1. Metrically approximate subgroups

Let G be a group with a metric d invariant under left and right translations. A (K, r)-*approximate subgroup* is a subset X of G containing 1, such that the product set XX is covered by at most K translates of XB_r, where B_r is the ball of radius r around 1. This notion is introduced by Terry Tao's in his blog entry [3] [1] In this note, we generalize to this setting some of the results of [6].The model-theoretic presentation uses additional assumptions; there may well be other routes. Here is the result under the strongest variant of the assumptions, a 'polynomial decay' condition on the entropy of X as a function of the scale. (See also Proposition 1.4, Proposition 1.5, Remark 1.6.)

Proposition 1.1. *Fix K, r, c. Then for some $M, m, m' \in \mathbb{N}$, the following statement holds for any metric group (G, d) as above. Assume X is a (K, r)-approximate subgroup of G, and in addition,*

$$\frac{N_r(X)}{N_{Mr}(X)} \leq M^c \tag{1}$$

then there exists Y with $1 \in Y = Y^{-1}$, $Y^k \subset (XX^{-1})^2 B_r$, and

$$N_r(Y) \geq \frac{1}{m'} N_r(X) \tag{2}$$

If we consider groups G of bounded exponent [2] *then we can also find $\widetilde{Y} \subset X^m$ with* (1) *and*

$$\widetilde{Y}^2 \subset \widetilde{Y} B_{mr} \tag{3}$$

so that $\widetilde{Y}$ is a $(1, mr)$-approximate subgroup of G.

Once the model-theoretic setting is in place, the proofs are straightforward transpositions of the previous ones. A few points of interest are nevertheless encountered:

1. Assume (K, r)-approximateness holds for a single K, but for a large number of scales r. In this case, we find again that a Lie group of dimension about $\log(K)$ is involved. But under somewhat weaker assumptions (Proposition 1.7), we encounter groups in the Ind-category of compact spaces, that are not locally compact. I do not know if any structure theory has been considered for such groups. Do they embed in pro-locally compact groups? Is it possible to have connected torsion groups in this category? If not, then Proposition 1.5 can be improved to include the same statement in the bounded exponent case as Lemma 1.4.
2. Under the assumption of Lemma 1.1 or Remark 1.6, the group is locally compact, involves a Lie group if it is not compact and I'd expect that a considerable further theory can be developed. The statements on the bounded torsion case are given here as tokens of this. Certainly analogues of partial Bourgain systems can also be found, but perhaps also nilpotence of the associated Lie group ([1]).
3. In the present setting, one disposes of an ideal resembling the ideal of measure zero sets, but no actual measure. Moreover at the level of generality of Proposition 1.7, the ideal is not $\bigwedge$-definable but only F_σ.

[1] In [3] the group is commutative.

[2] say $g^K = 1$, or just $d(1, g^K) < r_m$ for all $g \in G$

(4) The construction requires a seed of symmetry to begin with (the hypothesis (F) of Theorem 3.1). In [6], the same condition (F) was needed. One proof given there used Morley's omitting types theorem (and thus unncountably many cardinals); that proof would work here too. In [6], another, rather constructive construction was also given (Lemma 2.13 there). Immediately afterwards moreover, Sanders independently provided an entirely finite combinatorial proof. Here I was able to find Lemma 3.5 using $\aleph_1$, but at the moment I do not see a proof of Proposition 1.7 avoiding uncountable sets altogether.

(5) The Lie groups mentioned above are related only to K, i.e. to the need for a finite number of *translates*. No information is given on $(1,r)$-approximate groups. It becomes clear that the axis $(1,r)$ is a more or less autonomous world; one could call them quasi-subgroups, generalizing the graphs, or ranges, of quasi-morphisms. These include Turing's "Finite approximations to Lie groups" (Turing 1937), quasi-morpishms (cf. Burger-Iozzi-Wienhard, Annals 2010) , ϵ-representations (Kazhdan, IJM 1982). The counterexample to strong polynomial Freiman in [5] also fits here.

(6) It is likely that the two-sided invariance condition can be relaxed to a Lipshitz condition on translations. This much holds in a neighborhood of 1 in any real or p-adic Lie group. Moreover (1) is automatic in this case, so the hypotheses are valid for (K,r)-approximate subgroups of some neighborhood of 1. However in this setting, much more precise results were obtained by Nicolas de Saxce [10], generalizing a different method that applies in the linear case. I expect that even for $r=1$ or larger, one should have stronger results using the structure theory of Lie groups.

Consider triples (G,X,d), where G is a possibly non-commutative group G, $1 \in X = X^{-1} \subset G$, $d : G^2 \to \mathbb{R}$ is a metric, invariant under left and right translations. Let B_r denote the ball of radius r around 1.

I follow the notation and statements in [3]. In particular $N_r(E)$ is the minimum number of r-balls (not necessarily centered in E) needed to cover E. The metric entropy $N_r^{ent}(E)$ is the largest number of points one can find in E that are r-separated, i.e. pairwise of distance $\geq r$. We have:

$$N_{2r}^{ent}(E) \leq N_r^{pack}(E) \leq N_r(E) \leq N_r^{int}(E) \leq N_r^{ent}(E) \tag{4}$$

We will use the lemma below for $m=4$:

Lemma 1.2. *Fix $2 \leq m \in \mathbb{N}$, k > 0 and assume $N_r^{ent}(EE^{-1}E) \leq \kappa N_{(2m+1)r}^{ent}(E)$. Then:*

(1) For all $e \in E$, $N_r(B_{mr}e \cap E) \leq \kappa$

(2) For any $Y \subset E$, $N_r(Y) \leq \kappa N_{mr}(Y)$.

Proof. (1) Let Z be a $(2m+1)r$-separated subset of E with $|Z| = N_{(2m+1)r}^{ent}(E)$. For $z \in Z_r$, let $W_{r,z}$ be a maximal r-separated subset of $B_{mr}(z) \cap EE^{-1}E$. Then the $W_{r,z}$ are disjoint, and $\cup_{z\in Z} W_{r,z}$ is r-separated. Thus $|\cup_{z\in Z} W_{r,z}| \leq N_r^{ent}(EE^{-1}E)$. So

$$\frac{1}{|Z|}\sum_{z\in Z}|W_{r,z}| \leq \frac{N_r^{ent}(EE^{-1}E)}{N_{(2m+1)r}^{ent}(E)} \leq \kappa$$

So for some $a \in Z$ we have: $|W_{r,a}| \leq \kappa$. In particular, $B_{mr}(a) \cap EE^{-1}E$ can be covered by κ balls of radius r.

Now let $b \in E$ and let $f(x) = ab^{-1}x$. Then f maps $bB_{mr} \cap E$ isometrically into $aB_{mr} \cap EE^{-1}E$. Hence $bB_{mr} \cap E$ can be covered with κ balls of radius r, i.e. $N_r(B_{mr}b \cap E) \leq \kappa$.

(2) Y can be covered by $N_{mr}(Y)$ balls of radius mr, and each of these - when restricted to E - can be covered by κ balls of radius r; so Y can be covered by $\kappa N_{mr}(Y)$ balls of radius r.

□

Lemma 1.3. *Assume $N^{ent}_{r/2}(EE^{-1}E) \leq \kappa N^{ent}_{\frac{9r}{2}}(E)$. Let Z be a maximal r-separated subset of E. Then for any $Y \subset E$,*

$$N_r(Y) \leq |B_r Y \cap Z| \leq \kappa N_r(Y)$$

Proof. Any point of E, in particular of Y, has distance $< r$ from some point of Z, by maximality of Z. Hence $Y \subset \cup\{B_r(z) : z \in Z \cap B_r Y\}$, so Y is covered by $|Z \cap B_r Y|$ balls of radius r; by definition, $N_r(Y) \leq |Z \cap B_r Y|$. On the other hand, as Z is r-separated, no two points of Z lie in a single $r/2$-ball; hence

$$|Z \cap B_r Y| \leq N_{r/2}(Z \cap B_r Y)$$

By Lemma 1.2 (2) for $m = 4$, applied to the set $Z \cap B_r Y \subset E$, we have

$$N_{r/2}(Z \cap B_r Y) \leq \kappa N_{2r}(Z \cap B_r Y)$$

Now if a set of r-balls covers Y, and each is extended to a $2r$-ball, then the resulting set of $2r$-balls covers $B_r Y$. Thus $N_{2r}(B_r Y) \leq N_r(Y)$. A fortiori

$$N_{2r}(Z \cap B_r Y) \leq N_r(Y)$$

the lemma follows.

□

Let us assume that X resembles a metrically approximate subgroup not just at one scale r, but many. Say $r, r' > 0$ are *separated* if $r' > 2r$ or $r > 2r'$.

Proposition 1.4. *Fix K, and $k > 0$. Then for some $m \in \mathbb{N}$, for all triples (G, X, d), if there exist at least m separated values of $r > 0$ such that*

$$N_{r/8}(X^9) \leq K N_{9r/2}(X) < \infty$$

then there exists Y with $1 \in Y = Y^{-1}$, $Y^k \subset X^4 B_r$, and

$$N_r(Y) \geq \frac{1}{m} N_r(X) \tag{5}$$

holds for at least k among these scales r.

If we consider only triples (G, X, d) with G of bounded exponent, [3] *then we can also find $\widetilde{Y} \subset X^m$ with (5) and*

$$\widetilde{Y}^k \subset \widetilde{Y} B_{r_k} \tag{6}$$

For the first part of the statement, we can allow K to change with the scale.

Proposition 1.5. *Fix $K_1, K_2, \ldots$, and let $k > 0$. Then for some $m \in \mathbb{N}$, for all triples (G, X, d), and $r_m, \ldots, r_1$ with $2r_{i+1} < r_i$, if for each $i \leq m$ we have:*

$$N^{ent}_{r_i/2}(X^9) \leq K_i N^{ent}_{9r_i/2}(X) < \infty \tag{7}$$

then there exists $Y \subset X^4$ satisfying, for at least k values of i,

[3] or just $d(1, g^k) < r_m$ for all $g \in G$

$$(8) \qquad 1 \in Y = Y^{-1}, Y^k \subset X^4 B_{r_k},\ N_{r_i}(Y) \geq \frac{1}{m} N_{r_i}(X)$$

Remark 1.6. *In the bounded exponent case, the assertion* (6) *of Lemma 7 remains true under the assumptions of Lemma 1.5 if we further assume e.g. that* $X^{k+4}B_{r_i} \cap X^{k+16} \subset X^{k+8}$ *for* $k \leq m$*; see Lemma 1.11.*

The next remark proves Proposition 1.1.

Remark 1.7. Let $m = m(K,k)$ be as in Lemma 1.4. Assume X is (K,r)-approximate, and in addition, $N_r(X) \leq K^m N_{100^{2m}r}(X)$. Then the conclusion of Lemma 1.4 holds.

Proof. Indeed the hypothesis of Lemma 1.4 must hold for at least m of the $2m$ scales immediately above r, for K^2 in place of K. □

Example 1.8. *Let* $K_i = 5^{i+1}$. $r_1 > r_2 > \cdots > r_m > 0$ *be a descending sequence of positive reals, and* $0 < N_1 < N_2 < \cdots < N_M$ *a sequence of integers. Let* $G = \mathbb{R}$*, with the usual metric* d*, and*

$$X = X(r,N,m) = (\frac{\mathbb{Z}}{N_1} \cap [0,1])r_0 + (\frac{\mathbb{Z}}{N_2} \cap [0,1])r_1 + \cdots + (\frac{\mathbb{Z}}{N_m} \cap [0,1])r_m$$

Then X, G, d *fit the hypotheses of Lemma 1.5, but not of Lemma 1.4 or Lemma 1.7, or Corollary 1.11. The group obtained in the proof of Lemma 1.5 embeds into a pro-locally compact group, but is not locally compact.*

It would be interesting to know if non-pro-locally compact examples exist; this is related to the question of analyzing (K,r)-approximate subgroups at a single scale.

The proof of Lemma 1.5 will use the stabilizer theorem of [6], or Cor. 3.6 there. We need to apply it to a quotient of a definable group by an ∞-definable subgroup, namely the group of elements close to 1 (at an appropriate scale). Though originally formulated for definable groups, the extension to this case is routine. With the slight modifications needed, the stabilizer theorem and this corollary are attached below as Theorem 3.1 and Corollary 3.2.

1.9. **Proofs.** We prove Proposition 1.5. Suppose for contradiction that Lemma 1.5 is false. So $K_0, K_1, K_2, \ldots, k$ are fixed; and for each m we have a counterexample $(G(m), d(m), X(m), r_1(m), \ldots, r_m(m))$, with $2r_{i+1} < r_i$; i.e. there is no $Y \subset X(m)^4$ satisfying (8) for k values of i. Fix also a maximal r_i-separated subset $Z_i(m)$ of $X(m)^3$ (for $i \leq m$), and let $\nu_i(m)$ be the normalized counting measure on Z_i. Let

$$(G, d, X, r_1, r_2, \ldots, Z_1, Z_2, \ldots)$$

be an ultraproduct; here the r_i are nonstandard and d takes nonstandard values. (7) holds for all $i \in \mathbb{N}$. Note that $N_{r_i}(XXX)$ is defined as a nonstandard real (typically infinite, but not ∞.) On the other hand we will work with r_i only for $i \in \mathbb{N}$.

Let $\widetilde{G}$ be the group generated by X. Let

$$B_* = \cap_{k \in \mathbb{N}} B_{r_k} = \{g \in G : \bigwedge_{i \in \mathbb{N}} d(g,1) \leq r_i\}$$

B_* is an $\bigwedge$-definable subgroup of G; it is normalized by X by virtue of the invariance. So $B_* \cap \widetilde{G} \trianglelefteq \widetilde{G}$.

Let $o(1)$ denote the infinitesimal nonstandard reals, i.e. $|x| < 1/2, 1/3, \cdots$.

The ideal μ_r. Fix i for a moment, let $r = r_i, Z = Z_i, \nu = \nu_i$, so that (7) holds.

By (7), $N_r(X), N_{2r}(X), N_r(X^3)$ are of the same order of magnitude. Let

$$\varpi = \varpi_r = o(1)N_r(X) = o(1)N_{2r}(X) = o(1)N_r(X^3)$$

Define an ideal $\mu = \mu_r$ of definable subsets of G:

$$Y \in \mu \iff N_r(Y) \in \varpi$$

Since $N_r(Y)$ is 2-sided invariant, μ is 2-sided translation invariant. However, we will mostly be interested in subsets Y of X^3.

By Lemma 1.2 (2), $N_{r/2}(Y) \leq \kappa N_r(Y)$ for a standard integer κ. But clearly any covering of Y by $r/2$-balls yields (if these balls are inflated) a covering of $B_{r/2}Y$ by the same number of r-balls. Thus $N_{r/2}(Y), N_r(B_{r/2}Y), N_r(Y)$ differ by a bounded multiple. So: $Y \in \mu \iff N_{r/2}(Y) \in \varpi$, and

(9) $\qquad Y \in \mu \iff B_{r/2}Y \cap X^3 \in \mu$

Of course, this continues to hold if $r/2$ is replaced by any smaller positive number.

By (7) and Lemma 1.3, we have

(10) $\qquad$ For $Y \subset X^3,\ Y \in \mu \iff \nu(B_{r/2}Y \cap Z) = 0$

In particular, μ is invariant under automorphisms of $(G, \cdot, d, X, Z, \nu, r)$.

Note $X \notin \mu$. Also, (10) and the S1 property of ν immediately gives a metrically approximate version of S1 for μ: if $Y \subset X^3 \times P$, $Y(c) = \{y : (y, c) \in Y\}$ the section of c, and if (c_k) is an indiscernible sequence in P, and $Y(c_k) \notin \mu$, then

(11) $\qquad B_{r/2}Y(c_k) \cap B_{r/2}Y(c_l) \cap X^3 \notin \mu$ for $k < l$

The ideal μ_∞. We now restore the index i to the notation; the ideal μ_{r_i} above will be denoted μ_i.

Define an ideal μ_∞ on definable subsets of X^3 by:

$$Y \in \mu_\infty \iff \bigvee_{n \in \mathbb{N}} \bigwedge_{n \leq i \in \mathbb{N}} Y \in \mu_i$$

We extend μ_∞ to $\bigwedge$-definable sets by letting $\bigwedge_k E_k \in \mu_\infty$ iff some finite intersection of the E_k lies in μ_∞. This makes it a good ideal in the sense of § 2. Note μ_∞ is also invariant and left and right translations.

We let G act on products $G \times P$ by ignoring the P-coordinate, i.e. $g \cdot (h, y) = (gh, y)$. We say Y is B_*-invariant if $B_*Y = Y$.

Claim . μ_∞ has the S1-property for B_*-invariant subsets of B_*X^3: if $Y \subset B_*X^3 \times P$ is $\bigwedge$-definable, $B_*Y = Y$, and if $(c_1, c_2, \ldots)$ is an indiscernible sequence in P, and $Y(c_1) \cap X^3 \notin \mu_\infty$, then $Y(c_1) \cap Y(c_2) \cap X^3 \notin \mu_\infty$.

Proof. write $Y = \cap_j Y_j$ with Y_j definable; we may also arrange

(12) $\qquad B_{r_{j'}}Y_j \subset Y_{j-1}$

Where $j \mapsto j'$ is an increasing function. So for all j, $Y_j(c_1) \notin \mu_\infty$; thus there are arbitrarily large i with $Y_j(c_1) \notin \mu_i$. Fix j, and take $i > j'$ with $Y_j(c_1) \notin \mu_i$. By (11),

$$B_{r_i/2}Y_j(c_1) \cap B_{r_i/2}Y_j(c_2) \cap X^3 \notin \mu_i$$

Thus $B_{r_{j'}}Y_j(c_1) \cap B_{r_{j'}}Y_j(c_2) \cap X^3 \notin \mu_i$. By (12), $Y_{j-1}(c_1) \cap Y_{j-1}(c_2) \cap X^3 \notin \mu_i$. As this holds for infinitely many i, $Y_{j-1}(c_1) \cap Y_{j-1}(c_2) \cap X^3 \notin \mu_\infty$. Now that was proved for arbitary j, so $Y(c_1) \cap Y(c_2) \cap X^3 \notin \mu_\infty$. $\square$

Now let $\underline{G} = \widetilde{G}/(B_* \cap \widetilde{G})$, $\mathbf{h} : \widetilde{G} \to \underline{G}$ the quotient map. Then $\underline{G}$ is a piecewise hyperdefinable group (see § 2). Define μ on $\bigwedge$-definable subsets of $\underline{G}$: $Y \in \mu$ iff $\mathbf{h}^{-1}(Y) \in \mu_\infty$. Since μ_∞ is a good ideal, the same holds for μ; and by the above Claim, μ has the S1-property for subsets of $\underline{X}^3$. Let $M' \succ M$ be such that condition (F) of Theorem 3.1 holds. By Corollary 3.2, there exists an $\bigwedge$-definable over M' subgroup $\underline{S}$ of $\underline{G}$, with $\underline{G}/\underline{S}$ bounded. Moreover we may take $S = \mathbf{h}(W)$ where $W = (q^{-1}q)^2$.

Then $\mathbf{h}^{-1}\underline{S}$ is an $\bigwedge$-definable subgroup of $\widetilde{G}$, and $\mathbf{h}^{-1}\underline{S} \notin \mu_\infty$. As $(\mathbf{h}^{-1}\underline{S})^k = (\mathbf{h}^{-1}\underline{S}) \subset X^4B_\infty$, there exists a definable Y with $1 \in Y = Y^{-1}$, $Y^k \subset X^4B_{r_k}$, and $h^{-1}\underline{S} \subset Y$; so $Y \notin \mu_\infty$. Thus for infinitely many values of i, $Y \notin \mu_i$; choose k such values $i_1, \ldots, i_k$. For $i = i_1, \ldots, i_k$, we have $Y \notin \mu_i$, so for some m_0 we have $N_{r_i}(Y) \geq \frac{1}{m_0}N_{r_i}(X)$. But it follows that a set $Y(m)$ with the same properties exists in the factors $G(m)$ for infinitely many m; taking $m \geq m_0$, we have $N_{r_i}(Y(m)) \geq \frac{1}{m}N_{r_i}(X)$ for $i = i_1, \ldots, i_k$. This contradicts the initial assumption.

1.10. **Topologizing $\underline{G}/\underline{S}$.** The group $\underline{G}/\underline{S}$ is the union of subsets $\underline{X}^n$; each $\underline{X}^n$ is compact under the logic topology, where images of ∞-definable sets are closed. As $\underline{X}^n$ has bounded cardinality - it can be taken to be smaller than the saturation degree of the model - the topology is not sensitive to increasing the set of allowed parameters. For this reason it is also Hausdorff; and it follows that the product topology is the same as the logic topology on products.

One can topologize $\underline{G}/\underline{S}$ using the compactly generated topology, so that a set is open iff it has open intersection with $\underline{X}^n$ for each n. It is not clear that the resulting topology is Hausdorff; indeed Example 1.8 shows that $\underline{G}/\underline{S}$ need not be a locally compact group in general. However we do have:

Lemma 1.11. *The quotient $\underline{G}/\underline{S}$ of the proof of Lemma 1.5 is a locally compact topological group provided: for any k, for some k' and r_i, for all k'' we have $X^{k+4}B_{r_i} \cap X^{k''} \subset X^{k'}$,*

Proof. As $X^kB_{r_i} \cap X^{k''} \subset X^k$, we certainly have $X^kB_\infty \cap X^{k''} \subset X^k$. Now $(\mathbf{h}^{-1}\underline{S}) \subset X^4B_\infty$ so $X^k(\mathbf{h}^{-1}\underline{S}) \cap X^{k''} \subset X^k$, and

$$X^k \cap (\mathbf{h}^{-1}\underline{S})(X^{k''} \setminus X^{k'}) = \emptyset$$

Thus $\underline{X}^k \cap cl(\underline{X}^{k''} \setminus \underline{X}^{k'}) = \emptyset$, so $\underline{X}^k$ is contained in the interior of $\underline{X}^{k'}$ in $\underline{X}^{k''}$. This propagates: $\underline{X}^{k'}$ is contained in the interior of $\underline{X}^{k''}$ in $\underline{X}^{k'''}$; thus the interior of $\underline{X}^{k'}$ in $\underline{X}^{k''}$ and in $\underline{X}^{k'''}$ is the same; so $\underline{X}^k$ is contained in the interior of $\underline{X}^{k'}$ in $\underline{X}^{k'''}$, etc. Finally we see that $\underline{X}^k$ is contained in the interior of $\underline{X}^{k'}$ in $\underline{G}/\underline{S}$.

In particular, 1 lies in the the interior of $\underline{X}_{1'}$ in $\underline{G}/\underline{S}$. It follows that $\underline{G}/\underline{S}$ is locally compact. Similarly, if $1 \neq a \in \underline{G}/\underline{S}$ then $a \in int(\underline{X}^{k'})$ for some k; as $X^{k'}$ is Hausdorff it follows that $1, a$ are separated by open sets in $\underline{G}/\underline{S}$ is too. $\square$

Lemma 1.12. *Assume all K_i are equal (as in Lemma 1.4.) Then $\underline{G}/\underline{S}$ is a locally compact topological group.*

Proof. In the second case, we have a sequence r_i such that $N^{pack}_{r_i}(X^3) \leq K' N^{pack}_{r_i}(X)$. It follows that X^3 contains at most K' disjoint translates of $B_{r_i}X$; so it may be covered by K' translates of $X^2 B_{2r_i}$. Hence by compactness X^3 may be covered by K' translates of $X^2 B_*$. So $\underline{X}^3$ is covered by finitely many translates of $\underline{X}^2$; this propagates to $\underline{X}^n$. Cover $\underline{X}^n$ by finitely many translates of $\underline{X}^2$; those that meet $\underline{X}^k$ are contained in $\underline{X}^{k+4}$; thus $\underline{X}^n$ is covered by a finite union of compact sets not meeting $\underline{X}^k$, and by $\underline{X}^{k+4}$. So $\underline{X}^k$ lies in the interior of $\underline{X}^{k+4}$ in $\underline{X}^n$, and local compactness follows as above. □

Proof. of Lemma 1.4 It remains to prove the statement when G has bounded exponent. As in [6] 4.5, 4.16, the group $\underline{G}/\underline{S}$ (shown above to be locally compact) has a compact open subgroup. This subgroup, by definition of the topology on $\underline{G}/\underline{S}$, pulls back to an ∞-definable, co-∞-definable subset C of $\underline{X}^{m_1}$ for some m_1; and $\widetilde{Y} := \mathbf{h}^{-1}(C)$ is an ∞-definable, co-∞-definable and thus simply definable subset of $B_{r_1}X^{m_1}$. Since C is a subgroup and $\mathbf{h}$ a homomorphism we have $\mathbf{h}(\widetilde{Y}^k) = h(\widetilde{Y})^k \subset C$, so $\widetilde{Y}^k \subset \widetilde{Y}B_*$ and in particular $\widetilde{Y}^k \subset \widetilde{Y}B_{r_k}$. □

Remark 1.13. *The bounded torsion statement of Lemma 7 also holds in Lemma* **??***, under the assumption of Lemma 1.11 (up to X^m.)*

Remark 1.14. *To obtain a locally compact local group as in* [4]*, it would suffice in Lemma 1.11 to assume $X^4 B_{r_i} \cap X^{40} \subset X^{10}$.*

Remark 1.15. *Without the assumption of Lemma 1.11, in place of the compactly generated topology, it seems better to view $\underline{G}/\underline{S}$ as a group object in the Ind-category of compact spaces. The limit of the compact sets $[-n, n]^{\mathbb{N}}$, with the natural addition, is a simple example of a group of this type which is not locally compact.*

Remark 1.16.
1. In fact as in [6], in Lemma 1.4 we have a canonically associated Lie group with no compact normal subgroups, and the assumption of bounded exponent can be replaced by: Lie rank 0.
2. If the hypothesis of Lemma 1.4, holds at m consecutive scales, the ideals μ_r occurring in the proof will be all the same, so μ_* will be $\bigwedge$-definable.
3. In case $\underline{G}/\underline{S}$ is nilpotent, or becomes nilpotent after factoring out a compact normal subgroup, then in Lemma 1.5 one can find $Y \subset X^m$ satisfying (8) and such that the length k'- iterated commutator set $\widetilde{Y} := [Y, [Y, [\cdots, Y] \cdots]$ satisfies (6).
4. The group B_* should be $\bigwedge$-definable to use Cor. 3.6 of [6] for hyperdefinables; but there is no restriction on the complexity definition of the ideal (provided we don't mind a change of basis.)
5. With a little care it may be possible to relax right invariance to a uniform Lipshitz condition on right translations. Notably, in the proof of Lemma 1.5, translation invariance is used in order to achieve normality of the subgroup B_*, (at least within the group generated by X), and invariance (under X-translations) of the ideals μ_r. For both of these, a uniform Lipshitz condition on left and right translations by elements of X would suffice. The 9 could also be replaced by 3, with some adjustment of other parameters.

2. Appendix 1: Hyper-definable sets

We recall some basic definitions. We work in a sufficiently saturated and homogeneous structure $\mathbb{U}$; so that a 'small' intersection of definable subsets of $\mathbb{U}$ is nonempty, if every finite intersection is nonempty. For our present purposes 'small' can be taken to mean 'countable'. By an $\bigwedge$-definable set we mean a small intersection of definable sets.

We are given a definable set P_0 and an ∞-definable equivalence relation E on P_0. Let $\mathbf{h} : P_0 \to P := P_0/E$ be the quotient map. P is called a hyperimaginary set. A subset of P is called *$\bigwedge$-definable* if the pullback to P_0 is $\bigwedge$-definable. Note that the image of a definable subset of P is $\bigwedge$-definable. A product of hyperdefinable sets is viewed as hyperdefinable in the natural way.

We do not have a notion of a *definable* subset of P. But when $Q \subset P^m$ is $\bigwedge$-definable, we will call a function $f : Q \to P$ *definable* if the graph is ∞-definable. In this case there exists a definable $D \subset P_0^m$ and a definable function $F : D \to P_0$ whose projection is f. By a (complete) *type* over a base set M we mean a nonempty $M - \bigwedge -$ definable subset Y of P that is minimal under inclusion; thus for any $M - \bigwedge$-definable subset Z of P, either $Y \subset Z$ or $Y \cap Z = \emptyset$

By a *good ideal* on P (over A) we mean a family $\mathfrak{I}$ of ∞-definable subsets of P such that :

(1) $\mathfrak{I}$ is invariant under $Aut(\mathbb{U}/A)$.
(2) If $X, X' \in \mathfrak{I}$ and $Y \subset X \cup X'$ then $Y \in \mathfrak{I}$. Also $\mathfrak{I}$ is a proper ideal: $P \notin \mathfrak{I}$.
(3) If $X \notin \mathfrak{I}$, then there exists a complete type over $\mathbb{U}$ extending X, and including no element of $\mathfrak{I}$.

Definition 2.1. *A good ideal $\mathfrak{I}$ is S1 if whenever (a_i) is an indiscernible sequence (in some sort Q of the structure $\mathbb{U}$), $D \subset P \times Q$ is ∞-definable, and $D(a_i) \notin \mathfrak{I}$, then for some m, $D(a_i) \cap D(a_j) \notin \mathfrak{I}$.*

The ideals we will actually use have an additional property of *compactness*, namely, if $X = \cap_{i \in I} X_i$ is a small intersection, and $X \in \mathfrak{I}$, then for some finite $I_0 \subset I$ we have $\cap_{i \in I_0} \in \mathfrak{I}$.

When P is an ordinary sort carrying a measure μ, the ideal of measure zero $\bigwedge$-definable sets is not compact and does not satisfy (3), but the ideal generated by the definable measure zero sets is a good, compact ideal with S1.

Definition 2.2. *An $\bigwedge$ definable set X* divides *over M if X is defined over some (possibly infinite) tuple b, and there exists an indiscernible sequence $(b_i : i \in \mathbb{N})$ with corresponding sets X_i, $b_1 = b, X_i = X$, such that $\cap_{i \in \mathbb{N}} X_i = \emptyset$.*

The forking ideal $\mathfrak{f}_M$ is the ideal generated by the the ∞-definable subsets of P that divide over M.

Remark 2.3. *$\mathfrak{f}_M$ is a proper, M-invariant ideal. If X divides over M, then X clearly lies in any S1-ideal. Thus $\mathfrak{f}_M$ is contained in any S1-ideal.*

A *piecewise hyperdefinable group* is a strict Ind-object in the category of hyperdefinable sets. Explicitly, it is a group $G, \cdot, ^{-1}$ whose universe is $G = \cup_n G_n$,with each G_n hyperdefinable, and such that $G_n^{-1} = G_n$ and $G_n \cdot G_n \subset G_{n+1}$, and the graph of $\cdot$ restricted to $G_n^2 \times G_{n+1}$ is $\bigwedge$-definable. By an $\bigwedge$-definable subset of G, we mean such a subset of some G_n.

Key example: Consider a sufficiently satruated group H, say obtained as an ultraproduct, and B an $\bigwedge$-definable subgroup of H, i.e. B is an intersection $B = \cap_{n\in\mathbb{N}} B_n$ of definable sets, such that $B_n = B_n^{-1}$ and $B_{n+1}B_{n+1} \subseteq B_n$. Then H/B is a hyperdefinable group. An $\bigwedge-$ definable subgroup of H/B is just the image modulo B of an $\bigwedge$-definable subgroup of H.

If X is an $\bigwedge$-definable subset H, and B an $\bigwedge$-definable normal subgroup of H, we can also consider the subgroup of H/B generated by the image of X; then G is a piecewise hyperdefinable group. To define G, it suffices for B to be normalized by X rather than fully normal.

3. Appendix 2: The stabilizer theorem for hyperdefinable sets

We repeat here the stabilizer theorem of [6], noting that the proof carries over (routinely) for hyperdefinable groups. The text below is a lightly modified cut-and-paste from [6]. An easier proof was given for approximate equivalence relations as well, but that proof used measures whereas in [6] only the measure-zero *ideal* was used.

We are given a good ideal μ of $\bigwedge$-definable subsets of $\overline{G}$; An $\bigwedge$-definable subset of $\underline{G}$ is called *thin* if the image in $\overline{G}$ is in μ, *wide* otherwise.

We work over some base model M.

Theorem 3.1. *Let $\underline{G}$ be a piecewise hyperdefinable group, generated by a symmetric hyperdefinable set $\underline{X}$; μ is a good ideal, also invariant under left or right translations, with the S1 property on $\underline{X}^3$. Let $q \subset \underline{X}$ be a wide type over M. Assume:*

(F) There exist two realizations a, b of q such that $tp(b/Ma)$ does not fork over M and $tp(a/Mb)$ does not fork over M.

Then there exists a wide, M-$\bigwedge$-definable subgroup S of $\underline{G}$. We have $S = (q^{-1}q)^2$; the set $qq^{-1}q$ is a coset of S.

Moreover, S is normal in $\underline{G}$. It is the smallest $Aut(\mathbb{U}/M)$-invariant subgroup of $\underline{G}$ of small index.

Proof. We also write q to denote $\{a : tp(a/M) = q\}$; and $q^{-1} = \{a^{-1} : tp(a/M) = q\}$.

Given two subsets X, Y of $\underline{G}$, let

$$X \times_{nf} Y = \{(a, b) \in X \times Y : tp(b/M(a)) \text{ does not fork over } M\}$$

Let

$$Q = \{a^{-1}b : (a, b) \in q \times_{nf} q\}$$

$$Q' = \{a^{-1}b : a, b \in q, tp(b/Ma) \text{ is wide}\}$$

Note qq^{-1} is obviously wide by right-invariance, and similarly $q^{-1}q$ is wide assuming left-invariance.

Throughout this proof, we will use the fact that wideness of $qx \cap qy^{-1}$ is a stable relation between x and y. By Lemma 3.4, we have:

◇ *for any two types p_1, p_2, this relation holds for one pair $(a_1, a_2) \in p_1 \times_{div} p_2$ iff it holds for all pairs iff it holds for one or all pairs (a_2, a_1) in $p_2 \times_{div} p_1$.* Here $X \times_{div} Y = \{(a, b) \in X \times Y : tp(b/M(a)) \text{ does not divide over } M\}$.

Claim 1. $q^{-1}q \subseteq QQ$.

Proof. Let $a, b \in q$. Using (F), find $c \models q$ be such that $tp(a/Mc)$ does not fork over M, and $tp(c/Ma)$ does not fork over M. By extending $tp(c/Ma)$ to a type over $M(a,b)$ and realizing this type, we may assume $tp(c/Mab)$ does not fork over M. So we have $(b,c) \in q \times_{nf} q$, and $(c,a) \in q \times_{nf} q$. So $b^{-1}c, c^{-1}a \in Q$, hence $b^{-1}a \in QQ$. □

Claim 2. For all $(a,b) \in q \times_{nf} q$, $qa^{-1} \cap qb^{-1}$ is wide.

Proof. By $\diamond$, it suffices to show that for *some* $(a,b) \in q \times_{nf} q$, $qa^{-1} \cap qb^{-1}$ is wide. Let $a_1, a_2, \dots$ be an M- indiscernible sequence of elements of q, such that $tp(a_i/A \cup \{a_j : j < i\})$ does not fork over M. Then $(a_i, a_j) \in q \times_{nf} q$ for any $i < j$. It suffices to show that $qa_1^{-1} \cap qa_2^{-1}$ is wide. This is clear since μ is an S1-ideal, and by right-invariance, $qa_i^{-1} \notin I$. □

Claim 3'. For all $(c_1, c_2) \in (q^{-1}q) \times_{nf} Q'$, $qc_1^{-1} \cap qc_2^{-1}$ is wide.

Proof. Let $p_i = tp(c_i/M)$. As in Claim 2, it suffices to see that $qc_1^{-1} \cap qc_2^{-1}$ is wide for some $(c_1, c_2) \in p_1 \times_{nf} p_2$. Let $a_0 \models q$. Then there exists $a_1 \in q$ with $tp(a_0^{-1}a_1/M) = p_1$. Since $c_2 \in Q'$, there exists a_2' such that $r = tp(a_2'/M(a_0))$ is wide and $tp(a_0^{-1}a_2'/M) = p_2$; extend r to a wide type r' over $M(a_0, a_1)$, and let $a_2 \models r'$. We thus have $(a_0, a_1, a_2) \in (q \times q) \times_{nf} q$, with $tp(a_0^{-1}a_i/M) = p_i$ for $i = 1, 2$. Note also, using left invariance of μ, that $tp(a_0^{-1}a_2/M(a_0, a_1))$ is wide, hence so is $tp(a_0^{-1}a_2/M(a_0^{-1}a_1))$, so it does not fork over M.

By Claim 2 we have $qa_1^{-1} \cap qa_2^{-1}$ wide. By the right invariance of μ, $qa_1^{-1}a_0 \cap qa_2^{-1}a_0$ is wide. □

Claim 3. For all $(c,d) \in (q^{-1}q) \times_{nf} Q$, $qc^{-1} \cap qd^{-1}$ is wide.

Proof. Let $d = a^{-1}b$, with $tp(b/M(a))$ wide for the forking ideal over M. We have to show that $qc^{-1} \cap qb^{-1}a$ is wide. By $\diamond$, it suffices to show this for *one* instance (c, b, a) with $tp(b, a)$ specified and such that $tp(b, a/M(c))$ does not divide over M. We may thus take $tp(a/M(c))$ to be a nonforking extension of $q = tp(a/M)$, and $tp(b/M(a,c))$ to be a non-forking over M extension of $tp(b/M(a))$. The latter is possible using the assumption that $tp(b/M(a))$ does not fork over M.

By right-invariance, we need to show that $qc^{-1}a^{-1} \cap qb^{-1}$ is wide. We apply $\diamond$ to the pair (a,b) (viewed as a single tuple) and c. So it suffices to show that $qc^{-1}a^{-1} \cap q(b')^{-1}$ is wide, where $tp(b/M) = tp(b'/M)$ and $tp(b'/M(a,c))$ is wide. By left-invariance of μ, the type $tp(a^{-1}b'/M(a,c))$ is μ-wide, and hence $tp(a^{-1}b'/M(c))$ is μ-wide; so $tp(a^{-1}b'/M(c))$ does not fork over M. Also $tp(b'/M(a))$ is μ-wide, so $a^{-1}b' \in Q'$. By Claim 3', $qc^{-1} \cap q(a^{-1}b')^{-1}$ is wide. By right invariance, $qc^{-1}a^{-1} \cap q(b')^{-1}$ is wide, as required. □

Claim 4. Let $(b,a) \in Q \times_{nf} q^{-1}q$. Then $ab \in q^{-1}q$. In fact $qa \cap qb^{-1}$ is wide.

Proof. We have $a^{-1} \in q^{-1}q$. Since M is a model, $tp(a^{-1}/M)$ extends to a global type r finitely satisfiable type in M; so r is M-invariant. Use Lemma 3.4 (1), and Claim (3) to conclude that $qc^{-1} \cap qb^{-1}$ is wide if $c \models r|M(b)$. Now $tp(c/M(b))$ does not divide over M, so by $\diamond$, since $tp(a^{-1}/M(b))$ does not divide over M either, $qa \cap qb^{-1}$ is wide. In particular, for some $d, e \in q$ we have $da = eb^{-1}$. So $ab = d^{-1}e \in q^{-1}q$. □

Claim 5. Let $a \in q^{-1}q$, $b_1, \dots, b_n \in Q$ and assume $tp(a/M(b_1, \dots, b_n))$ is wide. Then $ab_1 \cdots b_n \in q^{-1}q$. In fact $qa \cap q(b_1 \cdots b_n)^{-1}$ is wide.

Proof. Since $tp(a/Mb_1)$ is wide, it does not fork over M (Remark 2.3). Hence by Claim 4 we have $ab_1 \in q^{-1}q$. By right-invariance of μ, $tp(ab_1/M(b_1, \dots, b_n))$ is wide, and in particular $tp(ab_1/M(b_2, \dots, b_n))$ is wide. By induction, $qab_1 \cap q(b_2 \cdots b_n)^{-1}$ is wide. Multiplying by b_1^{-1} on the right, $qa \cap q(b_1 b_2 \cdots b_n)^{-1}$ is wide. Hence as in Claim 4, $ab_1 \cdots b_n \in q^{-1}q$. ☐

In view of $\diamond$, Claim 5 is also valid assuming $tp(a/M)$ is wide, and $tp(a/M(b_1, \dots, b_n))$ does not fork over M. To show that $qq^{-1}q$ is a coset, we will later need a variant of Claim 5, proved in the same way:

Claim 5'. Let $a \in q^{-1}q$, $b_1, \dots, b_n \in Q$ and assume $tp(a^{-1}/M(b_1, \dots, b_n))$ is wide. Then $ab_1 \cdots b_n \in q^{-1}q$. In fact $qa \cap q(b_1 \cdots b_n)^{-1}$ is wide.

Proof. Since $tp(a^{-1}/Mb_1)$ is wide, it does not fork over M, and so $tp(a/Mb_1)$ does not fork over M. Hence by Claim 4 we have $ab_1 \in q^{-1}q$. By left-invariance of μ, $tp((ab_1)^{-1}/M(b_1, \dots, b_n))$ is wide, and in particular $tp((ab_1)^{-1}/M(b_2, \dots, b_n))$ is wide. By induction, $qab_1 \cap q(b_2 \cdots b_n)^{-1}$ is wide. Multiplying by b_1^{-1} on the right, $qa \cap q(b_1 b_2 \cdots b_n)^{-1}$ is wide. Hence as in Claim 4, $ab_1 \cdots b_n \in q^{-1}q$. ☐

Claim 6. $Q^n \subset q^{-1}qq^{-1}q$.

Proof. Let $b_1, \dots, b_n \in Q$. Let $a \in q^{-1}q$ with $tp(a/M(b_1, \dots, b_n))$ wide. Then $ab_1 \cdots b_n \in q^{-1}q$, so $b_1 \cdots b_n = a^{-1}(ab_1 \cdots b_n) \in q^{-1}qq^{-1}q$. ☐

It follows from Claim 1 that Q and $q^{-1}q$ generate the same subsemigroup, which is hence a group S. By Claim (6), this group is in fact equal to the $\bigwedge$-definable set $q^{-1}qq^{-1}q$.

Since $q^{-1}q \subseteq S$, we have $q \subseteq bS$ for any $b \in q$, and so $qq^{-1}q \subseteq bS$. Conversely, choose $b \in q$. Any element x of bS can be written $x = ba_1 \cdots a_4$ with $a_i \in Q$. Let $d \in q$ be such that $tp(d/M(a_1, \dots, a_4, b))$ is wide. Let $e = d^{-1}b$. Then $tp(e^{-1}/M(a_1, \cdots, a_4, b))$ and hence $tp(e^{-1}/M(a_1, \cdots, a_4))$ are wide. By Claim 5' we have $ea_1 \cdots a_4 \in q^{-1}q$. So $x = ba_1 \cdots a_4 \in dq^{-1}q \subset qq^{-1}q$. Thus $qq^{-1}q = bS$

S can have no proper $Aut(\mathbb{U}/M)$-invariant subgroups of bounded index. For suppose such a subgroup T exists. Let q' be an M-invariant type extending q. If $a, b \models q$, let $c \models q'|M(a, b)$. Then a, c must lie in the same coset of T (otherwise we may take any number of further realizations of q' to contradict the boundedness of T.) Similarly for b, c; so a, b lies in the same coset of T, i.e. all realizations of q lies in the same coset. But then $q^{-1}qq^{-1}q$, is contained in T; so $T = S$.

We know at this point that S is an $\bigwedge$-definable group over M, with no proper $\bigwedge$-definable over M (or even $Aut(\mathbb{U}/M)$-invariant) subgroups of bounded index. Let r be a type of elements of $X \cup X^{-1}$ over M. There cannot exist an unbounded family of cosets $a_i S$ with $a_i \in r$, for then the sets $a_i bq$ would also be disjoint for any $b \in q^{-1}$, contradicting the S1 property for μ within $rbX \subseteq (X \cup X^{-1})^3$. Thus r is contained in boundedly many left cosets of S. Now r extends to an M-invariant global type r'; r' must be contained in a single coset C_r of S; so C_r is M-definable, and hence the conjugate group $S^r = C_r^{-1} S C_r$ is M-definable.

For any $c \in X \cup X^{-1} \cup \{1\}$, $r = tp(c)$, the image of qc in G/S is bounded. Otherwise there is a large collection of disjoint sets of the form $a_i cS$, with $a_i \in q$. Pick $b_0 \in q$; then $q^{-1} b_0 \subseteq S$; the sets $a_i c S b_0^{-1}$ are also disjoint, hence so are the

a_icq^{-1}. But this contradicts the wideness of a_icq^{-1} and the S1 property within XcX^{-1}. Thus qc/S is bounded. It follows that q is contained in boundedly many cosets of $cSc^{-1} = S^r$. So q is contained in a single coset gS^r. It follows that $q^{-1}q \subseteq S^r$, so $S \subseteq S^r$. Similarly $S \subseteq S^{r^{-1}}$, so $S^r \subseteq S$ and $S^r = S$. This shows that $X \cup X^{-1}$ normalizes S, i.e. S is normal in $\underline{G}$.

We argued above that $q^{-1}q$ is wide; in particular S is wide. Also, there can be no large number of disjoint sets $aq^{-1}q$ with $a \in X$; in particular no large number of distinct cosets $aS, a \in \underline{X}$. Thus the image of $\underline{X}$ in $\underline{G}/\underline{S}$ is small. But $\underline{G}/\underline{S}$ is generated by the image of $\underline{X}$; so it is small, i.e. S has small index.

□

Corollary 3.2. *Let μ be a good ideal on $\underline{G}$, left, right - translation-invariant, and with the S1 property on subsets of $\underline{X}^3$. Assume M is $\aleph_1$- saturated. Then there exists wide, $\bigwedge$-definable over M subgroup S of G, with $\underline{G}/S$ bounded. For an appropriate complete type $q \subset \underline{X}$ over M we have $S = (q^{-1}q)^2$.*

Proof. Lemma 3.5 provides $M' \succ M$ and a type q over M' such that (F) holds, taking into account that μ contains the forking (over 0, hence over M') ideal. By Theorem 3.1 there exists an M'-$\bigwedge$-definable subgroup S' with $\underline{G}/S'$ bounded. We may find an M'-$\bigwedge$-definable subgroup S'' (containing S') with S'' defined over a countable set; by internalizing the parameters in M, we may also find such an $\bigwedge$-definable subgroup over M. □

Some remarks:

(1) (Locality). Inspection of the proof will show that for all assertions except the normality of S, we only use μ the S1 property only for subsets of $\underline{X}\underline{X}^{-1}\underline{X}$. To show normality S, we also require $\underline{X}a\underline{X}^{-1}$, where $a \in \underline{X}$ or $a \in \underline{X}^{-1}$. Moreover the group structure is used only up to $(\underline{X}^{-1}\underline{X})^3$. This is explicitly so everywhere except in Claim 5. There, note that $qc \subseteq \underline{X}\underline{X}^{-1}\underline{X}$. Hence $qc \cap Y \subseteq \underline{X}\underline{X}^{-1}\underline{X}$ for any set Y, and it makes sense to say that this intersection is wide. In the proof, by the time we use qab_1, we know that ab_1 is in $q^{-1}q$.
(2) The stronger statements on St_0 made in [6] probably go through as well.

Lemma 3.3. *Let I_z be an invariant S1-ideal. Let $P = P(x,z), Q = Q(y,z)$ be $\bigwedge$-definable sets. Define:*

$$R(a,b) \iff (P(a,z) \wedge Q(b,z)) \in I_z$$

Then R is a stable invariant relation.

Proof. We show indeed that R is equational: if $R(a_i,b_j)$ holds for $i < j$, where $(a_i,b_i)_i$ is indiscernible, then $R(a_i,b_i)$ holds too.

Otherwise, let $C_i = \{z : P(a_i,z) \wedge Q(b_i,z)\}$. Then $C_i \notin I_z$ but $\mu_z(C_i \cap C_j) = 0$. This contradicts the S1 property of I.

Since (a_{2i}, b_{2i-1}) is also indiscernible, by equationality we see that $R(a_{2i}, b_{2i-1})$ holds for each i, and so $R(a_i,b_j)$ holds for $i > j$ in general. □

Lemma 3.4. *Let $p(x)$ be a type over A, and $q(y)$ be a global, A-invariant type. Let R be a stable relation over A.*

(1) Assume $R(a,b)$ holds with $a \models p, b \models q|A(a)$. Then $R(a',b)$ holds whenever $a' \models p$ and $tp(a'/Ab)$ does not divide over A.

(2) Assume $tp(a/A) = tp(a'/A)$, $b \models q$, and neither $tp(a/Ab)$ nor $tp(a'/Ab)$ divides over A. Then $R(a,b)$ implies $R(a',b)$.

(3) Assume p too extends to a global, A-invariant type. Let $E = \{(a,b) : a \models p, b \models q|A\}$. Then the eight conditions:

$R(a,b)$ holds for some/all pairs $(a,b) \in E$ such that $tp(a/A(b))$ /$tp(b/A(a))$ does not fork / divide over A

are all equivalent.

Proof. (1) Suppose $R(a',b)$ fails to hold. Let $r = tp(a,b)$ and $r' = tp(a',b)$.

Define $a_1, \ldots, c_1, \ldots$ inductively: given $a_1 \ldots, a_{n-1}, c_1, \ldots, c_{n-1}$, choose c_n such that $c_n \models q|\{a_1, \ldots, a_{n-1}, c_1, \ldots, c_{n-1}\}$, and $a_n \models p$ chosen with $r'(x, c_i)$ for $i < n$. The latter choice is possible since by assumption, $p(x) \cup r'(x,y)$ does not divide over A. Then $r(a_i, c_j)$ holds if $i < j$, but $r'(a_i, c_j)$ holds when $i > j$. Hence $R(a_i, c_j)$ holds for $i < j$, but fails for $i > j$. This contradicts the stability of R.

(2) Let R' be the complement of R; it is also a stable relation. Let $c \models q|A(a)$. If $R(a,c)$ holds then by (1) we have $R(a',b)$ and $R(a,b)$. If $R(a,c)$ holds then similarly $R'(a',b)$ and $R'(a,b)$. In any case we have $R(a,b) \iff R(a',b)$, so the stated implication holds.

(3) Let E' be the set of pairs $(a,b) \in E$ such that $tp(a/A(b))$ does not divide over A, and E'' the set of pairs $(a,b) \in E$ such that $tp(b/A(a))$ does not divide over A. The equivalence between the four conditions for $tp(a/A(b))$ follows from (2): if $R(a,b)$ holds for some pair such that $tp(a/A(b))$ does not fork, then in particular it holds for a pair in E' (the same pair); by (2), it holds for al such pairs; hence certainly for all pairs for which $tp(a/A(b))$ does not fork over A.

Thus a single truth value for R is associated with pairs $(a,b) \in E'$. Similarly, as the conditions are symmetric, a single truth value for R is associated with pairs $(a,b) \in E''$. It remains to show that these truth values are equal. Replacing R be its complement if necessary, we may assume $R(a,b)$ holds in the situation of (1), where $b \models q|A(a)$. In particular $tp(b/A(a))$ does not fork over A; so $R(a',b')$ holds for all $(a',b') \in E''$. But (1) asserts that $R(a',b)$ holds for all $(a',b) \in E'$. Hence R holds for all pairs in $E' \cup E''$. □

Lemma 3.5. *Let L be a countable language. $I = I(x)$ be a proper ideal on a hyperdefinable set P, over A. Assume any I-wide type over a countable base set extends to an I- wide type over any bigger countable base set. There exists a model $M \geq A$, a type q over M, and $a,b \models q$ such that $tp(b/Ma)$ is I-wide and $tp(a/Mb)$ does not fork over M, and indeed is finitely satisfiable in M.*

Proof. We have $P = P_0/E$; we may pull back the ideal I to P_0; so we may assume here that P is definable. Let T_{sk} be a Skolemization of the theory, in a expansion L_{sk} of the language L; so the L_{sk}-substructure $< X >$ generated by a set X is an elementary submodel. Define a sequence of elements a_i ($i \leq \omega_1$), and sets $M_i =< \{a_j : j < i\} >$, p_i a type over A_i such that $p_i \subseteq p_j$ for $i < j$, $a_i \models p_i$, and with and with $tp_L(a_i/M_i)$ I-wide. Let $b = a_{\omega_1}$. Find i such that M_i is an elementary submodel of $\cup_{i<\omega_1} M_i$ in a language including a constant symbol for b. Then , $tp(a_i/M_i b)$ is finitely satisfiable in M_i, and we are done with $M = M_i, a = a_i$. □

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