples of value group elements, residue field elements or elements of the sorts S_n, T_n of [1]; Note that over the algebraic closure L of a local valued field, D is purely imaginary iff $D(L)$ is countable.

Proposition 1.7. *embed* Let G be a generically metastable group definable in the field sort of ACVF. Then there exists an algebraic group H over K and a definable homomorphism $f : G \rightarrow H(K)$ with purely imaginary kernel.

This reduces the study of a *definable* generically metastable group over K to that of generically metastable subgroups of algebraic groups H . There exists an exact sequence $1 \rightarrow A \rightarrow H \rightarrow_f L \rightarrow 1$, with A an Abelian variety and L an affine algebraic group. We show (Lemma 4.5) that a definable subgroup G of H is generically metastable if and only if $G \cap A$ and $f(G)$ are. The Abelian variety case falls under the general Lemma 5.1; this will be treated separately. For linear groups, we have:

Theorem 1.8. *structure* Let H be an affine algebraic group, and let G be a generically metastable definable subgroup of H . Then G is isomorphic to $H_1(R)$, H_1 an algebraic group scheme over $\mathcal{O}$.

If G is Zariski dense in H , H_1 can be taken to be K -isomorphic to H .

Examples include $GL_n(R); GL_n(R/aR)$ where R is the valuation ring; and “congruence subgroups” such as the kernel of $GL_n(R) \rightarrow GL_n(R/aR)$.

Parts of this text served as notes for a graduate seminar in the Hebrew University in Fall 2003; the participants have my warm thanks.

2. PRELIMINARIES

We recall some material from stability and stable domination, in a form suitable for our purposes.

Let $\mathbb{U}$ be a universal domain (a saturated model.) $\models \phi$ is shorthand for $\mathbb{U} \models \phi$.

By a $*$ -definable function f , we mean an indexed sequence (f_i) of definable functions. Similarly a $*$ -type.

Notation . Given a type p over C with a unique C -definable extension $\tilde{p}$ to $\mathbb{U}$, write:
 $a \downarrow_C b$ if $a \models \tilde{p}|acl(\{b, C\})$.

(This applies when $p, \tilde{p}$ are Δ -types; in this case, $\tilde{p}|E$ is defined to be the set of E -formulas, that follow from $\tilde{p}$.)

When the identity of C is clear, or when $C = acl(\emptyset)$, we write: $a \downarrow b$.

2.1. Definable types. Let Δ be a set of formulas $\phi(x; y)$. Here x is a fixed tuple of variables, while y may range over all variables. Let Δ_x be the Boolean algebra of formulas generated by those of the form $\phi(x; b)$, with $\phi \in \Delta$.

When not other specified, we assume Δ consists of all formulas in the appropriate variables. Though our applications will use this case, definability considerations will occasionally lead us to consider *finite* sets Δ .

By a Δ -definable type over C p we mean a homomorphism $\phi \mapsto (d_p)\phi$ carrying formulas $\phi(x; y) \in \Delta$ to formulas $(d_p)\phi(y)$ over C ; so that if M is a model, then

$$p|C = \{\phi(x, b) : \phi \in \Delta, M \models (d_p)\phi(b)\}$$

Sometimes to specify the variable, when $p = p(x)$, we will write: $d_p x \phi(x, y)$.

If r is a definable function, p a definable type (of elements of $\text{dom}(r)$), we define the push-forward $r_* p$ by:

$$(d_{r_* p} u) \phi(u, v) = (d_p x) \phi(r(x), v)$$

So that if $a \models p|C$ then $r(a) \models r_* p|C$.

2.1.1. Free products of definable types. Let $p(x), q(y)$ be definable types. Define $r(x, y) = p(x) \otimes r(y)$ by:

$$(d_{rxy}) \phi(xy, z) = (d_p x) (d_q y) \phi(xy, a)$$

For those formulas ϕ for which the right hand side is defined.

If p, q are complete (i.e. their domains are all formulas), then so is r .

Remark 2.2. Let ϕ be a formula in variables x, y, z ; the same formula may be viewed as $\phi(x; yz)$ or $\phi(y; xz)$ or $\phi(xy; z)$, defining three bipartite graphs. If $\phi(x; yz)$ lies in the domain of p , and $\phi(y; xz)$ lies in the domain of q and is stable, then $\phi(xy; z)$ lies in the domain of $p \otimes q$.

Proof. $(d_q y) \phi(y; xz)$ is equivalent to a Boolean combination of formulas $\phi(b_i, xz)$; all these lie in $\text{dom}(p)$ by assumption. $\square$

We will occasionally use a more general construction. Assume $p(x)$ is a definable type (with no parameters). Let $a \models p$ and let $q_a(y)$ be an definable type of the theory T_a .

Lemma 2.3. *concat* There exists a unique definable type $r(x, y)$ such that for any C , if $(a, b) \models r|C$ then $a \models p|C$ and $b \models q_a|C$.

Proof. Given a formula $\phi(xy, z)$, let $\phi^*(x, z)$ be a formula such that $\phi^*(a, z) = (d_{q_a} y) \phi(a, y, z)$. ϕ^* is not uniquely defined, but if ϕ', ϕ'' are two possibilities then $(d_p x)(\phi' \equiv \phi'')$. Therefore we can define:

$$(d_{rxy}) \phi(xy, z) = (d_p x) \phi^*(x, y, z)$$

It is easy to check that this definition scheme works. $\square$

2.4. Germs of definable functions. Let p be a definable type. Let $f = f_c(x)$ be a definable function, defined using parameters c .

When we consider definable functions $f(x, y), g(x, z)$, we will assume that the formula $f(x, y) = g(x, z)$ lies in Δ .

Definition 2.5. Two definable functions $f(x, b), g(x, b')$ are said to have the same p -germ if

$$\models (d_p x) f(x, b) = g(x, b')$$

We say that the p -germ of $f(x, b)$ is defined over C if whenever $tp(b/C) = tp(b'/C)$, $f(x, b), f(x, b')$ have the same p -germ. Note that the equivalence relation:

“ $b \sim b'$ iff $f(x, b), f(x, b')$ have the same p -germ”

is definable; the p -germ of $f(x, b)$ is defined over C iff $b/\sim \in dcl(C)$.

2.6. Stably dominated types.

Definition 2.7. A partial type P is stably dominated over C if there exist C -definable maps $\alpha_i : P \rightarrow D_i$, D stable, $\alpha = (\alpha_i)_i$, such that $\alpha(a) \downarrow b$ implies

$$tp(b/\alpha(a)) \models tp(b/a)$$

for any tuple b .

We can a definable set D stable if every formula $\phi(x; y)$ with $y = (y_1, \dots, y_m)$, such that ϕ implies $D(y_1) \& \dots \& D(y_m)$, is stable. This is often referred to as *stable, stably embedded* in the literature. See e.g. [?] for a treatment of basic stability.

A type over C is said to be *stably dominated* if it is stably dominated over C via some α .

Proposition 2.8. *stdomref1* Let p be a complete type over $C = acl(C)$. If p is stably dominated, it has a C -definable extension to $\mathbb{U}$, and this extension is unique.

Thus the $\downarrow$ - notation is applicable, as well as the notion of a p -germ.

Proposition 2.9. *stdomref2* Let $p = tp(a/C)$ be stably dominated.

- (1) (Symmetry) If $tp(b/C)$ is also stably dominated, $a \downarrow_C b$ iff $b \downarrow_C a$
- (2) (Transitivity) $a \downarrow_C bd$ iff $a \downarrow_C b$ and $a \downarrow_{acl(Cb)} d$.
- (3) (Base change) If $a \downarrow_C b$, then $tp(a/acl(Cb))$ is stably dominated.
- (4) If $tp(d/C)$ and $tp(b/acl(Cd))$ are stably dominated, then so is $tp(bd/C)$. Conversely if $a \in dcl(Cb)$ and $tp(b/C)$ is stably dominated, then so is $tp(a/C)$.
- (5) For any formula $\phi(x, y)$, $(d_p x \phi)$ is a positive Boolean combination of formulas $\phi(a_i, y)$, where the $a_1 \models p|C$, $a_2 \models p|Ca_1$, etc.

Proposition 2.10 (Descent). *stdomref3* Let p, q be $\text{Aut}(\mathbb{U}/C)$ -invariant $\star$ -types. Assume that whenever $b \models q|C$, the type $p|Cb$ is stably dominated. Then p is stably dominated.

Proposition 2.11 (The strong germ lemma). *strong* Let p be stably dominated. Assume p as well as the p -germ of $f(x, b)$ are defined over $C = acl(C)$. Then there exists a C -definable function g with the same p -germ as $f(x, b)$.

Proposition 2.12. *orth* A definable type p is stably dominated iff p is orthogonal to Γ .

See [2] for proofs of these three propositions. Proposition 2.9 (4) is Lemma 5.10 there. Symmetry, transitivity and base change are easy consequences of the corresponding facts in stable theories; but descent (Theorem 9.3 there) is more difficult, and uses the additional hypothesis of extendibility of types to invariant types.

Lemma 2.13. *(FD) fg Let M be a good base, and let d be a tuple. Then $\Gamma_M(c)$ and $St_M(c)$ are acl- finitely generated over M . Assuming (FD_ω) , they are finitely generated over M .*

Proof. Any tuple $d \in St_M(c)$ can be written $d = h(c)$ for some M -definable function. By the dimension bound in (FD), $\dim(d/M) \leq \dim_{st}^{def}(c/M)$. It follows that if $d_1, d_2, \dots \in St_M(c)$ then $d_n \in \text{acl}(M(d_1, \dots, d_{n-1}))$ for all sufficiently large n . So $St_M(c)$ is finitely generated in the sense of algebraic closure. Assuming (FD_ω) , $tp(c/M, St_M(c))$ is isolated; if the isolating formula has parameters $d \in \text{dcl}(M, St_M(c))$, it follows that $tp(c/M(d))$ implies $tp(c/M, St_M(c))$; since $St_M(c) \subseteq M(c)$ it follows that $St_M(c) \subseteq M(d)$. The same argument works for Γ in place of St . $\square$

Say $tp(a/C)$ is *strongly stably dominated* if there exists $\phi(x) \in tp(a/St_C(a))$ such that that for any tuple b with $St_C(a) \downarrow b$, $tp(a/Cb)$ is isolated via ϕ . Equivalently, $tp(a/C)$ is stably dominated via some definable h , and $tp(a/h(a))$ is isolated. (We then say that $tp(a/C)$ is strongly stably dominated via h .)

Lemma 2.14. *iso2 (FD $_\omega$) Let D be a formula over C_0 , $h : D \rightarrow S$ a definable map to a stable definable set of maximal possible dimension $\dim(S) = \dim_{st}(D)$. Then there exists M containing C_0 and $a \in D$ such that with $\Gamma_M(a) = C$ we have $\dim_{st}(D) = \dim St_C(a)/C = \dim St_M(a)/M$, and $tp(a/C)$ is strongly stably dominated via h . (In particular, $tp(a/St_C(a))$ is isolated.)*

Proof. We may assume h is defined over C_0 . Let $a_0 \in D$ be such that $\dim_{st}(D) = \dim h(a_0)/C_0$; extend C_0 to a good base M with $h(a_0)$ independent from M ; then $\dim_{st}(D) = \dim St_M(a_0)/M = \dim h(a_0)/M$. Choose a_0 such that $\dim_o \Gamma_M(a_0)/M$ is as large as possible (given the other constraints.) Let $C = \Gamma_M(a_0)$; then C is finitely generated over M , and $\Gamma_C(a_0) = C$. Let $C' = St_C(a_0)$. Let D' be a formula over C' implying D and such that for any a' with $D'(a')$, $C' \subseteq M(a')$ (Lemma 2.13.) By (FD_ω) there exists $a \in D'$ such that $tp(a/C')$ is isolated. By choice of D' we have $C' \subseteq St_C(a)$ and hence $St_{C'}(a) \subseteq St_C(a)$. But then $\dim(St_{C'}(a)/C) \leq \dim_{st}(D) = \dim(C'/C)$ so $St_C(a) \subseteq \text{acl}(C')$. Since $St_C(a)$ is finitely generated over C , $tp(St_C(a)/C')$ is algebraic and in particular $tp(St_C(a)/C'(a))$ is isolated. But $tp(a/C')$ is isolated, so $tp(a, St_C(a)/C'(a))$ is isolated and hence $tp(a/St_C(a))$ is isolated. Similarly by maximality of $\dim_o \Gamma_M(a_0)/M$ and the fact that $\Gamma_M(a_0) \subseteq C' \subseteq \Gamma_M(a)$ we have $\Gamma_M(a_0) = C$. By metastability, $tp(a/C)$ is stably dominated; since $tp(a/C')$ is isolated and $C' = St_C(a)$, $tp(a/C)$ is strongly stably dominated. $\square$

Lemma 2.15. *nfcP (FD) Let $f : P \rightarrow Q$ be a definable map between interpretable sets P, Q . Let $P_a = f^{-1}(a)$. Then there exists m such that if P_a is finite, then $|P_a| \leq m$.*

Proof. Say f, P, Q are 0-definable. By compactness, it suffices to show that if P_a is infinite then there exists a 0-definable set Q' with $a \in Q'$ and P_b infinite for all $b \in Q'$.

Claim. If P_a is infinite then either $\dim_{st}(P_a) > 0$ or $\dim_o(P_a) > 0$.

Proof. Let M be a good base, with $a \in M$. Let $c \in P_a \setminus M$. If $\Gamma_M(c) \neq M$ then $\dim_o(P_a) > 0$. Otherwise, by metastability, $tp(c/M)$ is stably dominated, say via f . If $f(c) \in M$, then $tp(b/M) \implies tp(b/Mc)$ for all b , and taking $b = c$ it follows that $c \in M$. Thus $f(c) \in St_M \setminus M$. It follows that $\dim_{st}(P_a) > 0$. $\square$

If $\dim_{st}(P_a) > 0$, then there exists a definable family of stable definable sets D_t with $\dim(D_t) = k > 0$, and a definable function $f(x, u)$ such that for some b and t , $f(P_a, b) = D_t$. Then the formula $(\exists u)(\exists t)f(P_y, u) = D_t$ is true of $y = a$, and implies that P_y is infinite. The case $\dim_o(P_a) > 0$ is similar. $\square$

2.16. Valued fields: imaginaries and resolution. Let K be an algebraically closed valued field, with valuation ring $\mathcal{O}_K$. The geometric language for valued fields has a sort for the valued field itself, and certain other sorts. In particular, there is a sort S_n such that $S_n(K)$ is the space of for the space of free $\mathcal{O}_K$ -modules in K^n , or $GL_n(K)/GL_n(\mathcal{O}_K)$.

We let $M(K)$ be the set of K -points of all these sorts.

By a *substructure*, we mean a subset of $M(K)$, closed under the quantifier-free definable functions.

A substructure A of $M(K)$ is called *resolved* if $K(A)$ is algebraically closed, and A is generated as a substructure of $M(K)$ by $K(A)$.

A *resolution* of a substructure A is a resolved substructure B containing A . B is a *minimal* resolution of A if no proper substructure of B is a resolution of A . A *prime resolution* is one that embeds over A into any resolution.

Proposition 2.17. resolution *Let A be a substructure of $M(K)$, finitely generated over a subfield L of K . Then, up to isomorphism over A , there exists a unique prime resolution $\tilde{\mathbf{A}}$ of A . $\tilde{\mathbf{A}}$ enjoys the following properties.*

- (1) $\tilde{\mathbf{A}}$ is a minimal resolution of A . Moreover it is the unique minimal resolution, up to isomorphism over A .
- (2) $St_L(\tilde{\mathbf{A}}) = St_L(A)$.
- (3) Let $A \leq A'$, with A'/A finitely generated. Then $\tilde{\mathbf{A}}$ embeds into $\tilde{\mathbf{A}}'$ over A . If $A \leq A' \leq \tilde{\mathbf{A}}$, then $\tilde{\mathbf{A}}$ is the prime resolution of A' .
- (4) Let $L'(A)$ be the structure generated by $L' \cup A$. Then $L'(\tilde{\mathbf{A}})^{alg}$ is a prime resolution of $L'(A)$.
- (5) If A/L is stably dominated, then $\tilde{\mathbf{A}}/L$ is stably dominated.
- (6) If A'/A is stably dominated, and $A' \downarrow_A \tilde{\mathbf{A}}$, then $\tilde{\mathbf{A}}'/\tilde{\mathbf{A}}$ is stably dominated.

Proof. The existence, uniqueness and minimality of $\tilde{\mathbf{A}}$ are [2], Theorem 9.12. It is also shown there that $k(\tilde{\mathbf{A}}) = k(\text{acl}(A))$ and $\Gamma(\tilde{\mathbf{A}}) = \Gamma(\text{acl}(A))$, where k is the residue field; and that $\tilde{\mathbf{A}}/A$ is atomic, i.e. $tp(c/A)$ is isolated for any tuple c from A .

Uniqueness of the minimal resolution: Let B be a minimal resolution of A . Then the prime resolution $\tilde{\mathbf{A}}$ embeds into B over A . As B is minimal, this embedding is an isomorphism.

Since L is a field, for any $B = \text{acl}(B)$ with $L \subset B$, $St_L(B) = dcl(B, k(B))$. This proves (2).

(3) is immediate from the definition of prime resolution: since $\tilde{\mathbf{A}}'$ is a resolution of A , $\tilde{\mathbf{A}}$ embeds into $\tilde{\mathbf{A}}'$. If $A \leq A' \leq \tilde{\mathbf{A}}$, then $\tilde{\mathbf{A}}$ is clearly a minimal resolution of A' ; hence by (1) it is the prime resolution.

(4) Let B be the prime resolution of $L'(A)$. Then $\tilde{\mathbf{A}}$ embeds into B . Within B , $L'(\tilde{\mathbf{A}})^{alg}$ is a resolution of $L'(A)$; by minimality of B , $B = L'(\tilde{\mathbf{A}})^{alg}$.

(5) We may assume $L = L^{alg}$. Let $\bar{L}$ be a maximal immediate extension of L . Choose it in such a way that $tp(\tilde{\mathbf{A}}/\bar{L})$ is a sequentially independent extension of $tp(\tilde{\mathbf{A}}/L)$.

Since A/L is stably dominated, $\Gamma(\bar{L}(A)^{alg}) = \Gamma(\bar{L})$. According to (4), $B\bar{L}(\tilde{\mathbf{A}})^{alg}$ is the prime resolution of $\bar{L}(A)^{alg}$, and so $\Gamma(\bar{L}(\tilde{\mathbf{A}})^{alg}) = \Gamma(\bar{L})$. Thus by [2], Theorem 10.12, $tp(\tilde{\mathbf{A}}/\bar{L})$ is stably dominated. Using descent, Proposition 2.10, $tp(\tilde{\mathbf{A}}/L)$ is stably dominated.

(6) Let $\tilde{\mathbf{A}}$ be a prime resolution of A , with $A' \downarrow_A \tilde{\mathbf{A}}$. Then $tp(A'/\tilde{\mathbf{A}})$ is stably dominated. By (5), $tp(B/\tilde{\mathbf{A}})$ is stably dominated, where B is a resolution of $\tilde{\mathbf{A}}(A')$. Since $\tilde{\mathbf{A}}'$ embeds into B over A' , $tp(\tilde{\mathbf{A}}'/\tilde{\mathbf{A}})$ is stably dominated. □

2.18. Valued fields: metastability.

Proposition 2.19. *ACVF is metastable, with (FD).*

Proof. Let P be a partial type, and let M be a maximally complete algebraically closed valued field, with P defined over M . Let $\gamma(a)$ enumerate $\Gamma(M, a) = \text{dcl}(M, a) \cap \Gamma$. Then $tp(a/\gamma(a))$ is stably dominated by [2], Theorem 10.12. $\square$

Lemma 2.20 (ACVF). *Let $M \models \text{ACVF}$, C finitely generated over M . Then isolated types over C are dense. Hence (FD_ω) holds for ACVF (imaginaries included.)*

Proof. The resolution N of C is a model of ACVF, and is atomic over M . (FD_ω) follows from Lemma 2.14 $\square$

Let EVDF be the model completion of the theory VDF of valued differential fields ([12]). EVDF extends ACVF and admits quantifier elimination. The residue field is stable as a sort in EVDF ; it has the induced structure of a model of the theory DCF of differentially closed fields.

Proposition 2.21. *vdf EVDF is metastable.*

Proof. Let $\mathbb{U}$ be a universal domain for EVDF . Let $C \leq \mathbb{U}$, and let $p(x_0)$ be a type over C . Add variables x_i denoting $D^i x$, obtaining a type $P(x_0, x_1, \dots)$. The new type P is generated by the formulas $Dx_i = x_{i+1}$, along with the restriction P_v of P to the valued field language. Let Q_v be an extendible type in the sense of ACVF. Given a VDF structure C' extending C , let $C'' = C'(c_0, c_1, \dots)$ be a valued field extension of C' , generated by a realization of $Q_v|C'$. Then $C(c_0, c_1, \dots)$ is linearly disjoint from C' over C . Therefore any derivation on $C(c_0, c_1, \dots)$ extending the given derivation on C extends to a derivation on C'' . It follows that $Q = Q_v \cup \{Dx_i = x_{i+1}\}$ is consistent. By quantifier elimination Q is a complete extendible type of EVDF . This shows that any type over C extends to an $\text{Aut}(\mathbb{U}/C)$ -invariant type.

Similarly, let M be a model of EVDF , maximally complete as a valued field. For any $a_0 \in \mathbb{U}$, let $a_0, a_1, \dots$ and P be as above. Let γ enumerate $\text{val}M(a_0, a_1, \dots)$, and let c enumerate the residue field of $M(a_0, a_1, \dots)$. Then P is dominated by $tp(c/M, \gamma)$ over M, γ , in the sense of ACVF. It follows as above that p is stably dominated. $\square$

Continuing with ACVF, we recall a characterization of independence in terms of a “maximum modulus” principle.

Proposition 2.22. *maxmod Let U, V be varieties over the algebraically closed valued field C . Let p, q be stably dominated types over C of elements of U, V respectively. Let F be a regular function on $U \times V$. Then there exists $\gamma \in \Gamma$ such that*

- (1) *If $(a, b) \models p \otimes q$ then $\text{val}F(a, b) = \gamma$.*
- (2) *For any $a \models p, b \models q$, we have $\text{val}F(a, b) \geq \gamma$.*
- (3) *Assume U, V are affine. If $a \models p, b \models q$ and $\text{val}F(a, b) = \gamma$ for all regular F on $U \times V$, then $(a, b) \models (p \otimes q)$.*

Proof. U admits a finite cover by open affines, and p concentrates on one of these affines; so we may assume U is affine, and similarly V . Choose embeddings $U \subseteq \mathbb{A}^n, V \subseteq \mathbb{A}^m$. Then F lifts to a polynomial on $\mathbb{A}^{n+m}$; conversely (in (3)) any polynomial on $\mathbb{A}^{n+m}$ restricts to a regular function on $U \times V$. Thus we may assume $U = \mathbb{A}^n, V = \mathbb{A}^m$. In this case the statement is part of [2] Theorem 13.10. $\square$

3. GROUPS WITH DEFINABLE GENERICS

Let G be a group, and let Δ be a set of formulas $\phi(x; y)$. Here x ranges over G , while y may range over G^n or elsewhere. Let Δ_x be the Boolean algebra of formulas generated by those of the form $\phi(x; b)$, with $\phi \in \Delta$. We assume Δ_x is closed under left translations and inversion.

If p is a definable type over C , we let ap (the left translate of p by a) be the definable type such that for any $C' \supset C \cup \{a\}$,

$$d \models p|C' \text{ iff } ad \models^a p|C'$$

Note that ${}^b p$ is Cb -definable.

Lemma 3.2. *bddtrans Right generics have boundedly many left translates. If Δ is finite, right Δ -generics have finitely many left-translates.*

Call a definable type p *symmetric* if whenever q is a definable type, p, q definable over C , $b \models q|C$, $a \models p|acl(C \cup \{b\})$, then $b \models q|acl(C \cup \{a\})$. (We will see that stably dominated types are symmetric in this sense.)

Proof. Let p be a left generic. Using the inverse map $x \mapsto x^{-1}$, we see that G also has right generics. Let q be any right generic of G . Let C be an elementary submodel with p, q defined over C . Let $a \models p|C$, $b \models q|acl(C \cup \{a\})$. Then $ab \models^a q|(C \cup \{a\})$, by definition of ${}^a q$. By right genericity of q , ${}^a q$ is C -definable. By symmetry, $a \models p|acl(C \cup \{b\})$. So $ab \models p^b|(C \cup \{b\})$. As p is left generic, p^b is C -definable. Thus ${}^a q|C = tp(ab/C) = p^b|C$; since C is an elementary submodel, these definable types are equal.

This shows that any two left generics differ by a left translation. In particular by 3.2 there are boundedly many left generics.

Let G^0 be the right stabilizer of all left generics. So G^0 is of bounded index in G . By translating into G^0 we see that it has left and right generics; the left generic is unique. G^0 can have no subgroups of finite index, since the left generic is unique. If p is the left generic of G^0 and q is a right generic, $a \models p$, $b \models q$, $a \not\models b$, then $tp(ab)$ is left generic so equals p . This shows q is also unique. Thus $p^b = p$ and $a^q = q$; since we saw above that $p^b = a^q$, we have $p = q$. When Δ is finite, G^0 is definable and hence of finite index. $\square$

The *principal* generic is the one lying in G^0 . If $G = G^0$ we say that G is *connected*.

Remark 3.4. *1genr* Let p be a symmetric definable generic type of elements of a definable group G . Then the following are equivalent:

- (1) p is the unique definable generic type of G .
- (2) For all $g \in G$, ${}^g p = p$.
- (3) $G = G^0$.

Proof. Assume (1). By definition of genericity, and associativity, for $g \in G$, ${}^g p$ is right generic. Hence by uniqueness, ${}^g p = p$.

Conversely given (2), let q be generic. Then by Lemma 3.3, ${}^g p = q$ for some g . By (1), $p = q$. The equivalence with (3) is immediate. $\square$

Lemma 3.5. *defbdd* Let T be a theory in a language L ; let H be a group ∞ -definable over C , with a symmetric definable generic type. Then H^0 is ∞ -definable over some $C_0 \subset C$, with $|C_0| \leq |L| + \aleph_0$. (We will say that C_0 is small.)

Proof. A definable type $q(x)$ is determined by the function $\phi \rightarrow (d_q x)\phi$, where $\phi = \phi(x, y) \in L$. Thus every definable type is C_0 -definable for some small $|C_0|$, in this sense. Apply this to the principal generic of H $\square$

3.6. Stabilizers. stabilizers

The stabilizer of a definable type can be viewed as an adjoint notion for the generic of a definable group. If p is a generic of G , then G^0 will be the stabilizer of p . If q is a definable type and is in some sense no bigger than $Stab(q)$, then it is a translate of a generic of $Stab(q)$.

Definition 3.7. Let p be a definable Δ -type of elements of a definable group G .

$$Stab(p) = \{c : {}^c p = p\}$$

The definition makes it clear that $Stab(p)$ is a subgroup of G . When Δ is generated by a formula ϕ , $q = {}^c p$, we have: $d_q \phi(x, y) = d_p \phi(cx, y)$; so $c \in Stab(p)$ iff $q = p$ iff $d_q \phi = d_p \phi$ iff

$$\models (\forall y)(d_p x)(\phi(x, y) \equiv \phi(cx, y))$$

Thus when Δ is finite, $Stab(p)$ is a definable group. In general, therefore, it is an intersection of definable groups.

Lemma 3.8. *stabgen* Let p be a definable type of elements of a definable group G . If ${}^a p = p$ for $a \in Q$, and Q generates G , then p is a generic of G . In particular if p is B -definable and $p|B \subset Stab(p)$, then p is a definable generic type of $Stab(p)$.

Proof. $Stab(p)$ is a subgroup, so if $Q \subset Stab(p)$ then $G \subset Stab(p)$; it follows that p is generic of G .

Let $g \in Stab(p)$. Let $a \models p|B$. Then $ga \models p|Ba$. In particular $ga \models p|B$, and $g = (ga)a^{-1}$. Thus $p|B$ generates $Stab(p)$. The second statement thus follows from the first. $\square$

3.8.1. Let p, q be two definable Δ -types, and let

$$Stab(p, q) = \{c : {}^c p = q\}$$

Then $Stab(p, q)$ may be empty. If nonempty, $Stab(p, q)$ is a regular torsor for both $Stab(p)$ and $Stab(q)$. In particular these two groups are then conjugate (by any element of $Stab(p, q)$.)

3.8.2. *A more general variant.* Let $p(x)$ be a complete definable type, and let h be a definable function, $h(a, x) = y$, with parameters a in $\mathbb{U}$. Then we have a definable type $q(y) =^h p(x)$:

$$(d_q y)\phi(y, u) \equiv (d_p x)\phi(h(a, x), u)$$

Let $\mathfrak{G}_0(p)$ be the family of $\mathbb{U}$ -definable functions h such that $^h p = p$. $\mathfrak{G}_0(p)$ forms a semi-group under composition; one has the quotient semi-group of p -germs of elements of $\mathfrak{G}_0(p)$. The invertible germs form a group, denoted $\mathfrak{G}(p)$.

If p is a type of elements of a group G , then the stabilizer $Stab(p)$ defined above embeds naturally into $\mathfrak{G}(p)$: c maps to the germ of left translation by c .

3.9. A general remark on interpreting groups. *interpreting* A structure M interprets a structure N if there exists a 0-definable set D over M , and a surjective function $f : D \rightarrow N$, such that for any 0-definable (in N) $R \subset N^k$, $f^{-1}R \subset D^k$ is 0-definable in M .

It suffices to have this for any basic R (in the language of N), if this is understood to include equality.

In order to interpret a group N , it suffices to find a function $f : D \rightarrow G$ and $m \in \mathbb{N}$ such that:

- (1) $f(D)$ generates G in m steps. I.e. any element of G is a product of $\leq m$ elements of $f(D)$ and their inverses.
- (2) $\{(d_0, \dots, d_{2m}) \in D^{2m+1} : f(d_0) \cdot \dots \cdot f(d_{2m}) = 1\}$ is a definable subset of D^{2m+1} .

Indeed in this case we obtain a surjective function $\mu : D^{\leq m} \rightarrow G$ defined by $(\mu(d_1, \dots, d_l) \mapsto f(d_1) \cdot \dots \cdot f(d_m)$. To see that the pullback multiplication is definable, show by induction on l that $\{(d, e, e') \in D^l \times D^{\leq m} \times D^{\leq m} : \mu(d)\mu(e) = \mu(e')\}$ is definable. For $l = 1$, this follows from assumption (ii). For $l + 1$, we have:

$$\mu(d_0 \frown d)\mu(e) = \mu(e') \equiv (\exists e'' \in D^{\leq m})(\mu(d)\mu(e) = \mu(e'') \& \mu(d_0)\mu(e'') = \mu(e'))$$

The pullback of equality can be viewed as a special case.

3.10. Group chunks. This idea, in the context of algebraic groups, is due to Weil.

Let G be a definable group, with principal generic type p . The $p \otimes p$ -germ of multiplication is called the *group chunk* corresponding to (G, p) .

An *abstract group chunk* is a C -definable type p and a C -definable function F (or a $p \otimes p$ -germ of such a function), such that:

- (1) If $a \models p|C$, and $b \models p|acl(Ca)$, then $F(a, b) \models p|acl(Ca)$, and $b \in dcl(a, F(a, b))$, and $a \in dcl(b, F(a, b))$.
- (2) $\models (d_p x)(d_p y)(d_p z)F(x, F(y, z)) = F(F(x, y), z)$

Proposition 3.11. *chunk* Let p, F be an abstract group chunk. Then (p, F) is (definably isomorphic to) the chunk of an ∞ -definable group G (in $\mathbb{U}^{eq}$.) In other words p can be identified with a generic type of G , in such a way that $(d_p x)(d_p y)F(x, y) = x \cdot y$.

Proof. Let $P = p|C$, and let $Q = P \times P$. For $a \in P$, The p -germ of $F(a, x)$ is invertible. The germ g_a of $F(a, x)$ is thus an element of the group $\mathfrak{G}(p)$ of § 3.6. Let $\mathbf{G}$ be the subgroup of $\mathfrak{G}(p)$ generated by the elements g_a and their inverses. By § 3.9, it suffices to see:

- (1) Any element of $\mathbf{G}$ is a product $g_a g_b^{-1}$.
- (2) $\{(a_0, \dots, a_m) : g_{a_0}(g_{a_1})^{-1} \dots g_{a_m} = 1\}$ is definable.

The second point is immediate from the definability of p , and the fact that the map $\mathfrak{G}_0(p) \rightarrow \mathfrak{G}(p)$ is a well-defined homomorphism. As for the first, it suffices to show that a product $g_a g_b^{-1} g_c g_d^{-1}$ has the required form. Note that by (iii), when $(a, b) \models p \otimes p$, $g_a g_b = g_{F(a, b)}$;

and thus also $g_a^{-1}g_b = g_c$ where $F(a, c) = b$. Applying this, Thus given any $a, b, c, d \in P$, let $e \models p|acl(Cabcd)$; then $g_d^{-1}g_e = g_{e'}$ for some e' , and $e' \models p|acl(Cabc)$; so $g_c g_{e'} = g_{e''}$ for some $e'' \models p|acl(Cab)$; continuing this way, we obtain $g_a g_b^{-1} g_c g_d^{-1} g_e = g_{e''''}$ and so $g_a g_b^{-1} g_c g_d^{-1} = g_{e''''} g_e^{-1}$ as required. $\square$

The uniqueness of G in 3.11 is guaranteed by the following:

Proposition 3.12. *chunk2 Let G_1, G_2 be ∞ -definable groups, and let p be the unique generic type of G_1 . Let $F : G_1 \rightarrow G_2$ be a partial definable map, such that $(d_p x)(d_p y)F(xy) = F(x)F(y)$. Then there exists a unique homomorphism $H : G_1 \rightarrow G_2$ such that $\models (d_p x)F(x) = H(x)$.*

Proof. Uniqueness of H is clear, since the solution set P to p generates G : if $g \in G$, $a \models p|acl(C, b)$, then $ag \in P$.

Existence is also clear, provided we show that $F(a)F(b) = F(c)F(d)$ when $a, b, c, d \in P$ and $ab = cd$. Let $e \models p|acl(Cabcd)$. It suffices to show that $F(a)F(b)F(e) = F(c)F(d)F(e)$. But $F(b)F(e) = F(be)$, $F(d)F(e) = F(de)$ by the property of F . Moreover, $be \models p|acl(Ca)$, and so $F(a)F(be) = F(abe)$. Similarly the right hand side equals $F(cde)$. Since $abe = cde$, equality holds.

H is definable by compactness. We can also write: $H(a) = b$ iff $(d_p x)F(ax) = bF(x)$. $\square$

The analogous statements and proofs work for *group actions*.

3.13. $\star$ -definable groups with definable generics are projective limits of definable groups. For simplicity, we will deal with ∞ -definable groups; the $\star$ -definable case differs only in that the inclusion maps $G_{n+1} \rightarrow G_n$ below are replaced by arbitrary definable maps. Thus we show that ∞ -definable groups with definable generics are intersections of definable groups.

Actually we will prove something more general, by weakening the assumption of a (complete) definable generic to what we will call a “largeness notion”.

Let us clarify the notion of an ∞ -definable group G . The required data is a sequence $G_1 \supset G_2 \supset \dots$ of definable sets and maps, and a definable map $m : (G_2)^2 \rightarrow G_1$, such that $m(G_{n+1})^2 \subset G_n$. Composition gives two maps $G_3 \rightarrow G_1$, $m(m(x, y), z)$ and $m(x, m(y, z))$; we assume they are equal. Further, there exists a unit element $1 \in \cap_n G_n$, $m(1, x) = m(x, 1) = 1$; and an inverse map $x \mapsto x^{-1}$ ($G_2 \rightarrow G_2$), such that $m(x, x^{-1}) = m(x^{-1}, x) = 1$. For $x_1, \dots, x_n \in G_n$, we can then write unambiguously $x_1 \cdot \dots \cdot x_n$ to denote their product (but not for more than n elements.)

Let $G = \cap_n G_n$: in the sense that in any model M , we set $G(M) = \cap_n G_n(M)$; so that $G(M)$ is a group (with multiplication m .) Two ∞ -definable groups G, H are considered equal if $H(M) = G(M)$ for all M ; equivalently, the $\{H_n\}, \{G_n\}$ are isomorphic as directed limit systems.

Fix a saturated model $\mathbb{U}$. By a *largeness notion* with support G , we mean a filter μ on the Boolean algebra of $\mathbb{U}$ -definable subsets of G_1 , such that:

- (1) If $Q_b = \{a : (a, b) \in R\}$ where R is a definable set, then for some definable set $d_\mu R$, $Q_b \in \mu$ iff $b \in d_\mu(R)(\mathbb{U})$.
- (2) Each $G_n \in \mu$.
- (3) (translation invariance) If $Q_b \in \mu$, $g \in G$, then $gQ_b \in \mu$.

Note using (2) that if $Q_b, Q_{b'}$ are two definable sets, and $Q_b \cap G = Q_{b'} \cap G$, then $Q_b \cap G_n = Q_{b'} \cap G_n$ for some n , and so $Q_b \in \mu$ iff $Q_b \cap G_n$ iff $Q_{b'} \cap G_n$ iff $Q_{b'} \in \mu$. Thus μ can be viewed as a filter on relatively definable subsets of G ; this explains (3).

If G has a definable generic type, then the principal generic gives a largeness notion.

Proposition 3.14. *infdefdef Let $G = \bigcap_{n \geq 1} P_n$ be an ∞ -definable group, supporting a largeness notion μ . Then G is the intersection of a sequence of definable subgroups of a definable group.*

Proof. Let P_n be as above. Let $Q_n = P_3 \cap d_\mu(xy^{-1} \in P_n)$. Thus for $a \in P_3(\mathbb{U})$, $a \in Q_n$ iff $aP_n \in \mu$.

Claim. $G = \bigcap_{n \geq 2} Q_n$

Proof. Let $a \in G$. Then $a^{-1} \in P_{n+1}$, so $a^{-1}P_{n+1} \subset P_n$, $P_{n+1} \subset aP_n$, and thus as $P_{n+1} \in \mu$ we have $aP_n \in \mu$. So $a \in Q_n$.

Conversely, let $a \in \bigcap_n Q_n$. Then each $aP_n \in \mu$. Since μ is closed under finite intersections, and $\mathbb{U}$ is saturated, there exists $b \in \bigcap_n aP_n \cap \bigcap_n P_n$. So $b \in G$, and $b = ac$, $c \in G$. It follows that $bc^{-1} = (ac)c^{-1} = a(cc^{-1}) = a \cdot 1 = a$ so $a \in G$. $\square$

Thus for some $n_0 \geq 3$, for all $n \geq n_0$, $Q_n \subset P_4$. Let $n > n_0$.

Claim. $GQ_n \subset Q_n$.

Proof. Let $g \in G, b \in Q_n$. As $b \in P_3$, $gb \in P_2$. Now $g(bP_n) = (gb)P_n$. Since $bP_n \in \mu$, also $g(bP_n) \in \mu$. Thus $(gb)P_n \in \mu$ and so $gb \in Q_n$. $\square$

Let $S_n = \{x \in P_3 : (\forall y \in Q_n)xy \in Q_n\}$. If $a, b \in S_n$, then for any $x \in Q_n$, $bx \in Q_n$, so $a(bx) \in Q_n$. In particular, taking $x = 1$, $ab \in Q_n$, so $ab \in P_3$. Thus by definition $ab \in S_n$. So S_n forms a semigroup under m . We have $S_n \subset Q_n$, and by the last claim, $G \subset S_n$.

Finally, let $H_n = \{x : x, x^{-1} \in S_n\}$. Then clearly H_n is a group under m , and $G = \bigcap_n H_n$. $\square$

Corollary 3.15. *infdefdef-rings Let $R = \bigcap P_n$ be an ∞ -definable ring, supporting a largeness notion μ invariant both for both additive translations and for multiplication by units of R . Assume every element of R is a sum of units. Then R is the intersection of a sequence of definable subrings of a definable ring.*

Proof. By Proposition 3.14 applied to the additive group, we may assume that the P_n have the structure of additive groups, with P_{n+1} a subgroup of P_n . By compactness, we can refine the sequence so that the multiplication map is defined and gives a bilinear map $m : P_n \rightarrow P_n \rightarrow P_{n-1}$. Now follow the proof of Proposition 3.14 with respect to the multiplicative group of units R^* . Let $Q_n = P_3 \cap d_\mu(xy^{-1} \in P_n)$. By the additive invariance of μ and the distributive law, Q_n is a subgroup of P_3 . As in the Claim, we have $R = \bigcap_n Q_n$. Let $R_n = \{x \in Q_n : m(x, Q_n) \subseteq Q_n\}$. Then R_n is a ring. Again as in the first Claim, but now multiplicatively, $R^* \subseteq R_n$; thus $R \subseteq R_n$. Conversely we have $\bigcap_n R_n \subseteq \bigcap_n Q_n = R$. $\square$

3.16. Products of types in definable groups. convolution

By a *piecewise definable* set, we mean a sequence H_n of definable sets, with injective definable maps $H_n \rightarrow H_{n+1}$, viewed as inclusion maps. The direct limit is thus identified with the union, and denoted H . A *piecewise definable group* is a piecewise definable set together with maps $m_n : H_n \times H_n \rightarrow H_{n+1}$, compatible with the inclusions, and inducing a group structure $m : H \times H \rightarrow H$.

By a definable (or ∞ -definable) subgroup G of H we mean a (∞) -definable subset $G \subset H_k$ for some definable piece H_k of H ; such that $m(G^2) \subset G$ and (G, m) is a subgroup of (H, m) .

Let $p_1, \dots, p_n$ be a definable type of elements of H , and let w be an element of the free group F on generators $\{1, \dots, n\}$. We construct a definable type $p_w = w_*(p_1, \dots, p_n)$. If w is the product of the generators 1, 2, we also write $p_1 * p_2$ for p_w . Let

$a_i \models p_i | acl(Ca_1 \dots a_{i-1})$. Let $a_w = w(a_1, \dots, a_n)$ be the image of w under the homomorphism $F \rightarrow H$ with $i \mapsto a_i$. Let $p_w | C = tp(a_w / C)$.

These are definable types.

The words of the free group will be denoted by expressions such as $1\bar{2}3$; $\bar{2}$ is the inverse of the generator 2.

If a single type p is given, rather than a sequence p_w will refer to the sequence $(p, p, \dots, p)$. Let p^{*n} denote $p_{123\dots n}$, $p^{\pm 2n[+1]} = p_{1\bar{2}3\bar{4}\dots(2\bar{n})[(2n+1)]}$.

3.17. Examples from ACVF: non-stably dominated generics. The definable generics that will interest us most are those arising from stable domination. Here we give some other examples.

Let K be an algebraically closed valued field; given an algebraic group K , look at the definable generic types of $G(K)$.

- (1) $G_m(K)$ has two definable generics: of elements of large absolute value, and of infinitesimal absolute value.
- (2) $G_a(K)$ has a unique definable generic: of elements of large absolute value.
- (3) Let G be the solvable group of upper diagonal matrices. Then G has one-sided definable left-generics, right-generics, as well as two-sided generics. One of the latter is given as follows:

Let (x_{ij}) be the matrix coefficients of an element of $x \in GL_n(K)$. A two-sided generic is determined by: $|x_{ij}| \gg |x_{i'j'}|$ when $(i, j) < (i', j')$ lexicographically (and $i < j$.)

- (4) For $n > 1$, $SL_n(K)$ (or $GL_n(K)$) has no definable left generic type. For suppose p is a definable type. Then $(d_p x)(x_{11}) \geq |x_{21}|$ or $(d_p x)(x_{11}) < |x_{21}|$. Multiplying on the left by the elementary matrix with 1's on the diagonal, and t (with $|t| > 1$) in the $(2, 1)$ -entry (respectively the $(1, 2)$ -entry), this situation becomes reversed. Thus p is not invariant under left multiplication. Since $GL_n(K)$ has no proper subgroups of finite index (definable or not), p cannot be a left generic.

$SL_n(\mathcal{O}_K)$ is generically metastable, with a unique definable generic.

4. GENERICALLY METASTABLE GROUPS

Definition 4.1. *gstdomdef* A generically metastable group is a group with a stably dominated left-generic.

Lemma 4.2. *gmpres* If A, B are generically metastable groups, so is $A \times B$. If A is generically metastable and $f : A \rightarrow C$ is a definable surjective homomorphism, then C is generically metastable.

Proof. Let p, q be the principal generic types of A, B . Then $p \otimes q, f_* p$ are the generics of $A \times, C$ respectively; the verifications are easy. $\square$

Lemma 4.3. *trans* Assume N is a generically metastable definable or ∞ -definable subgroup of G , and G/N has a stably-dominated type invariant under G -translations. Then G is generically metastable.

In particular, if $N \trianglelefteq G$ and $N, G/N$ are generically metastable, then so is G .

Proof. Let p_N be the principal generic type of N , the unique generic of the connected component N^0 of N , and let $p_{G/N}$ be a G -invariant stably dominated type of G/N .

Assume first that N is connected, $N = N^0$.

If $n \in N$ then ${}^n p = p$. Thus if $cN = dN$, so that $c = dn$ for some $n \in N$, then

$${}^c p_N = {}^{dn} p_N = {}^d ({}^n p_N) = {}^d p_N$$

Thus the definable type ${}^c p_N$ depends only on the coset $S = cN$; denote it p_S .

We obtain a definable type p_G of G : a realization of p_G over C is a realization of p_S over C, s where $s \models p_{G/N}|C$, and S is the coset of N corresponding to s . Here $p_{G/N}$ is the given G -invariant definable type of G/N .

Then p_S is stably dominated. (Descent: Proposition 2.10.) So is $p_{G/N}$. Hence by transitivity (Proposition 2.9 (4)) p_G is stably dominated. By construction it is translation invariant.

In general, the natural map $r : G/N^0 \rightarrow G/N$ has profinite fibers. Thus $p_{G/N}$ lifts to G/N^0 ; i.e. the set Q of definable types q of G/N^0 with $r_*q = p_{G/N}$ is nonempty. Moreover it is profinite. Now G acts on Q by translation. So a co-profinite ∞ -definable subgroup G_1 of G must fix some element $q \in Q$. The above proof now applies to G_1 , using q . So G_1 is generically metastable, and hence so is G . $\square$

Lemma 4.4. *subgroups* *Let G be a generically metastable group, N a definable subgroup, $X = N \backslash G$ the right coset space, $\eta : G \rightarrow X$ the map: $\eta(g) = Ng$. Assume there exists a definable $Y \subset G$ with $\eta|Y$ finite-to-one. Then N is generically metastable.*

Proof. Let p be the generic type of G^0 , and let $g \models p|C$ (where N, G, X, Y, η, p are C -definable.) Let $h = \eta(g)$. Let $g_1 \in Y$, $\eta(g_1) = h$. Then $g_1 \in \text{acl}(C, h) \subset \text{acl}(C, g)$. Thus $tp((g, g_1)/C)$ is stably dominated (Proposition 2.9 (4)). Let $n = gg_1^{-1} \in N$. Then $n \in \text{dcl}(C, g, g_1)$, so by Lemma 2.9, $q := tp(n/C)$ is stably dominated. Because of the finite ambiguity in the choice of g_1 , the above may not pick out a unique type q , but at worst a finite set Q of types meet the description.

We now show that Q is $N \cap G^0$ -invariant. Let $n' \in N \cap G^0$; take $g \models p|Cn'$. Let $h = \eta(g)$, $g_1 \in Y$, $\eta(g_1) = h$. Let $n = gg_1^{-1}$. By definition, $n \models q|C, q$ a typical element of Q , and $n'n \models^{n'} q|Cn'$. Note that $n'n = (n'g)g_1^{-1}$, and $g_1 \in Y$, $\eta(g_1) = h = \eta(n'g)$. Thus $tp(n'n/C) \in Q$ also. So Q is a finite, $N \cap G^0$ -invariant collection of definable types; hence all types in Q are generic. $\square$

Corollary 4.5. *vsubgroups* *Let G be an algebraic group, N an algebraic subgroup. Let H be a definable subgroup of G in $ACVF$, with H generically metastable. Then $H \cap N$ is generically metastable.*

Proof. Let X be the coset space $N \backslash G$; it is an algebraic variety. Let $\eta : G \rightarrow X$. Let $X_H = \eta(H) = N_H \backslash H$ (where $N_H = H \cap N$.) This is all defined over some $M \models ACVF$. Let $g \in H$ be generic, $h = \eta(g)$. Then $N = M(h)^{\text{alg}} \models ACVF$; so there exists $d \in H(N)$ with $\eta(d) = h$. Let Y' be a formula over M , true of (d, h) , with a finite-to-one projection to X_H ; let Y be the projection of Y' to G . Then the hypotheses of Lemma 4.4 hold true of $(H, N_H, X_H, \eta|H, Y)$. Thus N_H is generically metastable. $\square$

Proposition 4.6. *groupdom* *Let G be a generically metastable group. Then there exists a $*$ -definable stable group $\mathfrak{g}$, and a $*$ -definable homomorphism $g : G \rightarrow \mathfrak{g}$, such that the generics of G are stably dominated via g .*

Moreover there exists a maximal such g , in the sense that any homomorphism $g' : G \rightarrow \mathfrak{g}'$ into a stable group factors through g . The kernel of this maximal g is thus uniquely determined.

Assuming G is definable, and (FD), $\mathfrak{g}$ is a definable group and $g : G \rightarrow \mathfrak{g}$ is definable.

Proof. Let $\theta(a)$ enumerate $St_C(a)$. Consider the map $f_a, f_a(b) = \theta(ab)$. The p -germ is strong, and is in $St_C(a)$, so it factors through $\theta(a)$: $f_a = f'_{\theta a}$. By stable embeddedness, it factors through $\theta(b)$ too: let $c = f_a(b) = f'_{\theta a}(b)$. Since $c \in St_C, tp(\theta(a), c/C, \theta(b)) \implies tp(\theta(a), c/C, b)$. Thus $c \in \text{dcl}(\theta(a), \theta(b))$; i.e. $\theta(ab) = c = F(\theta(a), \theta(b))$ for $a \models p, b \models p|Ca$. Now the hypotheses of the group chunk theorem 3.11 are satisfied, so F is a restriction of the multiplication map on a $*$ -definable group $\mathfrak{g}$. θ is generically a homomorphism, and by Proposition 3.12, it extends to a group homomorphism.

If $g' : G \rightarrow \mathfrak{g}'$ is another such homomorphism, then $g(a)$ is part of $St_C(a)$, when a realizes the generic type of G . Thus for some definable function r , $g'(a) = r(g(a))$. r extends uniquely to a group homomorphism $R : \mathfrak{g} \rightarrow \mathfrak{g}'$, and $g' = R \circ g$.

By Proposition 3.14, $\mathfrak{g}$ is a projective limit $\varprojlim \mathfrak{g}_i$ of definable groups over some directed partially ordered index set I ; for $i > j$ we have a homomorphism $\pi_i : \mathfrak{g}_i \rightarrow \mathfrak{g}_j$. Let $g_i : G \rightarrow \mathfrak{g}_i$ be the natural homomorphism. Since there are no descending chains of definable subgroups of $\mathfrak{g}_i$, every ∞ -definable subgroup is definable; in particular the image of g_i is definable. Replacing $\mathfrak{g}_i$ by this image, we may assume g_i and hence all maps π_i are surjective.

Assume (FD). Let M be a good base and let a realize the generic type of G^0 over M . Then $St_M(a)$ is finitely generated over M ; say $St_M(a) = M(d)$. Then $\dim(d/M(g_i(a)))$ decreases as i increases; so it stabilizes at some i_0 . Similarly the Morley multiplicity of $tp(d/M(g_i(a)))$ stabilizes at some $i_1 \geq i_0$. Since all $g_i(a)$ are in $M(d)$, it follows that $g_i(a) \in M(g_{i_1}(a))$ for all $i \geq i_1$. Thus π_{i, i_1} is an isomorphism. At the limit we obtain an isomorphism $\mathfrak{g} \rightarrow \mathfrak{g}_{i_1}$. $\square$

Without the maximality condition, the homomorphism g is not uniquely determined by G ; even if G is stable, there may be a nontrivial homomorphism of this kind.

Lemma 4.7. *groupdomcor* Let G be a generically metastable group, with generics dominated by a surjective group homomorphism $g : G \rightarrow \mathfrak{g}$. Then $tp(a/C)$ is generic in G (i.e. $a \models r|C$ for some definable generic r of G) iff $tp(g(a)/C)$ is generic in $\mathfrak{g}$.

Proof. Assume $tp(g(a)/C)$ is generic in $\mathfrak{g}$. Let p be a generic type of G , stably dominated via $g : G \rightarrow \mathfrak{g}$. Let $b \models p|acl(Ca)$. Then $q =^a p$ is a definable generic. Hence q is stably dominated via g . Note that $ab \models q|C$. Since $g(a)$ is generic in $\mathfrak{g}$, we have $g(a)g(b) \downarrow_C g(b)$; since g is a homomorphism, $g(ab) \downarrow_C g(b)$. Since p, q are both stably dominated via g we have $ab \downarrow_C b$. Thus $ab \models q|acl(Cb)$. So $a \models r = q^{b^{-1}}|Cb$, and in particular $a \models r|C$. Since r is generic, the lemma is proved. $\square$

Corollary 4.8. *groupdom3* Let G be a generically metastable group, with generics dominated by a group homomorphism $g : G \rightarrow \mathfrak{g}$. Let H be a definable subgroup of G . If $g(H) = \mathfrak{g}$, then H has finite index in G .

Proof. Work over $C = acl(C)$. Let p be the principal generic type of G . Let $b \in H \cap G^0$ be such that $g(b)$ is generic in $\mathfrak{g}$ over C . By Lemma 4.7, $tp(b/C)$ is generic in G . Thus a generic of G lies in H , so H has finite index in G . $\square$

Corollary 4.9. *groupdom4* (FD $_{\omega}$) Let G be a generically metastable definable group, with generics dominated by a group homomorphism $g : G \rightarrow \mathfrak{g}$. Then G^0 is definable, i.e. $[G : G^0]$ is finite.

Proof. G/G^0 is profinite. Extend the base C so as to have a representative of each class of G/G^0 . As in Lemma 2.14 we may further extend the base, and then find $a \in D$, $C' = \Gamma_C(a)$, such that $g(a)$ has dimension n over C' , and $tp(a/C', g(a))$ is isolated. From the first fact it follows (Lemma 4.7) that $tp(a/C')$ is a generic of G . So it is a translate (by some element in $G(C) \subseteq G(C')$) of the generic p of G^0 . It follows that $p|C'$ is isolated.

On the other hand, $G = \cap_i G_i$ where $\{G_i\}$ is a bounded family of definable subgroups of finite index, closed under intersections. $p|C'$ is generated by all formulas $\neg g(x) \in E$ (with E a C' -definable non-generic subset of $\mathfrak{g}$) together with all formulas $G_i(x)$. So $tp(a/C', g(a))$ is generated by the formulas $G_i(x)$. Being isolated, it is generated by a single formula $G_i(x)$. It follows that every generic of G_i lies in G^0 . Go $G_i = G^0$, and G^0 is definable. $\square$

Here is a characterization of generics that does not explicitly mention g and $\mathfrak{g}$.

Corollary 4.10. *Let G be a generically metastable group, $\dim_{st}(G) < \infty$. Then the generic types of G over C are precisely the types $tp(c/C)$ maximizing, for generic h , $\dim(St(hc)/C(h))$.*

Proof. By Proposition 4.6 and Lemma 4.7: we have $\text{acl } St(hc) = \text{acl}(g(h)g(c))$ so $\dim(St(hc)/C(h)) = \dim(g(h)g(c)/C(h)) = \dim(g(c)/C(h)) \leq \dim(g(c))$; if this is maximal, then $g(c)$ is generic in $\mathfrak{g}$ over $C(h)$, and hence over C . $\square$

(We could have used the Lascar rank of the stable part, in place of Morley dimension.)

Note that the proof uses rather than proves the existence of a generic.

If G is piecewise-definable, in a superstable theory, and p is a type of maximal rank in G , then p is a translate of a generic of a definable subgroup of G , $Stab(p)$. For stably dominated types, the analog need not hold:

Example 4.11. (*ACVF*). Let $G = (G_a)^2$. Let $P = \{(x, y) : \text{val}(x - y^2) \geq v\}$ where v is a fixed positive element of the value group. Then the maximal dimension of the stable part of any element of G over any base equals 2. P also extends to a type p whose stable part has rank 2. But though p has maximal rank, it is not a generic type. (We do have: p^{*2} is generic.)

A certain converse to Proposition 4.6:

Proposition 4.12. *a2def Let G be a definable group in a metastable theory, with $\dim_{st}(G) = n < \infty$. Assume G admits a surjective definable homomorphism $g : G \rightarrow \mathfrak{g}$, with $\mathfrak{g}$ stable of dimension n . Then there exists an ∞ -definable normal subgroup T of G with:*

- 1) G/T almost internal to Γ
 - 2) T connected and generically metastable via $g|_T$.
 - 3) $g(T) = \mathfrak{g}^0$, and
- T is uniquely determined by (1,2) or by (2,3).

Proof. Let Z be the collection of definable types q (over some B) of elements of G with g_*q generic in $\mathfrak{g}$, and q stably dominated.

Claim 1. $Z \neq \emptyset$.

Proof. Let C be a good base for G . Let $q_1 = tp(a_1/C)$ be any type over C avoiding $g^{-1}(Y)$ for all non-generic C -definable $Y \subset \mathfrak{g}$. Then g_*q_1 is generic in $\mathfrak{g}$. Let $\gamma = \text{dcl}(C(a_1)) \cap \text{dcl}(\Gamma)$. Let $q = tp(a/\text{acl}(C, \gamma))$. Any $\text{acl}(C, \gamma)$ -definable subset of $\mathfrak{g}$ is already $\text{acl}(C)$ -definable, so q also avoids all pullbacks of non-generic sets, and g_*q is generic in $\mathfrak{g}$. By metastability, q is stably dominated, and extends to a unique definable type; thus $Z \neq \emptyset$. $\square$

Claim 2. Any two elements of Z are (left or right) translates of each other.

Proof. Let $q, r \in Z$; say both are B -definable. Let $a \models q|B$, $b \models r|B(a)$. Then $g(a), g(b)$ are B -independent generic elements of $\mathfrak{g}$; hence also $g(a), g(a)g(b)^{-1}$ and $g(b), g(a)g(b)^{-1}$ are independent pairs of elements of $\mathfrak{g}$ over B . Let $c = ab^{-1}$. Then $g(c) = g(a)g(b)^{-1}$. Since $\dim_{st}(G) = n = \dim(St_B(c)/B)$, we have $St_B(c) \subseteq \text{acl}(g(c))$. Hence $g(a) \downarrow_B c$. By stable domination, $a \models q|B(c)$. Similarly $b \models r|B(c)$. It follows that $q =^c r$. $\square$

It follows that $T := Stab(q)$ is an ∞ -definable group, not depending on $q \in Z$. Since Z is invariant under conjugation, T is normal. Moreover if a, b are independent realizations of q over B , then the proof of the second claim shows that $a \downarrow_B ab^{-1}$, hence $ab^{-1} \in S$. It follows that $r = q^{b^{-1}}$ is a type of elements of S , and must be a generic type of S . Thus S is generically metastable. Also $g(ab^{-1})$ is a B -generic element of $\mathfrak{g}$, so $g_*(r)$ is a generic of $\mathfrak{g}$.

By Lemma 3.14, or over a larger base directly from the definition of a stabilizer, T is the intersection of definable groups S_i .

Claim 3. G/S_i is almost Γ -internal.

Proof. Otherwise, fix S_i , a good base C , and a type $\bar{q} = tp(\bar{a}/C)$ of elements of G/S_i , with $\bar{q}$ not almost Γ -internal. The element $\bar{a}$ corresponds to a coset $U_{\bar{a}}$ of S_i . Pick $a \in U_{\bar{a}}$; if $c \in S$ is generic then $ac \in U_{\bar{a}}$ while $g(ac) = g(a)g(c)$, so replacing a by ac we may assume $g(a)$ is generic in $\mathfrak{g}$ over C . By the proof of Claims 1 and 2, for some γ from Γ , $q = tp(a/C(\gamma))$ is stably dominated, and $q \subseteq Sa$, i.e. q is contained in a single coset of S . It follows that each aS_j is algebraic over $C(\gamma)$, and in particular $\bar{a} = aS_i \in \text{acl}(C(\gamma))$; contradicting the choice of $\bar{q}$. $\square$

Let T' be another connected generically metastable group. Then $T'/(T \cap T')$ is almost Γ -internal but also connected generically metastable, hence is trivial; so $T' \subseteq T$. By Corollary 4.8 and Lemma 3.14, T/T' is profinite; but T is connected, so $T = T'$. $\square$

4.13. Groups internal to Γ . This subsection contains some lemmas on o-minimal partial orders and groups. The former will yield a definable generic type of limit metastable groups. The latter will be used to improve some statements from “almost internal” to “internal”.

Let P be a partial ordering, and p a definable type. We say p is cofinal if for any $c \in P$, $\models (d_p x)(x \geq c)$. Equivalently, for every non-cofinal parametrically definable $Q \subseteq P$, $\models \neg(d_p x)Q(x)$.

Lemma 4.14. *cofinal-def P a definable directed partial ordering in an o-minimal structure Γ . Then there exists a definable type p cofinal in P .*

Proof. We assume P is 0-definable, and work with 0-definable sets; we will find a 0-definable type with this property.

Note first that we may replace P with any 0-definable cofinal subset. Also if Q_1, Q_2 are non-cofinal subsets of P , there exist a_1, a_2 such that no element of Q_i lies above a_i ; but by directedness there exists $a \geq a_1, a_2$; so no element of $Q_1 \cup Q_2$ lies above a , i.e. $Q_1 \cup Q_2$ is not cofinal. In particular if $P = P' \cup P''$, at least one of P', P'' is cofinal in P (hence also directed.)

If $\dim(P) = 0$ then P is finite, so according to the above remarks we may assume it has one point; in which case the lemma is trivial. We use here the fact that in an o-minimal theory, any point of a finite 0-definable set is definable.

If $\dim(P) = n > 0$, we can divide P into finitely many 0-definable sets P_i , each admitting a map $f_i : P_i \rightarrow \Gamma$ with fibers of dimension $< n$ (where Γ is o-minimal.) We may thus assume that there exists a 0-definable $f : P \rightarrow \Gamma$ with fibers of dimension $< n$. Let $P(\gamma) = f^{-1}(\gamma)$, and $P(a, b) = f^{-1}(a, b)$.

Claim 1. One of the following holds:

- 1) For any $a \in \Gamma$, $P(a, \infty)$ is cofinal in P .
- 2) For some 0-definable $a \in \Gamma$, for all $b > a$, $P(a, b)$ is cofinal.
- 3) For some 0-definable $a \in \Gamma$, $P(a)$ is cofinal.
- 4) For some 0-definable $a \in \Gamma$, for all $b < a$, $P(b, a)$ is cofinal.
- 5) For all $a \in \Gamma$, $P(-\infty, a)$ is cofinal.

Proof. Suppose (1) and (5) fail. Then $P(a, \infty)$ is not cofinal in P for some a ; so $P(-\infty, b)$ must be cofinal, for any $b > a$. Since (5) fails, $\{b : P(-\infty, b) \text{ is cofinal}\}$ is a nonempty proper definable subset of Γ , closed upwards, hence of the form $[A, \infty)$ or (A, ∞) for some 0-definable $A \in \Gamma$. In the former case, $P(-\infty, A)$ is cofinal, but $P(-\infty, b)$ is not cofinal for $b < A$, so $P(b, A)$ is cofinal for any $b < A$; thus (4) holds. In the latter case, $(-\infty, b)$ is cofinal for any $b > A$, while $(-\infty, A)$ is not; so $P([A, b))$ is cofinal for any $b > A$. Thus either (2) or (3) hold. $\square$

Let p_1 be a 0-definable type of Γ , concentrating on sets X with $f^{-1}(X)$ cofinal. (For instance in case (1) p_1 concentrates on intervals (a, ∞) .)

Claim 2. For any $c \in P$, if $a \models p_1|\{c\}$ then there exists $d \in P(a)$ with $d \geq c$.

Proof. Let $Y(c) = \{x : (\exists y \in P(x))(y \geq c)\}$. Then the complement of $Y(c)$ is not cofinal in P , so it cannot be in the definable type p_1 . Hence $Y(c) \in p_1|\{c\}$. $\square$

Now let $M \models T$. Let $a \models p_1|M$. By induction, let q_a be an a -definable type, cofinal in $P(a)$, and let $b \models q_a|Ma$. Then $tp(ab/M)$ is definable. If $c \in M$ then by Claim 2, there exists $d \in P(a)$ with $d \geq c$. So $\{y \in P(a) : \neg(y \geq c)\}$ is not cofinal in $P(a)$. Therefore this set is not in q_a . Since $b \models q_a|Ma$, we have $b \geq c$. This shows that $tp(ab/M)$ is cofinal in P . $\square$

A definable set X is Γ -internal if $X \subseteq \text{dcl}(\Gamma, F)$ for some finite set F ; equivalently for any $M \prec M' \models T$, $X(M') \subseteq \text{dcl}(M \cup \Gamma(M'))$. The same condition with acl replacing dcl is called *almost internality*; thus X is almost Γ -internal if X is Γ' -internal for some Γ' defined over parameters F' , and admitting a definable m -to-one function $f : \Gamma' \rightarrow Y$ into some definable $Y \subseteq \text{dcl}(\Gamma^n)$, for some finite m .

Lemma 4.15. *almost-int* Let G be a definable group. Assume G is almost internal to a stably embedded definable set Γ . Then there exists a finite normal subgroup N of G with G/N internal to Γ .

Proof. (cf. [3], [6]). We work in a saturated model $\mathbb{U}$, possibly over parameters for a small elementary submodel. The assumption implies the existence of a definable finite-to-one function $f : G \rightarrow Y$, where $Y \subseteq \text{dcl}(\Gamma)$. Given a definable $Y' \subseteq Y$, let $m(Y')$ be the least integer m such that over further parameters, there exists a definable m -to-one map $f^{-1}(Y') \rightarrow Z$, for some definable $Z \subseteq \text{dcl}(\Gamma)$. Let I be the family of all definable subsets Y' of Y with $(Y = \emptyset$ or $m(Y') < m(Y))$. This is clearly an ideal (closed under finite unions, and definable subsets). Let $F = \{Y \setminus Y' : Y' \in I\}$ be the dual filter. For $g \in G$, let $D(g)$ be the set of $y \in Y$ such that for some (necessarily unique) y' , $gf^{-1}(y) = f^{-1}(y')$; and define $g_*y = y'$. The function $x \mapsto (f(y), f(gy))$ shows that $\{y : |f(gf^{-1}(y))| > 1\} \in I$; equivalently, $D(g) \in F$. Let $F|Z = \{W \cap Z : W \in F\}$. Let G_0 be the set of bijections $\phi : Y' \rightarrow Y''$ with $Y', Y'' \in F$, carrying the filter $F|Y'$ to $F|Y''$. Write $\phi \sim \phi'$ if ϕ, ϕ' agree on some common subset of their domains, lying in F ; and let $G' = G_0 / \sim$. Composition induces a group structure on G' . The function g_* on $D(g)$ lies in G_0 , and we obtain a homomorphism $G \rightarrow G'$, $g \mapsto g_* / \sim$. Let N be the kernel of this homomorphism. Clearly $\text{Aut}(\mathbb{U}/\Gamma)$ fixes G' and hence G/N pointwise. It remains only to show that N is finite. In fact $|N| \leq m!$. For suppose $n_0, \dots, n_{m!}$ are distinct elements of N . Then for some $y \in \cap D(n_i)$ we have $(n_i)_*(y) = y$. It follows that $n_i(f^{-1}(y)) = f^{-1}(y)$, and since $|f^{-1}(y)| = m$, for some $i \neq j$ we have $n_i g = n_j g$ for all $g \in f^{-1}(y)$. But then $n_i = n_j$, a contradiction. $\square$

Lemma 4.16. *almost-int-2* Let G be a definable group. Assume G is almost internal to an o -minimal definable set Γ . Then G is Γ -internal.

Proof. By Lemma 4.15, there exists a definable surjective homomorphism $f : G \rightarrow B$ with B a group definable over Γ , and $N = \ker(f)$ a group of finite size n .

Let B^0 be the connected component of the identity in B ; then B/B^0 is finite, and it suffices to prove the lemma for $f^{-1}(B^0)$. Assume therefore that B is connected.

If G has a proper definable subgroup G_1 of finite index, then $f(G_1) = B$ by connectedness of B . It follows that N is not contained in G_1 , so $N_1 = N \cap G_1$ has smaller size than N . Hence using induction on the size of the kernel, G_1 is Γ -internal; hence so is G . Thus we may assume G has no proper definable subgroups of finite index.

Since the action of G on N has kernel of finite index, N must be central.

Let $Y = \{g^n : g \in B\}$. By [17], Theorem 7.2, there exists a definable function $\alpha : Y \rightarrow B$ with $n\alpha(b) = b$. Define $h : Y \rightarrow A$ by $h(b) = na$ where $f(a) = \alpha(b)$; this does not depend on the choice of a , and we have $fh(b) = nf(a) = ng(b) = b$. It follows that $f^{-1}(Y) = Nh(Y) \subseteq \text{dcl}(N, \Gamma)$ is Γ -internal.

Similarly, let $[B, B] = \{[g, h] : g, h \in B\}$. Again there exists a definable $\alpha_1 : [B, B] \rightarrow B$ such that $(\exists y)[\alpha_1(b), y] = b$, and $\alpha_2 : [B, B] \rightarrow B$ such that $[\alpha_1(b), \alpha_2(b)] = b$. Define $h' : [B, B] \rightarrow A$ by $h'(b) = [a_1, a_2]$ where $f(a_i) = \alpha_i(b)$. Again h' is definable and well-defined, and shows that $h^{-1}[B, B]$ is Γ -internal.

Hence for any k , letting $X^{(k)} = \{x_1 \dots x_k : x_1, \dots, x_k \in X\}$, $(h^{-1}(Y \cup [B, B]))^{(k)} = h^{-1}((Y \cup [B, B])^{(k)})$ is Γ -internal. So we are done once we show:

Claim. Let B be any definably connected group definable in an o-minimal structure. Let $Y = Y_n(B) = \{g^n : g \in B\}$. Then for some $k \in \mathbb{N}$, $(Y \cup [B, B])^{(k)} = B$.

Proof If the Claim holds for a normal subgroup H of B with bound k' , and also for B/H (with bound k''), then it is easily seen to hold for B (with bound $k' + k''$).

We use induction on $\dim(B)$. If B has a nontrivial proper connected definable normal subgroup H , then the statement holds for H and for B/H . We may thus assume B has no such subgroups H .

If B is Abelian then in fact $Y = B$, by [14]. Otherwise the center Z of B is finite. In this case by the same argument as above, the claim holds with $k = k_{B/Z} + |Z|$. By connectedness B has no nontrivial finite normal subgroups. Hence B is definably simple.

Now by [16], B is elementarily equivalent to a Lie group. So we may assume B is a simple Lie group. In this case it is known that every element is the product of a bounded number of commutators.

□

4.17. Generically metastable subgroups of maximal rank. While not all generically metastable subgroups are definable, we will show that subgroups whose residual rank is $\dim_{st}(D)$ are.

Lemma 4.18. *a3def* Let G be a definable group in a metastable theory. Let T be an ∞ -definable subgroup of G with G/T almost internal to Γ , and T connected and generically metastable.

- 1) (FD) There exists a definable subgroup S of G with $S^0 = T$.
- 2) $(FD_\omega)T$ itself is definable.

Proof. By Lemma 3.14, T is the intersection of definable groups S_i ; and G/S_i is almost Γ -internal. Assume (FD). By Proposition 4.6, there exists a definable homomorphism $g : T \rightarrow \mathfrak{g}$, with $\mathfrak{g}$ stable, and T generically metastable via $g|_T$. By compactness, g extends to a definable homomorphism on some S_i ; replacing G by this S_i , we may assume $g : G \rightarrow \mathfrak{g}$.

1) By Lemma 4.15, some finite quotient of G/S is Γ -internal; define $\dim(G/S)$ to be the o-minimal dimension of any such quotient. By (FD), for any $a \in A$, the rank of $\Gamma_B(a)$ over B is finite, so the o-minimal dimension of G/S_i is bounded independently of i . Thus for some i , for all $j \geq i$, $\dim(G/S_j) = \dim(G/S_i)$. It follows that the natural map $G/S_j \rightarrow G/S_i$ has zero-dimensional fibers, hence it has finite kernel. This shows that for some i , for all $j \geq i$, S_i/S_j is finite.

Thus all S_j (with $j \geq i$) are generically metastable via g , and (1) holds.

2) Assume (FD_ω) . Then by Corollary 4.9, T is definable. □

Corollary 4.19. *maxrkdef* (FD_ω) Let H be a definable group, G be a connected ∞ -definable generically metastable subgroup of H , with a stable homomorphic image of dimension $n = \dim_{st}(H)$. Then G is definable.

Proof. By Proposition 4.6, there exists a definable homomorphism $g : G \rightarrow \mathfrak{g}$ with $\mathfrak{g}$ ω -stable, such that G is stably dominated via g . So $\dim(g(a)) = n$ for $a \models p$, p the generic of G . By Proposition 3.14, G is an intersection of definable groups G_i ; by compactness, g extends to a homomorphism $G_i \rightarrow \mathfrak{g}_i$, with $\mathfrak{g}_i$ a stable definable group (for some i .) Then $g(G_m)$ is a descending sequence of definable subgroups of $\mathfrak{g}_i$; by ω -stability of the stable part, it must stabilize; i.e. $\mathfrak{g}_m = \mathfrak{g}$ for large enough m . By Proposition 4.12 there exists an ∞ -definable T of H with H/T almost internal to Γ , T connected and generically metastable via $g|_T$, $g(T) = \mathfrak{g}^0$. By Lemma 4.18, T is definable. By the uniqueness in Proposition 4.12, $T = G$. $\square$

Corollary 4.20. *4def Let A be a definable Abelian group in a metastable theory.*

(FD) Any connected generically metastable ∞ -definable subgroup of A is contained in a definable generically metastable subgroup.

(FD $_\omega$) Any connected generically metastable ∞ -definable subgroup of A is contained in a definable connected generically metastable subgroup.

Proof. By Proposition 3.14, H is an intersection of definable groups H_i ; for simplicity assume $H = \bigcap_{i \in \mathbb{N}} H_i$, $H_0 \supset H_1 \supset \dots$. Then $\dim_{st}(A/H_i)$ is non-decreasing with i , and eventually stabilizes; we may assume it is constant, $\dim_{st}(A/H_i) = n$. Clearly $\dim_{st}(A/H) \leq n$; but also since A/H_0 is a homomorphic image of A/H , $\dim_{st}(A/H) \geq \dim_{st}(A/H_0) = n$. By Proposition 5.4, A/H contains a generically metastable ∞ -definable subgroup $\mathbf{S}$ with stable homomorphic image of dimension n . Let $\mathbf{S}^i$ be the image of G in A/H_i . So $\mathbf{S}^i$ is generically metastable, $\mathbf{S}$ is the inverse limit of the $\mathbf{S}_i$, and for large enough i (say for $i = 1$), $\mathbf{S}^1$ has a stable homomorphic image of dimension n . By Corollary 4.19, $\mathbf{S}^1$ is definable. Let S_1 be the pullback of $\mathbf{S}^1$ to A . So S_1 is definable, and for any $j \geq 1$, S_1/H_j has a stable homomorphic image of dimension $n = \dim_{st}(S_1/H_j)$. By Proposition 4.12 there exists a unique ∞ -definable group T_j with $H_j \leq T_j \leq S_1$, S_1/T_j almost Γ -internal, and T_j/H_j generically metastable and connected. By Lemma 4.18 T_j/H_j is definable, hence T_j is definable. If $k > j$ then $H_k \leq H_j$; since T_k/H_k is generically metastable connected, and S_1/T_j is almost Γ -internal, we have $T_k/H_k \leq T_j/H_k$, i.e. $T_k \leq T_j$. Since S_1/T_j is (almost) Γ -internal, the argument of Lemma 4.18 (1) shows that for large enough j (say for $j = 1$), for all $k \geq j$, T_k has finite index in T_1 ; i.e. the T_k all have the same connected component of 1, an ∞ -definable group T . Then $T/H_k \cap T$ is generically metastable for all k , so T/H is generically metastable. By Proposition 4.3, T is generically metastable. Hence so are the T_i . By Lemma 4.18, if (FD $_\omega$) holds, T is definable. $\square$

A connected generically metastable definable group H is called *certifiable* (over C) if it is an element of a uniformly definable family of definable groups $H_c : c \in Q$, with Q definable over C , such that each H_c is connected and generically metastable.

Lemma 4.21. *cert (FD $_\omega$) Let A be a definable Abelian group. There exists a base C and C -definable families S_t^ν of definable subgroups S_t^ν of A such that*

- (1) Any S_t^ν is connected, generically metastable.*
- (2) Any connected, generically metastable ∞ -definable subgroup of A (over any set of parameters) is contained in some S_t^ν .*

An equivalent statement is that any connected, generically metastable ∞ -definable subgroup of A is contained in a C -certifiable one. By Corollary ??, it suffices to show that any connected, generically metastable definable subgroup of A is contained in a C -certifiable one.

Proof. For a definable Abelian group B , define invariants n, k, l as follows: $n = \dim_{st}(B)$. Let $Z(B)$ be the collection of definable subgroups S of B with stable homomorphic images of dimension n ; by Lemma 5.4, $Z(B) \neq \emptyset$. Let $Z_2(B) = \{(T, K) : T \in Z(B), \dim_{st}(T) = n, K \leq T, T/K \text{ } \Gamma\text{-internal}\}$. Let $k = \max\{\dim(T/K) : (T, K) \in Z_2(B)\}$. (By (FD), such a

maximum exists.) If $(T, K) \in Z_2(B)$ then by Proposition 4.12 and Lemma 4.18, K is generically metastable, and by $[K : K^0]$ is finite. Since $Z_2(B)$ is uniformly definable, by Lemma 2.15 there exists a bound l on $[K : K^0]$, valid for all such K .

The set of triples (n, k, l) is ordered lexicographically.

Pick a definable generically metastable definable $B_0 \leq A$ such that $(n, k, l)(A/B_0)$ is as small as possible. (If $(T/B_0, K/B_0)$ attains the maximum for A/B_0 , then the pullbacks to A show that $(n, k, l)(A) \geq (n, k, l)(A/B_0)$. Thus increasing B_0 has the effect of decreasing $(n, k, l)(A/B_0)$.) For any generically metastable $H \leq A$, $H + B_0$ is also generically metastable; so it suffices to find a family $\{S_t\}$ for A/B_0 . Thus (after augmenting the base so that B_0 is definable) we may assume $(n, k, l)(A/B) = (n, k, l)(A)$ for any connected generically metastable definable $B \leq A$. Let $(n, k, l) = (n, k, l)(A)$. S

Claim. Let S be a definable subgroup of A admitting a surjective homomorphism $h : S \rightarrow \mathfrak{g}$ to a stable group of dimension n . Let W be a Γ -group, $\dim_o(W) = k$, and let $g : S \rightarrow W$ be a homomorphism with kernel L . Assume $[L : L^0] = l$. Then L^0 is connected, generically metastable, and certifiably so (over C .)

Proof. Say $S, h, \mathfrak{g}, g, W, L, L^0$ are all d -definable. Clearly they lie in a family $(S_t, h_t, g_t, \mathfrak{g}_t, W_t, L_t, L'_t)$ such that $\mathfrak{g}$ is stable, $\dim \mathfrak{g}_t = n(A)$, W_t is a Γ -group, $h_t : S_t \rightarrow \mathfrak{g}$ is a surjective homomorphism, $g_t : S_t \rightarrow W_t$ is a surjective homomorphism with kernel L_t , and L'_t is a definable subgroup of L_t of index l . By Proposition 4.12 and Lemma 4.18, L_t is generically metastable. By definition of l we have $[L_t : L'_t] \leq l$; so $L_t^0 = L'_t$. Thus L'_t is connected and generically metastable, for any element of the family. Let $\{L'_t\}$ be the projection to a family of subgroups of A . This is a definable family (using definability of Morley dimension and o-minimal dimension, (FD).) Any L'_t is connected and generically metastable, showing certifiability. $\square$

Let B be a generically metastable definable subgroup of A . Let $(T, K) \in Z_2(A/B)$ with $\dim_{st}(T) = n, \dim_0(T/K) = k, [K : K^0] = l$. Let (S, L, L') be the pullbacks to A , so that $B \leq L' \leq L \leq S$ and $T = S/B, K = L/B, K^0 = L'/B$. Clearly S admits a homomorphism onto a stable group of dimension n , and $\dim_o(S/L) = \dim_o(T/K) = k, [L : L'] = l$.

Since $(n, k, l)(A) = (n, k, l)$ it follows that L is generically metastable, and (since $[L : L^0] \leq l$) that $L' = L^0$. By the Claim, L' is certifiably connected generically metastable subgroup. Since $B \subseteq L'$, the Proposition is proved. $\square$

5. ABELIAN GROUPS

Lemma 5.1. *abelian1 Let H be a piecewise definable, or even piecewise $*$ -definable, Abelian group, p a symmetric definable type of elements of H . Assume H has p -weight $< 2n$, in the sense that:*

Whenever $b \in H, (a_1, \dots, a_{2n}) \models p^{\otimes 2n}, a_i \models p|b$ for some i .

Then there exists an ∞ -definable subgroup G of H with generic type $p^{\pm 2n}$. p is contained in a coset of G .

Proof. Let $(a_1, a_2, \dots, a_{2n}) \models p^{\otimes 2n}$, and let $b = a_1^{-1} a_2 \dots a_{2n}$.

By the weight assumption, $a_i \models p|b$ for some i . Say i is odd (the even case is similar.) Since the group is commutative, $tp(a_1, a_2, \dots, a_{2n}/b)$ is $Sym(n)$ -invariant, so $a_1 \models p|b$.

Let G be the stabilizer of $p^{\pm 2n}$, and $C = Stab(p^{\mp 2n-1}, p^{\pm 2n})$. Then $a_1^{-1} \in C$, so $p^{\pm 1}$ is a type of elements of C . It follows that $p^{\pm 2}$ and hence also $p^{\pm 2n}$ is a type of elements of G . By Lemma 3.8, it is the generic type. $\square$

Remark 5.2. *ab2* It follows that p generates a coset of the ∞ -definable group G , in finitely many iterations of the function: $(x, y, z) \mapsto xy^{-1}$. If p is stably dominated, it follows that G is connected and generically metastable.

For the rest of this section we work with a metastable theory.

Let us say that a set D has bounded weight if for some n , every stably dominated type of elements of D has weight $\leq n$. In ACVF, every definable set has bounded weight (if it is a subset of an algebraic variety V , the weight is bounded in fact by the dimension of V .) In EVDF, the same is true of a definable set of finite differential order.

Proposition 5.3. *sums* Let G be a definable Abelian group of bounded weight. Let A_i ($i \in I$) be connected generically metastable ∞ -definable subgroups of G (defined over a set C .) Then there exists a generically metastable ∞ -definable subgroup B containing all A_i .

Proof. The sum of two generically metastable groups is again generically metastable, by Lemma 4.2: $A + B$ is a homomorphic image of $A \times B$.

Let $\mathfrak{F}$ be the family of all C -definable functions into Γ . By metastability, enlarging C if necessary, we have for any $c \in G$: $tp(c/C, (f(c) : f \in \mathfrak{F}))$ is stably dominated.

Let p_i be the principal generic of A_i . Consider the partial type:

$$q_0 = \{(d_{p_i}y)f(x) = f(yx) : i \in I, f \in \mathfrak{F}\}$$

Claim 1. q_0 is consistent.

It suffices to show that any finite number of formulas, concerning say $p_1, \dots, p_n$, can be satisfied. As above, let $A = A_1 + \dots + A_n$, and let p be the principal generic of A .

If $(b, c) \models p_i \otimes p$, then by genericity of p , $bc \models p|b$. By symmetry for stably dominated types, $b \models p_i|c$. Now if $f \in \mathfrak{F}$, as p is stably dominated, and thus orthogonal to Γ , there exists γ_f such that for any $c' \models p|C$, $f(c') = \gamma_f$. In particular, $f(c) = \gamma_f = f(bc)$. Thus $\models (d_{p_i}y)f(c) = f(yx)$. Since this is true for each $i = 1, \dots, n$ and $f \in \mathfrak{F}$, q_0 is consistent.

Let $c \models q_0$, $C' = C \cup (f(c) : f \in \mathfrak{F})$, $C'' = \text{acl}(C')$.

Claim 2. $tp(c/C'')$ is generically p_i -invariant for each i .

Indeed if $a \models p_i|Cc$, then $tp(ac/C) = tp(c/C)$ (consider two-valued functions in $\mathfrak{F}$). Thus $tp(ac, (f(ac) : f \in \mathfrak{F})/C) = tp(c, (f(c) : f \in \mathfrak{F})/C)$. But $f(ac) = f(c)$; so $tp(ac, (f(c) : f \in \mathfrak{F})/C) = tp(c, (f(c) : f \in \mathfrak{F})/C)$. This shows $tp(c/C') = tp(ac/C')$. In fact

As noted, $tp(c/C')$ is stably dominated. By 5.2, $tp(c/C')$ generates a coset B' of a generically metastable group B . B' depends on the extension of $tp(c/C')$ to a type over C'' , but B does not. It follows that B is generically p_i -invariant, and hence $A_i \subseteq B$. $\square$

Lemma 5.4. *abelian-inertial-1* (FD). Let G be a $\star$ -definable Abelian group. Then G contains a generically metastable ∞ -definable subgroup S with stable homomorphic image of dimension $\dim_{st}(G)$.

Proof. Say $\dim_{st}(G) = n$. Let $C = \text{acl}(C)$ be a base substructure, $g \in G$ with $\dim(\text{St}_C(g)/C) = n$. Let $\gamma = \text{dcl}(C(g)) \cap \text{dcl}(\Gamma)$, $p = tp(g/C, \gamma)$. So p is stably dominated, with stable part p_{st} of dimension n .

A real valued function r on types is *subadditive* if $r(tp(ab/C)) \leq r(tp(b/C)) + r(tp(a/Cb))$.

Claim 1. $\dim(\text{St}_C(x)/C)$ is subadditive.

Proof.

$$\dim(\text{St}_C(a, b)/C) \leq \dim(\text{St}_C(a, b)/C, \text{St}_C(a)) + \dim(\text{St}_C(a)/C) = \dim(\text{St}_C(a, b)/Ca) + \dim(\text{St}_C(a)/C)$$

Claim 2. $p^{\pm m}$ is also stably dominated, with stable part of dimension n . More generally, if q is stably dominated, $\dim(p_{st}) = \dim(q_{st}) = n$, $a \models p|C$, $b \models q|Ca$, $c = ab$, then $\dim(St_C(c)/C) = n$, and $tp(c/C)$ is stably dominated.

Proof. $2n \leq \dim(St_C(a, b)/C) = \dim(St_C(a, c)/C) \leq \dim(St_{C_c}(a)/Cc) + \dim(St_C(c)/C) \leq n + \dim(St_C(c)/C)$. So $\dim(St_C(c)/C) \geq n$. But $\dim_{st}(G) = n$, so $\dim(St_C(c)/C) = n$. Since $tp(ab/C)$ is stably dominated, so is $tp(c/C)$.

By Lemma 5.1, G contains a generically metastable group S , with generic type $p^{\pm 2n}$. By the claims, the stable part of the generic of S has dimension n . □

5.5. Limit - metastable groups.

Definition 5.6. *limit* Let q be a $\star$ -type of Γ , over a small set C_0 . For $t \models q$, let S_t be an ∞ -definable subgroup of G : $S_t =_{def} \cap_n S_t^n$, where S_t^n is a Γ -definable family of subgroups of G . We call (S_t) a limit-metastable family for G if:

- (1) S_t is a connected generically metastable subgroup of G .
- (2) q carries a C_0 -invariant partial ordering $\leq$. Any small set has an upper bound in this ordering. If $t \leq t' \models q$, then $S_t \subset S_{t'}$.
- (3) If $W \subset G$ is connected and generically metastable, ∞ -definable over C , then $W \subseteq S_t$ for some $t \models q$.

$\cup_{t' \models q} S_{t'} = H$ is ∞ -definable (with no parameters.) We say that H is the limit-metastable group for G . If $G = H$ we say that G is limit-metastable.

Remark 5.7. *limitr*

- (1) In favorable cases, the partial ordering on q is definable. In particular this is the case when the S_t are definable groups. When this is so we will say that the system is definable.
- (2) We thus view H as the limit of a Γ -internal direct limit system of generically metastable groups. It is clearly independent of the particular limit-metastable family.
- (3) There are really two quite different behaviors here, according to whether or not the direct limit system has a maximal element. The latter is equivalent to the existence of a maximal connected generically metastable subgroup of G . (For linear algebraic groups in ACVF, this is related to the notion of a reductive group.)
- (4) When A has bounded weight, condition (3) is equivalent to: if q is a generically metastable type over C , $q \subset S_t$.

Lemma 5.8. *sums-cor* Let G be a definable Abelian group of bounded weight. There exists a limit-metastable ∞ -definable subgroup for G .

Proof. Let C_1 be an $\aleph_1$ -saturated elementary submodel of a large saturated model for the theory. Let $C_1 \subseteq C$ be such that metastability for G is witnessed over C .

Let $\mathfrak{F}$ be the family of all C -definable functions into Γ . Let $A_i (i \in I)$ be the family of all connected generically metastable ∞ -definable subgroups of G , defined over C . Let q_0 be the partial $\star$ -type found in the proof of Proposition 5.3. Let $c \models q_0$, $t = \mathfrak{F}(c)$, and let S_t be the generically metastable group found there.

By Lemma 3.5, we can take t to be a small tuple. Moreover we can find a small $C_0 \subset C$ such that S_t is $C_0(t)$ -definable. Let $q = tp(t/C_0)$.

If W is a connected generically metastable group, we must show that for *some* $t' \models q$, $W \subset S_{t'}$. For this purpose we can replace W by a conjugate, under the group of automorphisms of the universal domain over C_0 . Thus we may assume W is defined over C . In this case, $W \subset S_t$. This gives (3).

Define the partial ordering by: $S_t \subset S_{t'}$. The directedness follows from Proposition 5.3, together with (3). $\square$

Theorem 5.9. *ab Let A be a definable Abelian group of bounded weight.*

There exists an ∞ -definable limit- metastable subgroup K of A , with A/K almost internal to Γ .

If K is itself generically metastable, then A/K is a pro-definable Γ -group.

If (FD_ω) holds, then K is definable, has a definable generic type, and equals the kernel of a definable $\lambda : A \rightarrow \Lambda$ into a Γ -group.

Remark 5.10. *For the definability of Λ it suffices to assume, in place of (FD_ω) , that every ∞ -definable connected generically metastable subgroup of A is contained in a certifiable one.*

Proof. First suppose A contains no nontrivial generically metastable ∞ -definable subgroups. Then by Lemma 5.1, no type of elements of A is stably dominated. As the theory is metastable, for some base set C , there exist C -definable maps $f_j : A \rightarrow \Gamma$ such that the fibers of $f = (f_j)_j$ are stably dominated. Thus the common kernel of the maps f_j is finite. By compactness finitely many of the f_j have finite kernel. By Lemma 4.16, one can take λ to be an isomorphism.

This argument works for $\star$ -definable groups as well. Specifically if $B = \cap_i B_i$ is an intersection of definable subgroups of A , and $A/B = \varprojlim A/B_i$ has no nontrivial generically metastable ∞ -definable subgroups, then each A/B_i is isomorphic to a Γ -group; by stable embeddedness of Γ the group structures and the homomorphisms $A/B_i \rightarrow A/B_j$ for $j \geq i$ are also definable over Γ .

Next suppose A has a maximal ∞ -definable generically metastable subgroup B . By Proposition 3.14, we have $B = \cap_i B_i$ for certain definable subgroups B_i of A . We can view A/B as a $\star$ -definable group (a projective limit of the groups A/B_i .) By Lemma 4.3, A/B can have no nontrivial generically metastable ∞ -definable subgroups; otherwise the pullback to A would be a generically metastable ∞ -definable group. By the previous case, A/B is definably isomorphic to a pro-definable group Λ of Γ^{eq} . The isomorphism $A/B \rightarrow \Lambda$ gives an homomorphism $\lambda : A \rightarrow \Lambda \subset \Gamma^{eq}$ with kernel B .

In general, let B be the limit-metastable subgroup of A . B is ∞ -definable:

$$b \in B \iff (\exists t \models q)(b \in S_t) \iff (\exists t \models q) \bigwedge_n (b \in S_t^n)$$

If ψ ranges over all finite sets of formulas in q , we find

$$b \in B \iff \bigwedge_n \bigwedge_\psi (\exists t)(\psi(t) \& b \in S_t^n)$$

Let C be a good base for A (Definition 1.2).

Let $\mathfrak{F}$ be the family of all C -definable functions $f : A \rightarrow E$, where E is C -definable and $E \subseteq \text{acl}(C, \Gamma)$.

Claim. If $f(c) = f(d)$ for $f \in \mathfrak{F}$, then $cd^{-1} \in B$.

In fact every fiber of $\mathfrak{F}$ lies in a coset of some S_t . This is proved as in Proposition 5.3 Claim 2, and implies the claim. Thus A/B is definably isomorphic to $\mathfrak{F}(A)/E$ for some ∞ -definable equivalence relation E .

Now assume that among the connected generically metastable ∞ -definable subgroups of A , the certifiable ones are cofinal. Replacing S_t by a larger certifiable group, we can take $(S_t : t \models q)$ to be a uniformly definable family of connected generically metastable definable groups, such that every connected generically metastable definable group is contained in some S_t . Let Q be a definable set containing q , such that S_t is connected and generically metastable for $t \in Q$.

Then $\cup_{t \in Q} S_t = \cup_{t \models q} S_t$. (Any S_t with $t \in Q$ is connected generically metastable, and thus by cofinality of the family contained in some $S_{t'}$, $t \models q$.) Define a partial ordering on Q by $S_t \subseteq S_{t'}$. This is directed, since again if $t, t' \in Q$ there exists $t'' \models q$ with $t, t' \leq t''$. Thus the limit-metastable group is definable, and hence Λ is definable in Γ^{eq} .

By Lemma 4.14, Q has a cofinal definable type $\tilde{q}$. For $t \models \tilde{q}$ let p_t be the unique generic type of S_t . Let r be the definable type as in Lemma 2.3. If $c \in \Lambda$, and $t \models \tilde{q}|c$, then $c \in S_t$ (since $c \in S_{t'}$ for some $t' \in Q$, and $\tilde{q}$ is cofinal.) Hence if $a \models p_t|\{c, t\}$ then $ca \models p_t|\{c, t\}$. It follows that $a \models r|\{c\}$ and $ca \models r|\{c\}$; showing that r is Λ -translation invariant. $\square$

Corollary 5.11. *max-G-quotient* (FD_ω) *Let A be a definable Abelian group. There exists a universal pair (f, B) with B a Γ -internal definable group, and $f : A \rightarrow B$ a definable homomorphism. In other words for any (f', B') of this kind there exists a unique definable homomorphism $h : B \rightarrow B'$ with $f' = hg$.*

Proof. Let L be the limit-metastable subgroup of A . Then L is definable. For any pair (f', B') as above, f vanishes on any connected generically metastable group. Since L is a union of such groups, f vanishes on L . But A/L is Γ -internal, so the canonical homomorphism $A \rightarrow A/L$ clearly solves the universal problem. $\square$

In particular, in ACVF, for $0 < a \in \Gamma$, there cannot be a compatible sequence of homomorphisms $\psi_n : A \rightarrow C(n!a)$ for each n , where $C(b)$ is the definable subquotient $C(b) = \{c \in \Gamma : (\exists m \in \mathbb{N})(-mb < C < mb)\}/\mathbb{Z}b$ of $(\Gamma, +)$.

Corollary 5.12. *finite-cc* (FD_ω) *Let A be a definable Abelian group. There exists a smallest definable subgroup A^0 of A of finite index.*

Proof. As in Corollary 5.11, any subgroup A' of finite index must contain each connected generically metastable group, and hence the limit group L . The question thus reduces to the o-minimal group A/L , where it is known, [15]. $\square$

Lemma 5.13. *max-st-quotient* (FD_ω) *Let A be a definable Abelian group. There exists a universal pair (f, B) with B a stable definable group, and $f : A \rightarrow B$ a definable homomorphism. Equivalently, there exists a smallest definable subgroup K with A/K stable.*

Proof. Clearly if A/K and A/K' are stable, so is $A/(K \cap K')$. it suffices to show that there are no strictly descending chains $A \supset K_1 \supset K_2 \dots$ of such subgroups. By (FD) $\dim(A/K_n)$ is bounded; so we may assume it is constant, $\dim(A/K_n) = d$. Then K_n/K_{n+1} is finite. By Corollary 5.12, the chain stabilizes. $\square$

(Using (FD) in place of (FD_ω) , it is easy to see by a direct argument that there exists a universal homomorphism into a *connected* stable definable group.)

5.14. Metastable fields. * Under " ω -stability" assumptions; *or: sums-cor there exists a canonical *connected* limit-metastable ∞ -definable group, with pro- Γ -internal quotient. Of the entire field. By uniqueness it is an ideal, so must equal F . Hence F itself is a limit of connected groups. Multiplicative version:

From each group create an additively connected ring. $K^*/\lim R^*$ is Γ -internal. **If additive generic does not concentrate on units, still on an element of $K^*/\lim R^*$ with $2\gamma = \gamma$, but K^*/R^* is a group, so $\gamma = 0$. Hence eventually units. But any field element is a ratio of two additive generics. ? – if so then there will be a maximal stably dominated subring.

Lemma 5.15. *fields* *Let F be a metastable field of bounded weight. There exists an ∞ -definable subring D of F , with a stably dominated definable type p , such that p is generic for*

$(D, +)$ and also for $(D^*, \cdot)$, where D^* is the group of invertible elements of D . Moreover F is the field of fractions of D . F is algebraically closed.

Proof. Let n be the maximal weight of a stably dominated type of elements of F . Using Lemma 5.1, find a definable subgroup M of F^* with a stably dominated generic type p of weight n . Let D_0 be a subgroup of $(F, +)$ with generic type $p^{\pm 2n}$, generated by $p^{\pm 2}$ (additively). Then p clearly stabilizes D_0 multiplicatively. Hence $MD_0 \subseteq D_0$. Let $D = \{a \in F : aD_0 \subseteq D_0\}$. Then D is an ∞ -definable subring of F containing M . Fixing $0 \neq a_0 \in D_0$, we have an injection $D \rightarrow D_0, x \mapsto a_0x$.

Let F' be the field of fractions of D . Suppose $a \in F, a \notin F'$. Then $D + aD$ is definably isomorphic to D^2 , hence is metastable of weight $2n$; but this contradicts the weight bound n on stably dominated types of F . Thus $F = F'$ is the field of fractions of D .

Note also that a generic element of M is a unit, and every element is a sum of generics, so the units D^* generate D additively.

By Corollary 3.15, D is the intersection of definable subrings D_n .

Let N be the unique normal ∞ -definable subgroup of $(D, +)$ such that D/N is stable (cf. Proposition 4.6.) Then clearly N is multiplicatively invariant under the units of D . Thus N is actually an ideal of D , so D/N is a ring.

The image of D^* in D/N consists of invertible elements, so a generic element of D/N is a unit of D/N .

Ring of finite Morley rank. Modulo radical, a finite product of fields. If Artinian (bounded length of chains of f.g. ideals); Product of finitely many local rings. Local case: an additively generic element is invertible. Lift to obtain same for R . This gives statement.

Still why should every element of R have an n 'th root?—will be invertible in a higher ring.(?) $\square$

6. VALUED FIELDS: GENERICALLY METASTABLE SUBGROUPS OF ALGEBRAIC GROUPS

In this section, we work with definable groups in ACVF. Let K be an algebraically closed valued field; $\mathcal{O}$ denotes the valuation ring. Occasionally we will assume K to be saturated.

We repeat the statement of Lemma 1.7.

Lemma 6.1. *embed2 Let H be a definable or ∞ -definable group, with stably dominated generic type. Then there exists an algebraic group G and a definable homomorphism $f : H \rightarrow G$, with purely imaginary kernel.*

Proof. Let $M \models \text{ACVF}$, with H defined over M . Let p be the definable generic type of H ; let $(a_1, a_2) \models p^{\otimes 2} \upharpoonright M, a_3 = a_1a_2$.

Let $\tau = \{a_1, a_2, a_3, a_{12} = a_1a_2, a_{23} = a_2a_3, a_{123} = a_1a_2a_3\}$. We view this six-element set as a *matroid*, given by specifying the collection of algebraically dependent subsets of τ . This data is called a group configuration.

For $c \in \tau$, let $A(c) = \text{acl}(M(c)) \cap K$; this is the set of field elements in the algebraic closure of c . Pick a tuple $\alpha(c) \in A(c)$ such that $M(c) = M(\alpha(c))^{alg}$.

We view $c \mapsto \alpha(c)$ as a map on τ into the matroid of algebraic dependence in K over M . Then α preserves both dependence and independence. For independence this is clear. For dependence we need:

Claim 1. Let E_1, E_2 be two algebraically closed substructures of a model of ACVF, all sorts allowed. Let L_i be the set of field elements of E_i . If $c \in \text{acl}(E_1 \cup E_2)$ and $c \in K$, then $c \in (L_1L_2)^{alg}$.

Hence $\alpha(\tau)$ is isomorphic to τ . According to the group configuration theorem for stable theories, applied to the theory ACF, there exists an ACF-definable group G with parameters

in M , such that $a(\tau)$ is a group configuration for G ; in particular there exist $b_1, b_2 \in G$, $b_{12} = b_1 b_2$ such that $A(a_i) = M(b_i)^{alg}$. Since ACF-definable groups are (definably isomorphic to) algebraic groups, we can take G to be an algebraic group over M . Compare [7] (Proposition 3.1).

We work in the group $H \times G$. Let $c_i = (a_i, b_i)$, $q_i|_M = tp(c_i/M)$. By Proposition 2.9 (4), $q_i|_M$ is stably dominated, and so has a unique definable extension (denoted q_i). Let $C = Stab(q_2, q_3)$. This is a coset of the definable group $S = Stab(q_2, q_2)$; and q_1 is a generic type of C . (I.e. $q = (q_1)^{c_1^{-1}}$ is a generic type of S .) Let $J = \{h \in H : (h, 1) \in S\}$.

Claim 2. J is purely imaginary. *Proof*

If $h \in J$ then for $(a, b) \models q_2|_M(h)$, $(ha, b) \models q_2|_M$. Hence $h \in \text{acl}(a, ha, b)$. Now $tp(a/b)$ is purely imaginary, and so is $tp(ha/b)$. So $tp(h/b)$ is purely imaginary. But h, b are M -independent, so $tp(h/M)$ is purely imaginary.

Note that S projects onto H ; this is because the projection contains $a^{-1}a'$ for $(a, a') \models p_1$; in other words it contains a realization of $p_1|_M$; but p_1 generates H .

Thus J is a purely imaginary normal subgroup of H ; and S can be viewed as the graph of an injective homomorphism $s : H/J \rightarrow G$. $\square$

In place of Proposition 2.9 (4), we could have used the following general lemma:

Lemma 6.2. *def2* Let $M \leq N$ be models, and let $tp(a/N)$ be M -definable. Let $c \in \text{acl}(Ma)$. Then $tp(ac/N)$ is definable over M .

Proof. Indeed, $tp(a/N) \cup tp(ac/M) \models tp(ac/N)$. $\square$

Remark 6.3. *embedrem* Using Lemma 6.2, the proof of Lemma 6.1 goes through using only that H is a translation invariant symmetric definable type. However in our context this gains no generality, since in ACVF every translation invariant symmetric definable type is stably dominated.

The examples of § 6.14 show that some such hypothesis is necessary.

Without it, is it true that one can find an isogeny of H into a quotient of an Ind-algebraic group?

6.4. $\mathcal{O}$ -Varieties. By an $\mathcal{O}$ -variety (or variety over $\mathcal{O}$), we mean a flat, reduced scheme over $\text{Spec } \mathcal{O}$, admitting a finite open covering by schemes isomorphic to $\mathcal{O}[X_1, \dots, X_n]/I$. Since $\mathcal{O}$ is a valuation ring, $\text{Spec } R$ is flat over $\text{Spec } \mathcal{O}$ iff no nonzero element of $\mathcal{O}$ is a 0-divisor in R . So $\mathcal{O}[X_1, \dots, X_n]/I$ is flat iff $I = IK[X_1, \dots, X_n] \cap \mathcal{O}[X_1, \dots, X_n]$. Hence there are no infinite descending chains of $\mathcal{O}$ -subvarieties.

If V is an $\mathcal{O}$ -variety, we write $V(K)$ for $(V \otimes_{\text{Spec } \mathcal{O}} \text{Spec } (K))(K)$. Let $V_k = V \otimes_{\mathcal{O}} k$, and let $r : V(\mathcal{O}) \rightarrow V_k(k)$ be the natural map. By flatness, $\dim V_k = \dim V_K$.

For any set $Z \subseteq \mathcal{O}_K^n$, if $I = \{f \in \mathcal{O}[X_1, \dots, X_n] : f|_Z = 0\}$ and $R = \mathcal{O}[X_1, \dots, X_n]/I$, then $\text{Spec } R$ is flat over $\text{Spec } \mathcal{O}$. If $Z = V(K) \cap \mathcal{O}_K^n$ for some K -variety V , then $\text{Spec } R(\mathcal{O}) = Z$.

If W is a scheme over $\mathcal{O}$, and V' a subvariety of $W_K = W \otimes_{\mathcal{O}} K$, there exists a unique $\mathcal{O}$ -subvariety of W such that $V_K = V'$, and $V(\mathcal{O}) = V'(K) \cap W(\mathcal{O})$.

For example, if $V' \subset \mathbb{A}^n$ is an affine variety over K , defined by a radical ideal $P \subset \mathcal{O}[X]$, we let $V = \text{Spec } \mathcal{O}[X]/P \cap \mathcal{O}[X]$. Let $V_k \subset k^n$ denote the zero set of the image of $P \cap \mathcal{O}[X]$ in $k[X]$. In this case we denote the affine coordinate ring $K[X]/P$ by $K[V]$, and $\mathcal{O}[V] = \mathcal{O}[X]/(P \cap \mathcal{O}[X])$.

Lemma 6.5. *bdd* Let V be a scheme over $\mathcal{O}$, with $\dim V_K = n$. Let q be a K -definable type of elements of $V(\mathcal{O})$ with $r_* q$ of Morley rank $\geq n$. Assume V is defined over $B = \text{acl}(B)$; then so is q . In fact there are only finitely many q with this property.

Proof. Say q is defined over $B' \supset B$. Let $a \models q|B'$. Then $r(a) \downarrow_B B'$. $B(a)$ is a field extension of B of transcendence degree n , and also of residual transcendence degree n . (Indeed $\text{tr.deg.}_{B'} B'(r(a)) = n$, so $\text{tr.deg.}_B B(r(a)) \geq n$.) It follows that $a \downarrow_B B'$.

Since $\dim V_k \leq n$, the set of types in question is the set of q with r_*q a generic type of V_k . This is an elementary class. (If predicates are added to designate $(d_q x)\phi$ for all ϕ , then the set of q corresponds to the set of expansions to a certain partial theory.) Since it has boundedly many elements, it can only have finitely many. $\square$

Let G be a group scheme over $\mathcal{O}$, with generic fiber G_K and special fiber G_k . Then $G(\mathcal{O}) \subset G_K(K)$, and we have a definable group homomorphism $r : G(\mathcal{O}) \rightarrow G_k$.

Proposition 6.6. *genchar* Let p be a definable type of elements of $G(\mathcal{O})$. Assume $\dim G_K = \dim G_k = d$, and r_*p is a generic type of G_k . Then p is a generic type of $G(\mathcal{O})$.

When G is of finite type, $G(\mathcal{O})$ has finitely many generic types.

Proof. Consider translates $q =^g p$ of p , $g \in G(\mathcal{O})$. Clearly $r_*q =^{r(g)} r_*p$. By Lemma 6.5, q is defined over $\text{acl}(B)$, where B is a base of definition of G, p . So p is generic.

Since $G(\mathcal{O})$ has a stably dominated generic, all generics are translates of each other. Thus all generics q have r_*q generic. By Lemma 6.5 again, there are only finitely many generics. $\square$

We will see that all generically metastable ∞ -definable groups, with their generics, may be obtained this way; but Example 6.11 shows that we cannot take G to be of finite type.

Proposition 6.7. *genchar2* Let G be an algebraic group over K , H a Zariski dense definable subgroup of $G(K)$. Let p be a definable type of elements of $G(\mathcal{O})$. Then the following are equivalent:

- (1) p is the unique definable generic of H , and p is stably dominated.
- (2) For any open affine U of G and regular function f on U , p concentrates on U and attains the highest modulus of f on $H \cap U$; i.e for some γ_f , $\models (d_p x)(x \in U \& \text{val} f(x) = \gamma_f)$; and for any $x \in H \cap U$, $\text{val} f(x) \geq \gamma_f$.

Proof. (1) $\implies$ (2) : First we claim that p concentrates on any dense open affine of G . If p were contained in a subvariety U of G , then by genericity of p , finitely many translates of U cover H and hence, by Zariski density, all of G ; so $\dim(U) = \dim(G)$.

(Another argument: Pick some dense open U_0 such that p concentrates on U_0 . Then there exists a regular g on U_0 , vanishing on $U_0 \setminus U$. We will show below that p concentrates on the highest modulus of g ; in particular on $g(x) \neq 0$. Thus p concentrates on $U_0 \cap U$.)

Since p is stably dominated, for any regular f there exists γ_f with $(d_p x)(\text{val} f(x) = \gamma_f)$. If $(a, b) \models p \otimes p$ then $ab \models p$; so $\text{val} f(ab) = \gamma_f$. By Proposition 2.22, for any $a, b \models p$ we have $\text{val} f(ab) \geq \gamma_f$. But since p is the unique generic, any element c of H is a product of two realizations of p . Thus $\text{val} f(c) \geq \gamma_f$.

(2) $\implies$ (1): Note using quantifier elimination, that (2) characterizes p uniquely. Also, since γ_f does not depend on x , p is orthogonal to Γ , hence stably dominated by Proposition 2.12.

On the other hand, (2) is invariant under H -translations. Thus if (2) holds of p , it holds of every translate, so every translate equals p . By Remark 3.4, p is the unique generic type of H . $\square$

Note that connectedness of G follows from the assumptions (any function constant on connected components of G is regular.)

Let G be an algebraic group scheme over $\mathcal{O}$. $\text{res}G(\mathcal{O})$ is a definable, hence algebraic, subgroup of $(\text{res}G)(k)$. It may be a proper subgroup. Let $n = \dim(G_k)$. This in turn, when G is not of finite type, may be smaller than $\dim(G_K)$.

Proposition 6.8. *genchar2.1 Let G be an algebraic group scheme over $\mathcal{O}$. Assume:*

() if $f \in K[G]$ and $|f(x)| \leq 1$ for $x \models p$, then $f \in \mathcal{O}[G]$.*

*Let p be a definable type of elements of $G(\mathcal{O})$. Then p is the unique generic of $H = G(\mathcal{O})$ iff r_*p is the unique generic type of $\text{res } H$.*

Proof. In an algebraic group chunk, to show that a type is the unique generic is to show that any regular function f vanishing on the type vanishes on the whole chunk. If f over k vanishes on r_*p , lifting to $\mathcal{O}$ we obtain a regular function $F \in \mathcal{O}[G]$ with $\text{val}(F(a)) > 0$ for $a \models p$. By Proposition 6.7 (2), $\text{val}(F(a')) > 0$ for all $a' \in G(\mathcal{O})$. So F vanishes on $\text{res}(G(\mathcal{O}))$.

The converse uses (*). Let $F \in K[G]$. Since p is stably dominated, for some γ , for any $a \models p$, $|F(a)| = \gamma$. If $\gamma = 0$, then by assumption any K -multiple of F lies in $\mathcal{O}[G]$, so $F = 0 \in K[G]$. Otherwise, we may assume $\gamma = 1$. By assumption we may take $F \in \mathcal{O}[G]$. Suppose $|F(a')| = |c| > \gamma$, $a' \in G(\mathcal{O})$; (we may take $a' \in G(\mathcal{O}_0)$, a fixed submodel.) Then $c^{-1} \in \mathcal{O}$, and $|c^{-1}F(a)| < 1$ for $a \models p$, i.e. $\text{res}^{-1}F(a) = 0$. By Zariski density in G_k , $\text{res}^{-1}F$ vanishes on $\text{res}G(\mathcal{O})$; so $|c^{-1}F(a')| < 1$ for all $a' \in G(\mathcal{O})$; a contradiction. $\square$

Proposition 6.9. *genchar3 Let G be an affine pro-algebraic group over K . Let H be a Zariski dense ∞ -definable subgroup of G , with unique stably dominated generic type p . Then there exists a group scheme $\mathbb{H}$ over $\mathcal{O}$ and an isomorphism $\phi : G \rightarrow \mathbb{H}_K$, such that $\phi(H) = \mathbb{H}(\mathcal{O})$.*

Moreover, $\mathbb{H}$ is a pro-group variety over $\mathcal{O}$.

If H is definable, there exists an affine group variety H' over $\mathcal{O}$ such that $H \simeq H'(\mathcal{O})$.

Proof. p is Zariski dense in G (and G is connected) by Proposition 6.7.

G, H are defined over some subfield $K_0 = (K_0)^a$ of K . Let $R_0 := K_0[G]$ be the affine coordinate ring of G .

Define $R = \{f \in K_0[G] : (d_p x) \text{val} f(x) \geq 0\}$. This is an $\mathcal{O}$ -subalgebra of R_0 .

Claim 1. $R \otimes_{\mathcal{O}} K = R_0$.

Proof. Let $0 \neq r \in R_0$. Since p is stably dominated, it is orthogonal to Γ . Thus for some $c \in K_0$, for $a \models p|K_0$, $|r(a)| = |c|$. If $c = 0$, then r vanishes on p . Since p is Zariski dense in G , r vanishes on G , i.e. $r = 0 \in R_0$, contradicting the choice of r . So $c \neq 0$, and $c^{-1}r \in R$. This shows that the natural map $R \otimes_{\mathcal{O}} K \rightarrow R_0$ is surjective. Injectivity is clear since R has no $\mathcal{O}$ -torsion. $\square$ (Claim.)

Let $\mathbb{H} = \text{Spec } R$. So $\mathbb{H}_K := \mathbb{H} \times_{\text{Spec } \mathcal{O}} \text{Spec } K = \text{spec}(R \otimes_{\mathcal{O}} K) = G$. We identify $\mathbb{H}_K$ with G . So p is a type of elements of $G(K)$ and in fact, by definition of R , of $G(\mathcal{O})$.

The morphisms $x \mapsto x^{-1} : G \rightarrow G$ and $(x, y) \mapsto xy : G^2 \rightarrow G$ correspond to two operations:

$$i : R_0 \rightarrow R_0, i(r)(g) = r(g^{-1})$$

$$c : R_0 \rightarrow R_0 \otimes_{K_0} R_0, c(r) = \sum_{i=1}^n r_i \otimes s_j, r(gh) = \sum_{i=1}^n r_i(g) s_i(h)$$

Note that any $\mathcal{O}$ -subalgebra R' of R_0 is $\mathcal{O}$ -torsion-free, hence a flat $\mathcal{O}$ -module. Thus the maps $R' \otimes_{\mathcal{O}} R' \rightarrow R' \otimes_{\mathcal{O}} R \rightarrow R \otimes_{\mathcal{O}} R$ are injective. We identify $R' \otimes_{\mathcal{O}} R'$ with its image in $R \otimes_{\mathcal{O}} R$.

Let us say that an $\mathcal{O}$ -subalgebra R' of R_0 is *Hopf* if $i(R') \subseteq R'$ and $c(R') \subseteq R' \otimes_{\mathcal{O}} R'$.

Claim 2. R is Hopf.

Proof. : Co-inversion: Let $g = i(r), r \in R$. Clearly $\text{val}(g(x)) = \text{val}f(x^{-1}) \geq 0$ for $x \models p$. Hence $g \in R$.

Co-multiplication: Let $(a, b) \models p^2$, and $r \in R$. Write $r(ab) = \sum_j g_j(a) h_j(b)$ (a finite sum, with $g_j, h_j \in R_0$.) By [2] (Lemma 12.4; see Claim there), we may assume $|r(ab)| =$

$\max_j |g_j(a)| |h_j(b)|$. Note no g_j is zero; and then by renormalizing g_j and h_j , we may assume $|g_j(a)| = 1$ (as in the first Claim.) But then

$$\max_j |h_j(b)| = |r(ab)| \leq 1$$

Since both a and b realize $p|K_0$, for generic $a' \models p|K_0$ we have $|g_j(a')| = 1$, $|h_j(a')| = |c_j|$ with $|c_j| \leq 1$. As in Claim 1, we can take $c_j \in K_0$. Thus $g_j, h_j \in R$. $\square$ (Claim).

Claim 3. p is the unique generic type of $\mathbb{H}(\mathcal{O})$.

Proof. Proposition 6.7 (2) and the definition of R . $\square$

It follows that $H = \mathbb{H}(\mathcal{O})$. Let $\mathcal{F}$ be the family of finitely generated $\mathcal{O}$ -subalgebras of R that are Hopf. If $R' \in \mathcal{F}$ then $\text{Spec} R'$ is a group $\mathcal{O}$ -variety. To show that $\mathbb{H}$ is a pro-group variety over $\mathcal{O}$, it suffices to show that R is the direct limit (i.e. the union) of $\mathcal{F}$.

Note that if S is generated by R', R'' as an $\mathcal{O}$ -algebra, then then S is closed under c if R', R'' are. Indeed $c : R \rightarrow R \otimes_{\mathcal{O}} R$ is an $\mathcal{O}$ -algebra homomorphism, so $\{r : c(r) \in S \otimes_{\mathcal{O}} S\}$ is an $\mathcal{O}$ -subalgebra of S , hence equal to S since it contains R', R'' . Moreover i is an automorphism of R of order 2; if $c(R') \subseteq R' \otimes R'$, then the same holds for the $\mathcal{O}$ -algebra $i(R')$; since the $\mathcal{O}$ -algebra generated by $R' \cup i(R')$ is closed under i , it is Hopf. Hence it suffices, given $r \in R$, to find a finitely generated $\mathcal{O}$ -subalgebra R' of R with $r \in R'$ and $c(R') \subseteq R' \otimes R'$.

Claim 4. Let $r, a_i, b_i \in R$. If $(g, h) \models p^2$ and $r(gh) = \sum_{i=1}^n a_i(g)b_i(h)$, then $c(r) = \sum a_i \otimes b_i$.

Proof. Since p is Zariski dense in G , $p^{(2)}$ is Zariski dense in G^2 . So if $r(gh) = \sum_{i=1}^n a_i(g)b_i(h)$ for $(g, h) \models p^2$ then this holds for all $(g, h) \in G^2$. $\square$

By virtue of this claim, all elements x, y, g_j, h considered below can be taken to be independent realizations of p .

Fix $r \in R$.

Write $c(r) = \sum_{i=1}^n a_i \otimes b_i$, with n least possible, and $a_1, \dots, a_n, b_1, \dots, b_n \in R$. (Proof of Claim 2.)

The expression $r(xy) = \sum_{i=1}^n a_i(x)b_i(y)$ shows that $\{r(gy) : g \in G\}$ spans a finite-dimensional K -space. Similarly $\{r(gyh) : g, h \in G\}$ spans a finite-dimensional K -subspace V of $K[G]$. Note that $V \cap R$ is a lattice in V . Let R' be the $\mathcal{O}$ -algebra generated by $V \cap R$. We will show that $a_i, b_i \in R'$, so that $r \in A = \{x : c(x) \in R' \otimes_{\mathcal{O}} R'\}$. By construction, R' and hence A are left, right G -invariant. It follows that A contains V and hence R' . So R' is Hopf.

We saw in Claim 2 that the a_i, b_i lie in R . Thus it suffices to show that they lie in V . By symmetry, it suffices to show that $a_i \in V$.

Let $(g_1, \dots, g_n) \models p^{(n)}|K_0$. Then $r(xg_j) = \sum_i a_i(x)b_i(g_j)$.

Claim 5. The matrix $b = (b_i(g_j))_{1 \leq i, j \leq n}$ is invertible over K .

Proof. Suppose otherwise. Then there exists a nonzero vector $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha \cdot b = 0$. We may assume $a_n = 1$. Since $g_1, \dots, g_n$ are independent, some g_j must be independent from α (which has weight $n-1$). Now $\sum \alpha_i b_i(g) = 0$ for generic $g \models p|K_0(\alpha)$. By Claim (4), $\sum \alpha_i b_i = 0$; thus $b_1, \dots, b_n$ are K -linearly dependent, contradicting the minimality of n . $\square$

Now the $r(xg_j)$ are in V by definition. By Claim 5, $a_i(x) \in V$. $\square$

This gives Theorem 1.8.

Lemma 6.10. *zdense* Let $V \subset \mathbb{A}^n$ be an irreducible affine K -variety, and assume $V(\mathcal{O}) := V(K) \cap \mathcal{O}^n \neq \emptyset$. Then $V(\mathcal{O}) \cap \mathcal{O}^n$ is Zariski dense in V .

Proof. (A similar argument was given by Scanlon, in a slightly different context.) Let L be a large algebraically closed field extending K . Let $a \in V(L)$ be a K -generic point. Thus for any $b \in V(K)$ there exists a K -algebra homomorphism $h : K[a] \rightarrow K$ with $h(a) = b$. In particular, there exists such an h with $h(a) \in V(\mathcal{O})$. It follows that $\mathcal{M}_K$ generates a proper ideal of $\mathcal{O}_K[a]$. (Otherwise, for some $m \in \mathcal{M}_K$ and $f \in \mathcal{O}_K[X]$, $mf(a) = 1$; but then applying h we would have $mf(h(a)) = 1$.) Thus $\mathcal{M}_K$ extends to a maximal ideal M' of $\mathcal{O}_K[a]$, and thence to a maximal ideal M'' of some valuation ring $\mathcal{O}_L$ of L , with $\mathcal{O}_K[a] \subset \mathcal{O}_L$. Thus the valuation on K can be extended to L in such a way that $a \in \mathcal{O}(L)$. By model completeness of ACVF, there exists $a' \in V(K)$ outside any given proper K -subvariety of V , with coordinates in $\mathcal{O}$. $\square$

Example 6.11. *noninertial* An ∞ -definable generically metastable group that is not definable, not purely inertial, and not the connected component of a definable group.

Let $K_0 = \mathbb{C}(t)^a$, with $\text{val}(t) > 0$. Let $H_n = \{(x, y) \in G_a \times G_m : \text{val}(y - \sum_{i=1}^n 1/n!(tx)^n) \geq \text{val}(t^{n+1})\}$. H_n is a definable subgroup of $(G_a \times G_m)(\mathcal{O})$, isomorphic to $(G_a \times G_m)(\mathcal{O})$. $H = \cap H_n$ is also generically metastable; but it is dominated by the map $(x, y) \mapsto \text{res}(x)$. The Zariski closure of H has dimension 2, but the generic type of H has a residual part of transcendence degree one. $\square$

Lemma 6.12. *conn* Let G be a group scheme over $\mathcal{O}$. For each n , let $\phi_n(g) = g^n$, and assume $\phi_n : \phi_n^{-1}(V) \rightarrow V$ is a finite morphism. Then $G(\mathcal{O})$ is connected.

Proof. By properness, $\phi_n : G(\mathcal{O}) \rightarrow G(\mathcal{O})$ is surjective. So any finite quotient has order prime to n . This holds for all n , so $G(\mathcal{O})$ has no finite quotients. $\square$

Question 6.13. *in ACVF*, is G/G^0 always finite?

The answer is presumably yes (cf. Lemma 5.12. This would imply that Lemma 5.11 holds for non-Abelian groups too.

6.14. Examples: versions of tori. *versions-of-tori* The value group of an algebraically closed valued field is an ordered $\mathbb{Q}$ -vector space, admitting quantifier elimination. Every group definable over the value field has a chunk in common with the group $V = \Gamma^n$. Interesting variants of Γ^n are known. If Δ is a finitely generated subgroup of Γ^n , the convex hull $C(\Delta)$ of Δ (with respect to the product partial ordering of V) is an Ind-definable subgroup (a non-definable direct limit of definable sets). Δ is of course itself not-definable. But the quotient $C(\Delta)/\Delta$ is canonically isomorphic to a group interpretable over Γ . See [10].

These examples lift to the tori $T = G_m^n$ over algebraically closed valued fields. Let $r : T \rightarrow V$ be the homomorphism induced by the residue map. Let Δ be a finitely generated subgroup of T . Then $C(\Delta) = r^{-1}(C(r\Delta))$ is Ind-definable, and $C(\Delta)/\Delta$ is interpretable in ACVF. For example, fix $t \in K$ with $\text{val}(t) = \tau > 0$, and define a group structure on $A(t) = \{x : \text{val}(x)0 \leq \text{val}(x) \leq \tau\}$ by: $x * y = xy$ if $xy \in A(t)$, $x * y = xy/t$ otherwise. This group admits a homomorphism into a Γ -interpretable group, with generically metastable kernel.

6.15. Abelian varieties. If G be an Abelian scheme over $\mathcal{O}_K$ with good reduction, then by Lemma 6.5, Lemma 6.6, $G(K) = G(\mathcal{O}_K)$ is generically metastable. Since $G(K)$ is divisible, it has a unique definable generic. The case of bad reduction will be taken up elsewhere. In this case one has a nontrivial homomorphism $\phi : G(K) \rightarrow C$ where C is a definably compact group over the value group (such as the circle group, obtained by identifying the endpoints of an interval $[0, a]$.) The kernel of this homomorphism is generically metastable, via a homomorphism $g : \text{Ker}(\phi) \rightarrow A$, with A stable. Note for an Abelian variety defined over a field $F \leq K$, ϕ, g are also defined over F . If F is locally compact, then $\phi(G(F))$ will be a *finite* subgroup of the definably connected group C . After appropriate base change, A becomes isomorphic to a group

over the residue field. It is interesting to compare this with the classical theory of Neron models and semi-stable reduction.

6.16. Residually Abelian groups.

Example 6.17. *In ACVF, there exists a stably dominated non-Abelian group, with Abelian stable part.*

Proof. Let $A = \{G_a\}^2$, and let $\beta : A^2 \rightarrow G_a$ be a non-symmetric bilinear map; defined over the prime field. E.g. $\beta((a_1, a_2), (a'_1, a'_2)) = a_1 a'_2 - a_2 a'_1$. Let t be an element with $\text{val}(t) > 0$. Define on A^2 :

$$(a, b) \star (a', b') = (a + a', b + b' + t\beta(a, a'))$$

□

In fact, Simonetta found a C-minimal example:

Example 6.18. *[xmpl](#) If we work in $G_a(\mathcal{O}/t^2\mathcal{O})$, where $\text{val}(t) > 0$, we can also just take $a \star b = a + b + t\beta(a, b)$. cf. [13].*

The group in the above example does lift to an algebraic group structure on $\mathcal{O}$.

We may still ask if a G be connected metastable of weight 1 is nilpotent.

We believe that a definably simple group interpretable in ACVF is either finite or an algebraic group over the valued field or over the residue field. The existing metastable technology yields a proof of the Abelian case.

Proposition 6.19. *Let A be a nonzero Abelian group interpretable in ACVF. Then there exist definable subgroups $B \leq C \leq A$, with $B \neq C$, and C/B definably isomorphic (with parameters) to an algebraic group over the residue field or a definable group over the value group.*

Proof. Let $\lambda : A \rightarrow \Lambda$ be as in Theorem 5.9. If $\lambda(A) \neq 0$ we can take $C = A, B = \ker(\lambda)$. If $\lambda(A) = 0$, then A is limit-metastable. In particular A contains a nonzero generically metastable definable Abelian group C . C has a k -algebraic group as a nontrivial quotient.

□

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