RESEARCH ARTICLE

Asymmetric first-price auctions with uniform distributions: analytic solutions to the general case

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Abstract   In 1961, Vickrey posed the problem of finding an analytic solution to a first-price auction with two buyers having valuations uniformly distributed on $[\underline{v}_1, \overline{v}_1]$ and $[\underline{v}_2, \overline{v}_2]$. To date, only special cases of the problem have been solved. In this paper, we solve this general problem and in addition allow for the possibility of a binding minimum bid. Several interesting examples are presented, including a class where the two bid functions are linear.

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1 Introduction

In his 1961 paper, "Counterspeculation, Auctions, and Competitive Sealed Tenders," Vickrey introduced to the literature first-price auctions with incomplete information. He solved the symmetric case for the uniform distribution and attempted to solve the more general case of two asymmetric bids for the uniform case; however, he claimed that the problem "resists solution by analytical methods." He did offer a solution to the asymmetric environment where one bidder's value is commonly known. Later Griesmer et al. (1967) tackle the same problem and manage to solve for the case where the lower end of the two supports are the same. Although in the past 40 years since, research on auctions, including asymmetric auctions, has grown significantly, the general case with different lower ends of the supports has remained an open problem. In this paper, we complete this work and solve for the general uniform case with no restricting assumptions and add to it the possibility of a binding minimum bid. We accomplish this using standard mathematical techniques for solving differential equations.

Besides the mathematical challenge and historical significance in solving this problem, obtaining a general solution is important in its own right for several reasons. First, the uniform distribution plays a central role in Bayesian statistics ("in the veil of ignorance") in the same manner that the normal distribution plays a central role in econometrics. Furthermore, in view of the continuity result of Lebrun (2002), it is a reasonable approximation if the true distributions are not far from uniform.

Second, the assumption that the two lower ends of the supports are the same is substantial: it restricts the asymmetry between the buyers to one dimension, namely, one distribution being a "stretch" of the other while ignoring other aspects of strength like "shifts", two notions coined by Maskin and Riley (2000a). For example, if the distribution of one buyer's value is $U[0, 1]$, then the second buyer could be considered stronger for having $U[0, 2]$ or for having distribution $U[1/2, 1]$. In the first case (stretch), he is stronger in the sense that he may have higher values (in $[1, 2]$), while in the second case (shifts) he is stronger in the sense that he cannot have low values (in $[0, 1/2]$). As Maskin and Riley (2000a) observe, these two notions of strength may affect the equilibrium in different ways.

Likewise, a binding minimum bid is yet another dimension missing in the existing results. As we shall show, the presence of a minimum bid may qualitatively affect the equilibrium. For instance, the two equilibrium bid functions may intersect both at the minimum bid and at an internal point (see Example 1). This means that the bidding of the buyers cannot be ordered so that one is more "aggressive" than the other. Rather, one is more aggressive in a part of the common region of values while the other buyer is more aggressive in the other region. This is despite the fact that the 'effective' distribution of values (that is, starting from the minimum bid) are ordered by stochastic dominance. This is the first example of such a phenomenon that we are aware of. This example highlights the fact that while stochastic dominance of the distribution of values is a necessary condition for non-crossing of the bid functions (Kirkegaard 2009) it is not a sufficient condition. The sufficient condition for non-crossing is conditional stochastic dominance (see Maskin and Riley 2000a).

Finally, there are very few analytic solutions in the auction literature: the afore-mentioned Vickrey (1961) and Griesmer et al. (1967) papers. Plum (1992) covers the power distribution $F_1(x) = x^\mu$ and $F_2(x) = (\frac{x}{\beta})^\mu$ again with the restriction that the same lower end of the support of the two distributions. And more recently Cheng (2006) adds the case of $F_1(x) = x^\mu$ and $F_2(x) = (\frac{x}{\beta})^\gamma$ with even tighter restrictions on both the lower and top ends of the supports (in addition to starting at the same lower end, the top end is specified by $\beta = \frac{\gamma(\mu+1)}{\mu(\gamma+1)}$). These existing analytic solutions continue to serve as tools. For instance, Maskin and Riley (2000a) and Cantillon (2008) make use of the solutions provided by Griesmer et al. (1967) and Hafalir and Krishna (2009) compare the solution of first-price auctions with resale from Hafalir and Krishna (2008) to the analytical solutions to first-price auctions without resale provided by Plum (1992) and Cheng (2006). We expect that our additional analytic solutions will enhance this toolset.

The solutions we obtain on general $[\underline{v}_1, \overline{v}_1]$ and $[\underline{v}_2, \overline{v}_2]$ and minimum bid m are consistent with previously known solutions of auctions with uniform distributions. As we explain later, our solution also covers the general case of uniform distributions with atoms at the lower end of the interval. The mathematical expressions vary in the different regions of the parameters $\underline{v}_1, \overline{v}_1, \underline{v}_2, \overline{v}_2$ and m. While one change occurs when the minimum bid ceases to bind, surprisingly, we find another change occurs when $m = \max\{\underline{v}_1, \underline{v}_2\}$. Furthermore, as a function of the distributions, changes of the solution occur when a distribution shrinks to a single point (that buyer's value becomes commonly known) whereupon that buyer uses a mixed strategy in equilibrium. This case (without a minimum bid) was first analyzed by Vickrey (1961). Using Lebrun (2002, 2006) results, we prove that despite these changes the solutions are still continuous in the parameters and we verify this directly by taking the appropriate limit. A consequence of the continuity is that the profits are also continuous in the parameters.

Several interesting examples are presented, including a class where both bid functions are linear. In particular, given any minimum bid $m \geq 0$, there is a class of uniform distributions (where buyer 2's distribution stochastically dominates buyer 1's distribution) for which the bid functions $b_1(v) = \frac{v}{2} + \frac{m}{2}$ and $b_2(v) = \frac{v}{2} + \frac{m}{4}$ form the equilibrium; a nice generalization of the symmetric case without minimum bids where $b_1(v) = b_2(v) = \frac{v}{2}$. This provides a handy textbook example of asymmetric auctions with both linear equilibrium bidding and uniform distributions. Furthermore, we characterize the environments with uniform distributions that yield linear bid functions and provide a more general formula that becomes linear in those environments. By obtaining analytic solutions to an important class of asymmetric first-price auctions, we believe that our results will improve our understanding of auctions and serve as a useful tool for future research on auctions. We also hope that this paper will serve as a useful reference (as Bagnoli and Bergstrom 2005 provide for log-concavity and Baye et al. 1996, 2010 provide for all-pay auctions with complete information) and help or at least inspire others to find analytic solutions of other classes of auction models.

In Sect. 2, we describe the model and provide initial results about the equilibrium and boundary conditions. We then derive the differential equations resulting from

the first-order conditions of the equilibrium and appropriate boundary conditions. We show that this and the second-order conditions can be reduced to a single differential equation. In Sect. 3, we make use of these results to provide solutions that are distinct on the various regions of the parameters. In Sect. 4, we show that our solution is continuous in the parameters, which we also verify directly. Some examples are then provided in Sect. 5 along with a short discussion in Sect. 6. Several of the proofs are given in the Appendix.

2 The model and the equilibrium conditions

We consider a first-price, independent, private-value auction for an indivisible object with two buyers having two general uniform distributions: $U[\underline{v}_1, \overline{v}_1]$ for buyer 1 and $U[\underline{v}_2, \overline{v}_2]$ for buyer 2 (where $-\infty < \underline{v}_1, \overline{v}_1, \underline{v}_2, \overline{v}_2 < \infty$, as a uniform distribution has a bounded support). Without loss of generality, we assume that $\underline{v}_1 \leq \underline{v}_2$.

As usual, we are interested in the Bayes-Nash equilibrium of this game with incomplete information, that is, a pair of continuous and monotone bidding strategies that are best replies to each other, given the beliefs of the buyers about the values of the object.

We assume there is a finite minimum bid m (reserve price) such that all bids below m are ignored.[1] The fact that non-ignored bids are bounded from below implies that a buyer never wins when bidding less than $\underline{v}_1$ (the argument here is similar to the one made in Kaplan and Wettstein (2000).[2] In particular, in equilibrium, if $m \leq \underline{v}_1$, there is no bid b lower than $\underline{v}_1$. Consequently, without loss of generality, we shall consider minimum bids satisfying $m \geq \underline{v}_1$ since a minimum bid is not binding for $m < \underline{v}_1$. Also, we assume that (in equilibrium) a buyer with zero probability of winning never bids above his value (this includes any value below m).[3]

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[1] Kaplan and Wettstein (2000) and Baye and Morgan (2002) show that when $\underline{v}_1 = \overline{v}_1 = \underline{v}_2 = \overline{v}_2$ then without a lower bound on bids there exists a positive profit equilibrium with mixed-strategies (with no lower bound).

[2] The argument is by contradiction along the following lines. Assume that there is a minimum bid m and that there is a buyer bidding below $\underline{v}_1$ and winning with positive probability. From this it follows that both buyers must have strictly positive expected profits for all values, including $\underline{v}_1$. Denote b^* as the infimum of all winning bids in the equilibrium. Then, $m \leq b^* < \underline{v}_1$. By continuity of the bidding strategies, there must be a buyer bidding b^*. Also bidding b^* must have zero probability of winning since if not, the other buyer must be bidding b^* with strictly positive probability. Then, a slight increase of the bid would yield a discrete jump in probability of winning. Since the buyer bidding b^* has zero probability of winning, he also has zero expected profits, providing the contradiction.

[3] Without this assumption a bidder with value v, who in equilibrium has zero probability of winning, can sometimes bid more than his value. Formally, this could still be part of a Bayes–Nash equilibrium and have a different allocation than other Bayes–Nash equilibria. For example, if $m, \underline{v}_1, \overline{v}_1, \underline{v}_2, \overline{v}_2$ equals $0, 0, 1, 4, 5$ respectively, then without our assumption two possible equilibria are $b_1(v) = v, b_2(v) = 1$ and $b_1(v) = 2v, b_2(v) = 2$. (While only the winning price changed in this example, it is possible to create other examples where the allocation of the object also changes.) Note that these extraneous equilibria are equivalent to having a minimum bid at 1 and 2, respectively. More generally, each of the additional equilibria are equivalent to an equilibrium under our assumption with a minimum bid m (equivalent in allocation and strategies for values above m). Such equilibria can be eliminated, for example, by a trembling-hand argument, i.e., by assuming that each bidder i bids with positive density on $[\min_i \underline{v}_i, \max_i \overline{v}_i]$. While a bidder bidding below his value when he has zero probability of winning can also be supported in a

Notice that when $m \geq \min\{\overline{v}_1, \overline{v}_2\}$, the only equilibrium is the trivial equilibrium of one buyer placing a bid at m.[4] In addition, if $\underline{v}_2 \geq 2\overline{v}_1 - \underline{v}_1$, then any Nash equilibrium must have buyer 2 always bidding $\overline{v}_1$ (and hence always winning the object at price $\overline{v}_1$).[5] Such an equilibrium reflects what Maskin and Riley (2000a) refers to as the *Getty effect* where one buyer (the J. Paul Getty Museum) is so dominant that it always wins (in art auctions). This discussion yields the following lemma which specifies the non-trivial cases left to analyze.

Lemma 1 *The set of parameters $\underline{v}_1, \underline{v}_2, \overline{v}_1, \overline{v}_2$, and m for which non-trivial equilibria may exist is defined by the following constraints:*

(i) $\underline{v}_1 \leq \overline{v}_1$,
(ii) $\underline{v}_1 \leq \underline{v}_2 \leq \overline{v}_2$,
(iii) $\underline{v}_2 < 2\overline{v}_1 - \underline{v}_1$,
(iv) $m < \min\{\overline{v}_1, \overline{v}_2\}$.

When $\underline{v}_1 = \overline{v}_1$ and $\underline{v}_2 = \overline{v}_2$, the winning bid is at $\underline{v}_1$. The case when exclusively either $\underline{v}_1 = \overline{v}_1$ or $\underline{v}_2 = \overline{v}_2$ is treated in Sect. 3.4. When (i)–(iv) hold and $\underline{v}_1 < \overline{v}_1$ and $\underline{v}_2 < \overline{v}_2$, an equilibrium would consist of strictly monotone, differentiable bid functions $b_1(v)$ and $b_2(v)$. Denote the inverses of these bid functions as $v_1(b)$ with support $[\underline{b}_1, \overline{b}_1]$ and $v_2(b)$ with support $[\underline{b}_2, \overline{b}_2]$. In Lemma 4 of Appendix A.1, we prove that the closure of the set of equilibrium bids in which buyer i has a positive probability of winning is a subinterval of $[\underline{b}_i, \overline{b}_i]$ and is the same for both buyers. We denote this common interval of winning bids by $[\underline{b}, \overline{b}]$.

In the next section, we present the two differential equations whose solutions form an equilibrium. Since we will find a unique solution to the set of equations, it will be the unique solution in the class of monotonic and differentiable bid functions.[6] Furthermore, Griesmer et al. (1967) show that any non-trivial equilibrium must consist of monotonic and differential bid functions, at least without a minimum bid, and Lebrun (2006) shows uniqueness with a minimum bid, making our solution unique overall.[7]

Footnote 3 continued

Bayes–Nash equilibrium, the allocation is the same as the Bayes–Nash equilibrium where he bids his value. For simplicity, we may eliminate such equilibria.

[4] When $\underline{v}_1 = \overline{v}_1 = \underline{v}_2 = \overline{v}_2$, both buyers place a bid at m otherwise at most one buyer will place a bid at m.

[5] Let us denote v_1^* as the highest value of buyer 1 for which he wins with zero probability. If $v_1^* = \overline{v}_1$, then the equilibrium is as stated. If $v_1^* < \overline{v}_1$, then by our assumption buyer 1 bids his value for all $v < v_1^*$. Since buyer 1 wins with some probability for all $v > v_1^*$, then for some v_2 buyer 2 must bid v_1^* with positive probability. For this to be a part of an equilibrium, bidding v_1^* and winning with positive probability must have a profit at least as high as bidding $\overline{v}_1$, which would guarantee winning. Hence, we must have $\frac{v_1^* - \underline{v}_1}{\overline{v}_1 - \underline{v}_1}(v_2 - v_1^*) \geq (v_2 - \overline{v}_1)$. If $\underline{v}_2 \geq 2\overline{v}_1 - \underline{v}_1$ is satisfied, then the LHS is strictly increasing in v_1^* (for $v_1^* < \overline{v}_1$). However, when the $v_1^* = \overline{v}_1$, the LHS equals the RHS, providing a contradiction.

[6] See Lizzeri and Persico (2000) for a general proof of uniqueness and existence in auctions with interdependent values and minimum bids for this class of solutions with monotonic and differentiable bid functions.

[7] The uniqueness requires our assumption that (in equilibrium) a buyer with zero probability of winning bids his value (see Footnote 3).

2.1 The differential equations

In the interval $[\underline{b}, \overline{b}]$, the functions $v_1(b)$ and $v_2(b)$ must satisfy (by the first-order conditions of the maximization problems)

$$
\begin{aligned}
v_1'(b)(v_2(b) - b) &= v_1(b) - \underline{v}_1, \\
v_2'(b)(v_1(b) - b) &= v_2(b) - \underline{v}_2.
\end{aligned}
\tag{1}
$$

The boundary conditions as follows (proof in Appendix A.2).

B1 $v_1(\underline{b}) = \underline{b}$.
B2 $v_2(\underline{b}) = \max\{\underline{v}_2, m\}$.
B3 $v_1(\overline{b}) = \overline{v}_1$ and $v_2(\overline{b}) = \overline{v}_2$.
   Adding the equations in (1) together yields

$$
v_1'(b)v_2(b) + v_2'(b)v_1(b) = [(v_1(b) + v_2(b) - (\underline{v}_1 + \underline{v}_2))b]'.
$$

By integrating, we have

$$
v_1(b) \cdot v_2(b) = b(v_1(b) + v_2(b)) - (\underline{v}_1 + \underline{v}_2) \cdot b + c.
\tag{2}
$$

where c is the constant of integration.

Lemma 2 *In equilibrium,*

$$
\underline{b} = \max\left\{\frac{\underline{v}_1 + \underline{v}_2}{2}, m\right\}.
\tag{3}
$$

Proof The proof is in Appendix A.3.

Lemma 3 *The upper bound of the bid functions, $\overline{b}$, is given by*

$$
\overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - c}{(\overline{v}_1 - \underline{v}_1) + (\overline{v}_2 - \underline{v}_2)},
\tag{4}
$$

where

$$
c = \begin{cases}
\frac{(\underline{v}_1 + \underline{v}_2)^2}{4} & \text{if } \frac{\underline{v}_1 + \underline{v}_2}{2} \geq m, \\
(\underline{v}_1 + \underline{v}_2)m - m^2 & \text{otherwise.}
\end{cases}
\tag{5}
$$

Proof Substituting the lower boundary condition B1 into (2) yields

$$
v_2(\underline{b})\underline{b} = \underline{b}(v_2(\underline{b}) + \underline{b}) - (\underline{v}_1 + \underline{v}_2)\underline{b} + c.
$$

This simplifies to[8]

$$c = (\underline{v}_1 + \underline{v}_2)\underline{b} - \underline{b}^2.$$

From (3), we have (5) (note that c, as a function of m, reaches its maximum at $m = \frac{v_1 + v_2}{2}$). Using B3 and (2) we have

$$\overline{v}_1 \cdot \overline{v}_2 = \overline{b}(\overline{v}_1 + \overline{v}_2) - (\underline{v}_1 + \underline{v}_2) \cdot \overline{b} + c,$$

which yields (4).

2.1.1 Reduction to a single differential equation

We can use (2) to find $v_2(b)$ in terms of $v_1(b)$ as follows:

$$v_2(b) = \frac{b v_1(b) - (\underline{v}_1 + \underline{v}_2)b + c}{v_1(b) - b}. \tag{6}$$

We can then rewrite the differential equation (1) as

$$v_1'(b) \cdot \left(\frac{b v_1(b) - b(\underline{v}_1 + \underline{v}_2) + c}{v_1(b) - b} - b \right) = v_1(b) - \underline{v}_1$$

or

$$v_1'(b) \cdot (-b(\underline{v}_1 + \underline{v}_2) + c + b^2) = (v_1(b) - \underline{v}_1)(v_1(b) - b). \tag{7}$$

Equations (5) and (7) and boundary condition $v_1(\overline{b}) = \overline{v}_1$ are used to find a solution for $v_1(b)$. With the solution of $v_1(b)$, Eqs. (5) and (6) are then used to find $v_2(b)$. We note that using similar steps we can arrive at a symmetric equation to (7) with boundary condition $v_2(\overline{b}) = \overline{v}_2$; the difference being that any subscript representing buyer 1 is replaced with buyer 2 and vice-versa. For instance, $v_1(b)$, $v_1'(b)$, $\underline{v}_1$, and $\underline{v}_2$ are replaced by $v_2(b)$, $v_2'(b)$, $\underline{v}_2$, and $\underline{v}_1$, respectively. This implies that the solution will be symmetric in the same way.

Although the differential equation is derived from the first-order conditions, any solution to it also satisfies the second-order conditions and hence is an equilibrium bid function. For the uniform distribution without the minimum bid, the second-order conditions was proved by Griesmer et al. (1967). A simple more general proof that first-order conditions are sufficient for the second-order conditions is found in Wolfstetter (1996) and readily applies to our framework. In the following section, we find the equilibrium bid functions by solving the differential equations.

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[8] Notice that no where do we use boundary condition B2. Nonetheless, B2 will hold in our solutions. This is because it is, in fact, redundant and a consequence of the differential equations and boundary condition B1. Likewise, boundary condition B1 is implied by B2 and the differential equations.

3 Solutions

3.1 Auction without a binding minimum bid

The auction without a minimum bid has the same solution as an auction with a minimum bid m that satisfies $m \leq \frac{\underline{v}_1 + \underline{v}_2}{2}$.

Proposition 1 *When* $m \leq (\underline{v}_1 + \underline{v}_2)/2$, *the equilibrium inverse bid functions are given by*

$$v_1(b) = \underline{v}_1 + \frac{(\underline{v}_2 - \underline{v}_1)^2}{(\underline{v}_2 + \underline{v}_1 - 2b)c_1 e^{\frac{\underline{v}_2 - \underline{v}_1}{\underline{v}_2 + \underline{v}_1 - 2b}} + 4(\underline{v}_2 - b)}, \tag{8}$$

$$v_2(b) = \underline{v}_2 + \frac{(\underline{v}_2 - \underline{v}_1)^2}{(\underline{v}_1 + \underline{v}_2 - 2b)c_2 e^{\frac{\underline{v}_1 - \underline{v}_2}{\underline{v}_1 + \underline{v}_2 - 2b}} + 4(\underline{v}_1 - b)} \tag{9}$$

where

$$c_1 = \frac{\frac{(\underline{v}_2 - \underline{v}_1)^2}{\overline{v}_1 - \underline{v}_1} + 4(\overline{b} - \underline{v}_2)}{-2(\overline{b} - \underline{b})} e^{\frac{\underline{v}_2 - \underline{v}_1}{2(\overline{b} - \underline{b})}}, \tag{10}$$

$$c_2 = \frac{\frac{(\underline{v}_2 - \underline{v}_1)^2}{\overline{v}_2 - \underline{v}_2} + 4(\overline{b} - \underline{v}_1)}{-2(\overline{b} - \underline{b})} e^{\frac{\underline{v}_1 - \underline{v}_2}{2(\overline{b} - \underline{b})}} \tag{11}$$

and

$$\underline{b} = \frac{\underline{v}_1 + \underline{v}_2}{2}, \quad \overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - (\frac{\underline{v}_1 + \underline{v}_2}{2})^2}{(\overline{v}_1 - \underline{v}_1) + (\overline{v}_2 - \underline{v}_2)}. \tag{12}$$

Proof In solving differential equation (7), we first have (by (5) and (4)) $c = \frac{(\underline{v}_1 + \underline{v}_2)^2}{4}$ and

$$\overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - (\frac{\underline{v}_1 + \underline{v}_2}{2})^2}{(\overline{v}_1 - \underline{v}_1) + (\overline{v}_2 - \underline{v}_2)}.$$

Rewrite Eq. (7) as

$$v_1'(b) \cdot (\underline{v}_1 + \underline{v}_2 - 2b)^2 = 4(v_1(b) - \underline{v}_1)(v_1(b) - b).$$

Define now $\alpha \equiv \underline{v}_1 + \underline{v}_2 - 2\underline{v}_1 = \underline{v}_2 - \underline{v}_1$, $x \equiv b - \underline{v}_1$ and $D(x)$ such that

$$v_1(x + \underline{v}_1) = \frac{\alpha^2}{D(x)} + \underline{v}_1. \tag{13}$$

We then have $v_1'(x + \underline{v}_1) = -\frac{\alpha^2}{D(x)^2} D'(x)$, and Eq. (7) becomes

$$
\begin{aligned}
D'(x) \cdot (\alpha - 2x)^2 &= 4(D(x)x - \alpha^2), \\
D'(x) \cdot (\alpha - 2x)^2 &= 4D(x)x - 16x(\alpha - x) - 4(\alpha - 2x)^2, \\
(D'(x) + 4) \cdot (\alpha - 2x)^2 &= 4x(D(x) - 4(\alpha - x)),
\end{aligned}
$$

$$
\begin{aligned}
\frac{D'(x) + 4}{D(x) - 4(\alpha - x)} &= \frac{4x}{(\alpha - 2x)^2} \\
&= \frac{2\alpha}{(\alpha - 2x)^2} - \frac{2}{\alpha - 2x}.
\end{aligned}
$$

By integrating both sides, we obtain

$$
\ln(D(x) - 4(\alpha - x)) = \frac{\alpha}{\alpha - 2x} + \ln(\alpha - 2x) + \ln c_1,
$$

and taking the exponent of both sides yields

$$
\begin{aligned}
D(x) - 4(\alpha - x) &= (\alpha - 2x)c_1 e^{\frac{\alpha}{\alpha - 2x}}, \\
D(x) &= (\alpha - 2x)c_1 e^{\frac{\alpha}{\alpha - 2x}} + 4(\alpha - x).
\end{aligned}
\tag{14}
$$

The upper boundary condition $v_1(\bar{b}) = \bar{v}_1$ determines c_1. When $b = \bar{b}$, we have $x = \bar{x} \equiv \bar{b} - \underline{v}_1$. From our definition we have $D(\bar{x}) = \frac{\alpha^2}{\bar{v}_1 - \underline{v}_1}$. Hence the boundary condition becomes

$$
c_1 = \frac{\frac{\alpha^2}{\bar{v}_1 - \underline{v}_1} - 4(\alpha - (\bar{b} - \underline{v}_1))}{(\alpha - 2(\bar{b} - \underline{v}_1))} e^{-\frac{\alpha}{\alpha - 2(\bar{b} - \underline{v}_1)}},
$$

which can be rewritten as (recall that in this case $\underline{b} = \frac{\underline{v}_1 + \underline{v}_2}{2}$)

$$
c_1 = \frac{\frac{(\underline{v}_2 - \underline{v}_1)^2}{\bar{v}_1 - \underline{v}_1} + 4(\bar{b} - \underline{v}_2)}{-2(\bar{b} - \underline{b})} e^{\frac{\underline{v}_2 - \underline{v}_1}{2(\bar{b} - \underline{b})}}.
$$

Note that this depends only on the constants of the game $\underline{v}_i, \bar{v}_i$, since

$$
\bar{b} - \underline{v}_2 = \frac{\bar{v}_1 \cdot \bar{v}_2 - \frac{(\underline{v}_1 + \underline{v}_2)^2}{4}}{(\bar{v}_1 + \bar{v}_2) - (\underline{v}_1 + \underline{v}_2)} - \underline{v}_2
$$

and

$$
\bar{b} - \underline{b} = \frac{\bar{v}_1 \cdot \bar{v}_2 - \frac{(\underline{v}_1 + \underline{v}_2)^2}{4}}{(\bar{v}_1 + \bar{v}_2) - (\underline{v}_1 + \underline{v}_2)} - \frac{\underline{v}_1 + \underline{v}_2}{2}.
$$

Equations (8) and (10) are obtained from Eqs. (13) and (14) and the definitions of α, x. Finally, Eqs. (9) and (11) are obtained from Eqs. (8) and (10), respectively, by reversing the roles of $\underline{v}_1, \overline{v}_1$ with those of $\underline{v}_2, \overline{v}_2$.

Note that Eqs. (8)–(12) imply, as expected, that the solution is invariant under shifts by z_i to $[\underline{v}_i + z_i, \overline{v}_i + z_i]$.

3.2 Auction with a binding minimum bid

When the minimum bid is binding, as in the case where $m > (\underline{v}_1 + \underline{v}_2)/2$, Eq. (5) becomes $c = (\underline{v}_1 + \underline{v}_2)m - m^2$ and (4) becomes $\overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - (\underline{v}_1 + \underline{v}_2)m + m^2}{(\overline{v}_1 - \underline{v}_1) + (\overline{v}_2 - \underline{v}_2)}$. Also, since $\underline{v}_1 \leq \underline{v}_2$, we have $m > \underline{v}_1$. Now, we can rewrite the differential Eq. (7) as

$$v_1'(b) \cdot (b - m)(b + m - \underline{v}_1 - \underline{v}_2) = (v_1(b) - \underline{v}_1)(v_1(b) - b). \tag{15}$$

Notice that since $b \geq m$ and $2m > \underline{v}_1 + \underline{v}_2$, the coefficient of $v_1'(b)$ on the left-hand side of the above equation is positive. This leads to the following proposition:

Proposition 2 *The equilibrium inverse bid function for buyer* 1 *with minimum bid m such that* $m > (\underline{v}_1 + \underline{v}_2)/2$ *and* $m \neq \underline{v}_2$ *is given by*

$$v_1(b) = \underline{v}_1 + \frac{(m - \underline{v}_1)(m - \underline{v}_2)}{b - \underline{v}_2 - c_3(b - m)^\theta (b + m - \underline{v}_1 - \underline{v}_2)^{1-\theta}}, \tag{16}$$

where

$$c_3 \equiv \frac{(\overline{v}_2 - \underline{v}_2)}{(\overline{v}_1 - \underline{v}_1)} \frac{\left(\frac{\overline{v}_1 - m}{m - \underline{v}_1 + \overline{v}_2 - \underline{v}_2} \right)^{1-\theta}}{\left(\frac{\overline{v}_2 - m}{m - \underline{v}_2 + \overline{v}_1 - \underline{v}_1} \right)^{\theta}}. \tag{17}$$

and $\theta \equiv \frac{m - \underline{v}_1}{(m - \underline{v}_1) + (m - \underline{v}_2)}$. *Buyer 2's inverse bid function* $v_2(b)$ *is obtained from* $v_1(b)$ *by interchanging the roles of* $\underline{v}_1, \overline{v}_1$ *and* $\underline{v}_2, \overline{v}_2$. *We denote* c_4 *as the corresponding constant for* $v_2(b)$. *The bounds of the bid functions are* $\underline{b} = m$ *and* $\overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - (\underline{v}_1 + \underline{v}_2)m + m^2}{(\overline{v}_1 - \underline{v}_1) + (\overline{v}_2 - \underline{v}_2)}$.
Proof The derivation of this solution of Eq. (15) is given in Appendix A.4.

3.2.1 Special case when $\underline{v}_1 = \underline{v}_2 = 0$

Corollary 1 *The equilibrium inverse bid function for buyer* 1 *with minimum bid* $m > 0$ *and* $\underline{v}_1 = \underline{v}_2 = 0$ *is given by*

$$v_1(b) = \frac{m^2}{b - c_3 \sqrt{b^2 - m^2}}, \tag{18}$$

$$c_3 = \frac{\overline{v}_2}{\overline{v}_1} \left(\frac{\overline{v}_1^2 - m^2}{\overline{v}_2^2 - m^2} \right)^{1/2}. \tag{19}$$

Proof Substituting $\underline{v}_1 = \underline{v}_2 = 0$ into the solution, Eqs. (16) and (17) yields the result.

This is a special case that comes directly from substitution in our formula of Proposition 2.

We are not aware of this solution elsewhere for general $m > 0$. As a prelude to Sect. 4, we show that the limit of this solution converges to that in Griesmer et al. (1967) as $m \to 0$. Indeed, taking the limit as $m \to 0$ of $v_1(b)$, given by Eqs. (18) and (19), and applying L'Hopital's rule yields

$$v_1(b) = \frac{2b\overline{v}_1^2\overline{v}_2^2}{\overline{v}_1^2\overline{v}_2^2 + b^2(\overline{v}_2^2 - \overline{v}_1^2)}. \tag{20}$$

Reversing the roles of $\overline{v}_1$ and $\overline{v}_2$ gives us

$$v_2(b) = \frac{2b\overline{v}_1^2\overline{v}_2^2}{\overline{v}_1^2\overline{v}_2^2 - b^2(\overline{v}_2^2 - \overline{v}_1^2)}, \tag{21}$$

which is the result in Griesmer et al. (1967).

Furthermore, setting $\overline{v}_1 = \overline{v}_2 = 1$ in Eqs. (18) and (19) yields the symmetric case with a minimum bid:

$$v_1(b) = \frac{m^2}{b - c_3\sqrt{b^2 - m^2}},$$

$$c_3 = \frac{(1 - m)^{\frac{(m+1)}{(1-m)}}}{(m + 1)} = 1.$$

The limit as $m \to 0$ is $v_1(b) = 2b$, which agrees with the standard result for the symmetric case.

3.3 The case when $m = \underline{v}_2$

Looking at the solution for the case of a minimum bid, the expressions $(m - \underline{v}_1)$ and $(m - \underline{v}_2)$ appear in the denominator (in the constant). Since we are in the case where $m \geq (\underline{v}_1 + \underline{v}_2)/2$ and $\underline{v}_2 \geq \underline{v}_1$, we have $m = \underline{v}_1$ only when $\underline{v}_1 = \underline{v}_2 = m$, which reduces to the case of no minimum bid. We are thus left to provide a solution for the case with a minimum bid equal to $\underline{v}_2$. The reason for this transition at $m = \underline{v}_2$ is that boundary condition B2, $v_2(\underline{b}) = \max\{\underline{v}_2, m\}$, is at the border between $v_2(\underline{b}) = \underline{v}_2$ and $v_2(\underline{b}) = m$. It is interesting that the equations in Proposition 2 do double duty for both when $m < \underline{v}_2$ and $m > \underline{v}_2$.

Proposition 3 *The equilibrium inverse bid function for buyer* 1 *with minimum bid m such that* $m = \underline{v}_2$ *and* $\underline{v}_2 > \underline{v}_1$ *is given by*

$$v_1(b) = \underline{v}_1 + \frac{\underline{v}_2 - \underline{v}_1}{1 - \left(\frac{b-\underline{v}_2}{\underline{v}_2-\underline{v}_1}\right)\left[c_5 + \log\left(\frac{b-\underline{v}_1}{b-\underline{v}_2}\right)\right]} \tag{22}$$

where

$$c_5 \equiv \frac{(\overline{v}_1 - \underline{v}_2)(\underline{v}_2 - \underline{v}_1)}{(\overline{v}_1 - \underline{v}_1)(\overline{b} - \underline{v}_2)} - \log\left(\frac{\overline{b} - \underline{v}_1}{\overline{b} - \underline{v}_2}\right)$$

$$= \frac{(\overline{v}_1 - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)(\underline{v}_2 - \underline{v}_1)}{(\overline{v}_1 - \underline{v}_1)(\overline{v}_2 - \underline{v}_2)} - \log\left(\frac{(\overline{v}_2 - \underline{v}_1)(\overline{v}_1 - \underline{v}_1)}{(\overline{v}_2 - \underline{v}_2)(\overline{v}_1 - \underline{v}_2)}\right). \quad (23)$$

Again, the buyer 2's function $v_2(b)$ is obtained from $v_1(b)$ by interchanging the roles of $\underline{v}_1, \overline{v}_1$ and $\underline{v}_2, \overline{v}_2$. The bounds of the bid functions are $\underline{b} = m$ and $\overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - \underline{v}_1 \cdot \underline{v}_2}{(\overline{v}_1 - \underline{v}_1) + (\overline{v}_2 - \underline{v}_2)}$.

Proof See the Appendix A.5.

3.4 The case where one valuation is commonly known

The case where $\underline{v}_1 = \overline{v}_1 \equiv v_1$ or $\underline{v}_2 = \overline{v}_2 \equiv v_2$, namely, the situation in which the value of at least one of the two buyers is common knowledge cannot be obtained directly from Eqs. (8) and (9). The case when there is no minimum bid was treated in Appendix 3 of Vickrey (1961) (see also Kaplan and Zamir 2000; Martínez-Pardina 2006). For simplicity, we normalize this situation to $[\underline{v}_1, \overline{v}_1] = [0, 1]$ and $\underline{v}_2 = \overline{v}_2 = \beta$ where $0 < \beta < 2$ (when $\beta > 2$, the equilibrium is that buyer 2 bids 1 and wins with certainty).[9]

For this situation, Vickrey found that in the equilibrium of the first-price auction, buyer 1's inverse bid function is

$$v_1(b) = \frac{\beta^2}{4(\beta - b)}, \quad (24)$$

while buyer 2, whose value is known to be β, uses a mixed strategy given by the following cumulative probability distribution (with support from $\underline{b} = \frac{\beta}{2}$ to $\overline{b} = \beta - \frac{\beta^2}{4}$) :

$$F(b) = \frac{(2-\beta)\beta}{2(2b-\beta)} e^{-\frac{\beta}{2b-\beta} - \frac{2}{\beta-2}}. \quad (25)$$

The case with a binding minimum bid

In the presence of a binding minimum bid, the counterpart of the previous result is the following.

Proposition 4 *When $\beta/2 \le m < \min\{1, \beta\}$, the unique equilibrium when buyer 2's value is known is where buyer 1 has inverse bid function $v_1(b)$ and buyer 2 bids with a mixed-strategy given by the cumulative distribution $G(b)$, with support $\underline{b} = m$ to $\overline{b} = \beta - m(\beta - m)$, described by*

[9] When buyer 1's value is commonly known, the equilibrium is that buyer 2 wins the auction at buyer 1's value and buyer 1 uses a mixed strategy. (Thus, we relax the assumption that buyer 1 bids his value when his probability of winning is zero).

$$v_1(b) = \frac{m(\beta - m)}{\beta - b}$$

and

$$G(b) = \frac{((1-m)(\beta - m))^{\frac{m-\beta}{2m-\beta}} (m(m - \beta + 1))^{\frac{m}{2m-\beta}}}{(b - m)^{\frac{m-\beta}{2m-\beta}} (b + m - \beta)^{\frac{m}{2m-\beta}}}. \qquad (26)$$

Proof Since buyer 2 bids a mixed strategy, he must be indifferent to every point in his support including the minimum bid m (which is in the support if the minimum bid is binding). The following formula represents this.

$$v_1(b)(\beta - b) = m(\beta - m).$$

Hence, we have

$$v_1(b) = \frac{m(\beta - m)}{\beta - b}. \qquad (27)$$

If the cumulative distribution of buyer 2's mixed strategy is $G(b)$, then the first-order conditions of buyer 1 yields

$$G'(b)(v_1(b) - b) = G(b).$$

By rewriting this equation and substituting for $v_1(b)$ using Eq. (27), we have

$$\frac{G'(b)}{G(b)} = \frac{1}{v_1(b) - b} = \frac{\beta - b}{m(\beta - m) - b(\beta - b)}.$$

We can solve this differential equation through the following steps:

$$\int \frac{G'(b)}{G(b)} db = \int \frac{\beta - b}{m(\beta - m) - b(\beta - b)} db = \int \frac{\beta - b}{(b + m - \beta)(b - m)} db,$$

$$\ln(G(b)) = \frac{(\beta - m)\ln(b - m) - m\ln(b + m - \beta)}{2m - \beta} + \ln(C), \qquad (28)$$

$$G(b) = C(b - m)^{\frac{\beta - m}{2m - \beta}} (b + m - \beta)^{-\frac{m}{2m - \beta}}.$$

Note that for $\bar{b}$, we have

$$v_1(\bar{b})(\beta - \bar{b}) = m(\beta - m) = (\beta - \bar{b}),$$

which implies

$$\bar{b} = \beta - m(\beta - m).$$

This determines C by using the equality $G(\bar{b}) = 1$:

$$1 = C(\bar{b} - m)^{\frac{\beta - m}{2m - \beta}} (\bar{b} + m - \beta)^{-\frac{m}{2m - \beta}}.$$

Rewriting this gives us the expression for C:

$$C = ((1 - m)(\beta - m))^{-\frac{\beta - m}{2m - \beta}} (m(m + 1 - \beta))^{\frac{m}{2m - \beta}}.$$

Substituting this value of C into (28) yields (26).

We note that when $m \to \beta(<1)$, the solution approaches the equilibrium that buyer 2 stays out of the auction and buyer 1 wins the auction (for all values above m). Also, it can be shown that when $m \to \beta/2$, this goes to the solution in Sect. 3.4 (when the minimum bid is not binding).

3.5 Summary of the solution

– When $\underline{v}_2 = \bar{v}_2 \equiv \beta$ and for convenience $[\underline{v}_1, \bar{v}_1] = [0, 1]$,
  ○ $m \leq \frac{\beta}{2}$ (Eqs. (24) and (25))

$$v_1(b) = \frac{\beta^2}{4(\beta - b)},$$

while buyer 2 uses a mixed-strategy (with support from $\underline{b} = \frac{\beta}{2}$ to $\bar{b} = \beta - \frac{\beta^2}{4}$)

$$F(b) = \frac{(2 - \beta)\beta}{2(2b - \beta)} e^{-\frac{\beta}{2b - \beta} - \frac{2}{\beta - 2}}.$$

  ○ $m > \frac{\beta}{2}$ (Proposition 4)

$$v_1(b) = \frac{m(\beta - m)}{\beta - b},$$

$$G(b) = \frac{((1 - m)(\beta - m))^{\frac{m - \beta}{2m - \beta}} (m(m - \beta + 1))^{\frac{m}{2m - \beta}}}{(b - m)^{\frac{m - \beta}{2m - \beta}} (b + m - \beta)^{\frac{m}{2m - \beta}}},$$

on support from $\underline{b} = m$ to $\bar{b} = \beta - m(\beta - m)$.
– When $\bar{v}_2 > \underline{v}_2$,
  ○ $m \leq \frac{\underline{v}_1 + \underline{v}_2}{2}$ (Proposition 1)

$$v_1(b) = \underline{v}_1 + \frac{1}{\ell_1(b)e^{\frac{1}{\ell_2(b)}} + \ell_3(b)},$$

○ $m > \frac{\underline{v}_1 + \underline{v}_2}{2}$ and $m \neq \underline{v}_2$ (Proposition 2)

$$v_1(b) = \underline{v}_1 + \frac{1}{\ell_4(b) - \ell_5(b)^\theta \cdot \ell_6(b)^{1-\theta}},$$

○ $m = \underline{v}_2 > \underline{v}_1$ (Proposition 3)

$$v_1(b) = \underline{v}_1 + \frac{\underline{v}_2 - \underline{v}_1}{1 - \left(\frac{b - \underline{v}_2}{\underline{v}_2 - \underline{v}_1}\right)\left[c_5 + \log\left(\frac{b - \underline{v}_1}{b - \underline{v}_2}\right)\right]},$$

where $\ell_1, \ldots, \ell_6$ are linear functions of b and θ, c_1, c_3, c_5 are constants. These are defined as follows:

$$\ell_1(b) \equiv \frac{(\underline{v}_2 + \underline{v}_1 - 2b)c_1}{(\underline{v}_2 - \underline{v}_1)^2},$$

$$\ell_2(b) \equiv \frac{\underline{v}_2 + \underline{v}_1 - 2b}{\underline{v}_2 - \underline{v}_1},$$

$$\ell_3(b) \equiv \frac{4(\underline{v}_2 - b)}{(\underline{v}_2 - \underline{v}_1)^2},$$

$$\ell_4(b) \equiv \frac{b - \underline{v}_2}{(m - \underline{v}_1)(m - \underline{v}_2)},$$

$$\ell_5(b) \equiv \frac{(b - m)c_3}{(m - \underline{v}_1)(m - \underline{v}_2)},$$

$$\ell_6(b) \equiv \frac{(b + m - \underline{v}_1 - \underline{v}_2)c_3}{(m - \underline{v}_1)(m - \underline{v}_2)},$$

$$\theta \equiv \frac{m - \underline{v}_1}{(m - \underline{v}_1) + (m - \underline{v}_2)},$$

$$c_1 \equiv \frac{\frac{(\underline{v}_2 - \underline{v}_1)^2}{\overline{v}_1 - \underline{v}_1} + 4(\overline{b} - \underline{v}_2)}{-2(\overline{b} - \underline{b})} e^{\frac{\underline{v}_2 - \underline{v}_1}{2(\overline{b} - \underline{b})}},$$

$$c_3 \equiv \frac{(\overline{v}_2 - \underline{v}_2)}{(\overline{v}_1 - \underline{v}_1)} \frac{\left(\frac{\overline{v}_1 - m}{m - \underline{v}_1 + \overline{v}_2 - \underline{v}_2}\right)^{1-\theta}}{\left(\frac{\overline{v}_2 - m}{m - \underline{v}_2 + \overline{v}_1 - \underline{v}_1}\right)^\theta},$$

$$c_5 \equiv \frac{(\overline{v}_1 - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)(\underline{v}_2 - \underline{v}_1)}{(\overline{v}_1 - \underline{v}_1)(\overline{v}_2 - \underline{v}_2)} - \log\left(\frac{(\overline{v}_2 - \underline{v}_1)(\overline{v}_1 - \underline{v}_1)}{(\overline{v}_2 - \underline{v}_2)(\overline{v}_1 - \underline{v}_1)}\right).$$

Buyer 2's inverse bid function $v_2(b)$ is obtained from $v_1(b)$ by interchanging the roles of $\underline{v}_1, \overline{v}_1$ and $\underline{v}_2, \overline{v}_2$.

4 Continuity of the equilibrium

In addition to verifying that our solution is consistent with all known results (symmetric case, Griesmer et al. 1967) we ask the more general question: Is our solution continuous in the parameters of the problem? That is, are the bid functions continuous in the space of functions w.r.t. $\underline{v}_1, \overline{v}_1, \underline{v}_2, \overline{v}_2$ and m? To answer this, we make use of a proposition by Lebrun (2002) for equilibrium bid strategies in first-price, private-value auctions without a minimum bid.

Proposition 5 *The equilibrium bid strategies are continuous in the distributions of valuations (endowed with the weak topology).*

Proof By (Lebrun 2006, Theorem 1), which applies to our case, the equilibrium is unique. By (Lebrun 2002, Corollary 1), the equilibrium correspondence is upper hemi-continuous, which in conjunction with uniqueness implies continuity.

We observe that the distributions $U[\underline{v}_1, \overline{v}_1]$ and $U[\underline{v}_2, \overline{v}_2]$ are continuous in the *weak topology* in the space of probability measures (on a bounded interval), with respect to $\underline{v}_1, \overline{v}_1, \underline{v}_2, \overline{v}_2$.[10] When the minimum bid is binding and the case is not trivial, $\frac{\underline{v}_1 + \underline{v}_2}{2} < m < \min\{\overline{v}_1, \overline{v}_2\}$, the effect on the bid functions is equivalent to modifying the distributions- $U[\underline{v}_i, \overline{v}_i]$ to distributions $F_{i,m}$ on $[\tilde{\underline{v}}_i, \overline{v}_i]$ where $\tilde{\underline{v}}_i = \max\{m, \underline{v}_i\}$ and $F_{i,m}$ is uniform on $(\tilde{\underline{v}}_i, \overline{v}_i)$ with an atom $\delta = \frac{\tilde{\underline{v}}_i - \underline{v}_i}{\overline{v}_i - \underline{v}_i}$ at $\tilde{\underline{v}}_i$. The modified distributions $F_{i,m}$ are continuous (in the weak topology) in the minimum bid m. Thus, we can use Proposition 5 also for the continuity in m and arrive at the following corollary.

Corollary 2 *The equilibrium bid strategies are continuous in the parameters* $\underline{v}_1, \overline{v}_1, \underline{v}_2, \overline{v}_2$ *and* m.

We shall verify directly the continuity of the equilibrium (inverse) bid strategies. Outside the case of $\overline{v}_i \to \underline{v}_i$, we have found the equilibrium bid functions on four regions of the minimum bid m:

(1) For $m \le (\underline{v}_1 + \underline{v}_2)/2$. This was the case of "no minimum bid" , that is, the minimum bid is not binding in equilibrium. This equilibrium, given in Eqs. (8) and (10), thus does not depend on m.
(2) For $m > (\underline{v}_1 + \underline{v}_2)/2$ and $m \ne \underline{v}_2$. The minimum bid is binding in equilibrium, and this equilibrium depends on m. It is given in Eqs. (16) and (17).
(3) For $m > (\underline{v}_1 + \underline{v}_2)/2$ and $m = \underline{v}_2$. This equilibrium is listed in Eqs. (22) and (23).
(4) For $\overline{v}_i \le m \le \overline{v}_j$. Buyer j bids m for all $m \le v_j$ while buyer i bids his value v_i.

It follows from the explicit forms that, within each of these regions, the solution is continuous in $\underline{v}_1, \overline{v}_1, \underline{v}_2, \overline{v}_2$, and m. Hence, to check the continuity of the equilibrium, we need to check continuity between regions in the following cases:

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[10] A sequence of probability measures (on Ω) $(F_n)_{n=1}^{\infty}$ converges in the weak topology to the probability F if $\lim_{n \to \infty} \int_{\Omega} g \, dF_n = \int_{\Omega} g \, dF$ for all bounded and continuous functions $g : \Omega \to \mathbb{R}$. Equivalently, if $\lim_{n \to \infty} F_n(x) = F(x)$ for all $x \in \mathbb{R}$ at which F is continuous.

(A) between regions 1 and 2 ($m = (\underline{v}_1 + \underline{v}_2)/2$).

(B) between regions 2 and 3 ($m = \underline{v}_2$ and $m > (\underline{v}_1 + \underline{v}_2)/2$).

(C) between regions 2 and 4 ($m = \min\{\overline{v}_1, \overline{v}_2\}$).

In addition, we need to check continuity as $\overline{v}_i \to \underline{v}_i$, which we denote as case (D).

(A) Continuity at $m = (\underline{v}_1 + \underline{v}_2)/2$

The proof is in Appendix A.6.

(B) Continuity at $m = \underline{v}_2$ (in the region $m > (\underline{v}_1 + \underline{v}_2)/2$)

To prove continuity at $m = \underline{v}_2$, we have to show that the limit of the functions in (16) and (17) as $m \to \underline{v}_2$ converges to the functions in (22) and (23) in Proposition 3. This we prove in Appendix A.9, which completes our proof of the continuity in m.

(C) Continuity at $m = \min\{\overline{v}_1, \overline{v}_2\}$ (between regions 2 and 4)

Here we examine the case where $\overline{v}_1 \geq \overline{v}_2$. In region 2, when $m \to \overline{v}_2$, by Eq. (17), $c_3 \to -\infty$. By examining Eq. (16). The only way that $v_1(b) > \underline{v}_1$ is for $b \to m$. Similarly, we can show the same for $v_2(b)$ when $\overline{v}_1 \leq \overline{v}_2$.

(D) Continuity as $\overline{v}_i \to \underline{v}_i$

In Appendix A.7 and A.8, we prove continuity when $\overline{v}_2 \to \underline{v}_2$ with and without a binding minimum bid m. When $\overline{v}_1 \to \underline{v}_1$ (and $m \leq \underline{v}_1$), the solution goes to the equilibrium where buyer 2 is bidding $\overline{v}_1$ and winning the auction.

We note that continuity of the bid functions in the parameters also implies that the profits are continuous in the parameters (if $\underline{v}_1 < \overline{v}_1$ and $\underline{v}_2 < \overline{v}_2$).[11] To see this notice that given this continuity of the bid functions in the parameters, a discrete change in the profits requires both an atom in the distribution of equilibrium bids and that this atom transverses the minimum bid. An atom in the distribution of equilibrium bids can only occur if $\min\{\overline{v}_1, \overline{v}_2\} < m < \max\{\overline{v}_1, \overline{v}_2\}$. In such a case, the atom is always at m and thus the atom doesn't transverse m.

5 Some new examples

In this section, we provide a few examples of interest that were not solved analytically before. In looking at these examples, we note the minimum bid m provides a way to model distributions of values with atoms at the lower end of the intervals. In fact, when $V_i \sim U[\underline{v}_i, \overline{v}_i]$ and m is in $(\underline{v}_i, \overline{v}_i)$, then this is equivalent to a distribution with an atom $\delta_i = \frac{(m - \underline{v}_i)}{(\overline{v}_i - \underline{v}_i)}$ at m and a uniform distribution on $[m, \overline{v}_i]$ with the remaining probability.[12] This may be called the 'effective' distribution of V_i with the presence of a minimum bid.

Thus, our analytic solution for the general uniform case with a minimum bid also covers the case of two buyers with distributions that are uniform on intervals when only one has an atom at the lower end of his interval while the other has his interval above this atom or they both have atoms at the common lower end of their intervals.

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[11] The reason why we need $\underline{v}_1 < \overline{v}_1$ and $\underline{v}_2 < \overline{v}_2$ is, for instance, if $\overline{v}_1 < \underline{v}_2 = \overline{v}_2$, there will be a discrete jump in profits as we lower m from above $\overline{v}_2$ to below $\overline{v}_2$.

[12] In the distribution with atoms, we either have to relax the assumption that a buyer bids his value when he has zero probability of winning, break any ties with messages sent by the bidders as in Lebrun (1996) or break ties by holding a subsequent second-price auction as in Maskin and Riley (2000b).

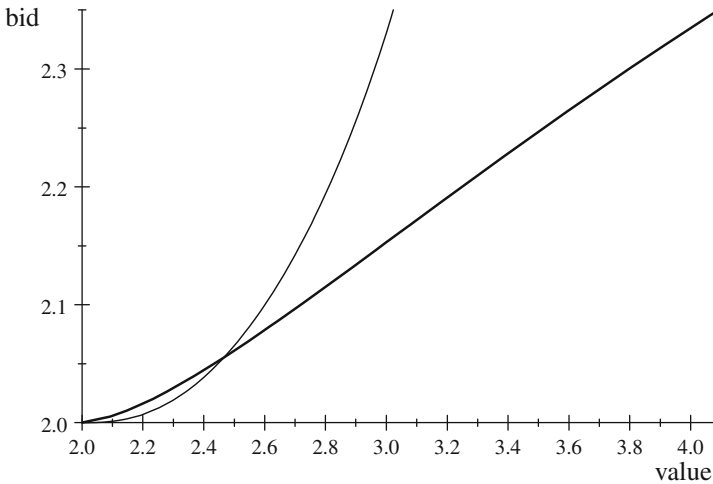**Fig. 1** Solution to Example 1, $\underline{v}_1 = 0, \underline{v}_2 = 1, m = 2, \overline{v}_2 = 3, \overline{v}_1 = 4$. The *thick line* is $v_1(b)$

We generate examples using the solution with a minimum bid given by Eqs. (16) and (17).

Example 1 $\underline{v}_1 = 0, \underline{v}_2 = 1, m = 2, \overline{v}_2 = 3, \overline{v}_1 = 4$ (see Fig. 1). Applying Proposition 2, we have

$$v_1(b) = \frac{2}{b - 1 + (b-2)^{\frac{2}{3}}(b+1)^{\frac{1}{3}}c_3}, \quad c_3 = \frac{(10)^{\frac{2}{3}}}{(-4)},$$

$$v_2(b) = \frac{2}{b + (b-2)^{\frac{1}{3}}(b+1)^{\frac{2}{3}}c_4} + 1, \quad c_4 = \frac{2(10)^{\frac{1}{3}}}{(-5)}.$$

These bid functions, as can be seen in Fig. 1 contrast with the results of Maskin and Riley (2000a) (and Lebrun 1999, Corollary 3 (ii) and (iii)) who prove that a strong buyer's bid function is lower than the weak buyer's bid function (that is, the weak buyer bids more aggressively); however, for that result to hold the value distribution of the strong buyer has to *conditionally stochastically dominate* (CSD) the value distribution of the weak buyer.[13,14] The subtlety of this example is that with the presence

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[13] Cumulative distribution F_1 stochastically dominates (SD) F_2 if $F_1(v) \leq F_2(v)$ for all v in the union of the supports of F_1 and F_2. Cumulative distribution F_1 conditionally stochastically dominates (CSD) F_2 if $F_1(v)/F_2(v)$ weakly increases in the union of the supports of F_1 and F_2. Note it can be shown that CSD implies SD (but SD does not imply CSD).

[14] Also, (Lebrun 1999, Corollary 3 (i)) and (Milgrom 2004, page 151) show that if one bidder has a higher distribution of values, then in equilibrium he also has a higher distribution of bids.
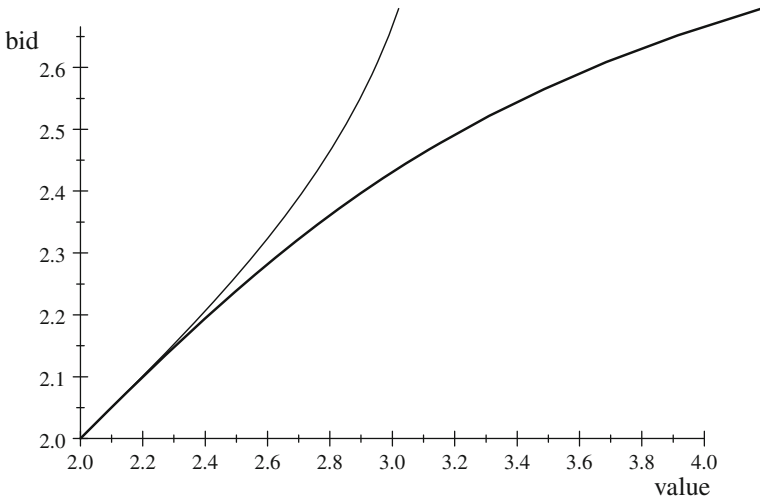
Fig. 2 Solution to modified Example 1 when $\underline{v}_1 = 2$, $\underline{v}_2 = 2$, $\overline{v}_2 = 3$, $\overline{v}_1 = 4$. The *thicker line* is $v_1(b)$

of a binding minimum bid the "effective" value distributions are:

$$F_1(v) = \frac{v}{4}; \ 2 \le v \le 4, \quad F_2(v) = \begin{cases} \frac{v}{2} - \frac{1}{2} & 2 \le v \le 3 \\ 1 & 3 \le v \le 4 \end{cases}$$

and hence, while F_1 stochastically dominates F_2, it does not CSD F_2. In addition, this example proves that while stochastic dominance is necessary for bidding function not to cross (Kirkegaard 2009), it is not sufficient. As a matter fact, this is the first analytical solution of intersecting bid functions that we are aware of.

It is interesting to compare this with the same conditional value distributions above 2 (without the atoms at $m = 2$), namely, $V_1 \sim U[2, 4]$ and $V_2 \sim U[2, 3]$. This is given in Fig. 2 and can be obtained from Eqs. (20) and (21) by shifting the lower bound from 0 to 2 (hence, it is a case treated by Griesmer et al. 1967). In this case, the distribution of V_1 conditionally stochastically dominates the distribution of V_2 and indeed $b_1(v) \le b_2(v)$ in accordance with Maskin and Riley's result.

Summing up, the presence of a minimum bid, even though it is at the center of both distributions, changes the equilibrium qualitatively by introducing the crossing of the bid functions. This example generalizes for a range of minimum bids.

Example 2 $\underline{v}_1 = 0, \underline{v}_2 = 1, 1/2 < m < 3, \overline{v}_2 = 3, \overline{v}_1 = 4$. By (16) and (17) of Proposition 2, we have

$$v_1(b) = \frac{m(m-1)}{b - 1 + (b - m)^{\frac{m}{2m-1}}(b + m - 1)^{\frac{m-1}{2m-1}}c_3},$$

$$c_3 = -\frac{(4-m)\left(\frac{(m+3)(m+2)}{(3-m)(4-m)}\right)^{\frac{m}{2m-1}}}{2(m+2)},$$

$$\bar{b} = \frac{\bar{v}_1 \cdot \bar{v}_2 + m^2 - m(\underline{v}_1 + \underline{v}_2)}{(\bar{v}_1 + \bar{v}_2) - (\underline{v}_1 + \underline{v}_2)} = \frac{12 + m^2 - m}{6},$$

$$v_2(b) = 1 + \frac{m(m-1)}{b + (b-m)^{\frac{m-1}{2m-1}}(b+m-1)^{\frac{m}{2m-1}}c_4},$$

$$c_4 = -\frac{2(3-m)\left(\frac{(m+2)(m+3)}{(3-m)(4-m)}\right)^{\frac{m}{2m-1}}}{m+3}.$$

We have found by numerical computation of the solution that the crossing occurs for different values of m in the range.[15]

In the following example we characterize a family of auctions with uniform distributions with linear equilibrium bid functions.

Example 3 $\underline{v}_1 = 0, \bar{v}_1 = m + z, \underline{v}_2 = 3m/2, \bar{v}_2 = 3m/2 + z$ (where $z > 0$). Here we obtain from (16) and (17) that

$$v_1(b) = 2(b-m) + m = 2b - m,$$
$$v_2(b) = 2(b-m) + 3m/2 = 2b - m/2,$$
$$b_1(v) = \frac{v}{2} + \frac{m}{2}, \quad b_2(v) = \frac{v}{2} + \frac{m}{4}.$$

Notice that these bid functions are independent of z and linear. Furthermore, the measure of values where a bid is submitted above the minimum is the same for both buyers, namely, z. Also notice that when $m \to 0$, this goes to the standard symmetric uniform case of uniformly distributed values on $[0, z]$.

It turns out that linear bid functions appear only in this special case, as we see in the following proposition (Fig. 3).

Proposition 6 *The bid functions are linear[16] if and only if $m = (2\underline{v}_2 + \underline{v}_1)/3$ (the minimum bid is two-thirds of the way from the lower end of the support of buyer 1's values to the lower end of the support of buyer 2's values) and $\bar{v}_1 - m = \bar{v}_2 - \underline{v}_2 \equiv z$ (the range of values above the minimum bid is the same length for both buyers). In this case, $b_1(v) = \frac{v}{2} + \frac{m}{2}$ and $b_2(v) = \frac{v}{2} + \frac{m}{4} + \frac{\underline{v}_1}{4}$.*

Proof See Appendix A.10.

Corollary 3 *When the equilibrium in the first-price auction is linear, the first-price auction generates higher revenue to the seller than the second-price auction.*

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[15] One method is by way of computation to show that $v_1(m + \varepsilon) < v_2(m + \varepsilon)$. This then implies that the bid functions must cross in order to arrive at their respective endpoints.

[16] While as given $b_1(v)$ is linear, it is not a unique equilibrium bid function below m. If we assume that for these values buyer 1 bids his value or zero, then buyer 1's overall bid function will be piecewise linear.
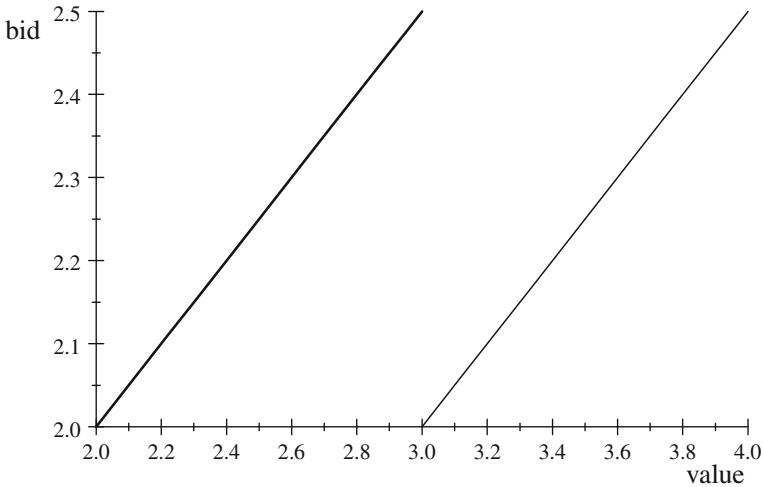
Fig. 3 Solution to Example 3 when $\underline{v}_1 = 0$, $\underline{v}_2 = 3$, $m = 2$, $\bar{v}_1 = 3$, $\bar{v}_2 = 4$. The *thicker line* is $v_1(b)$

Proof We note that in this class of auctions, the revenue for the first-price auction (for $\underline{v}_1 = 0$) is

$$R_{\text{FP}} = \frac{12m^2 + 15mz + 4z^2}{12(m + z)},$$

and the revenue for the second-price auction is

$$R_{\text{SP}} = \begin{cases} \frac{m^3 + 42m^2z + 60mz^2 + 16z^3}{48z(m+z)} & \text{if } z > m/2, \\ \frac{2m^2 + 2mz + z^2}{2(m+z)} & \text{if } z \leq m/2. \end{cases}$$

In both cases, the first-price auction has higher revenue (it is higher by $\frac{m^2(6z-m)}{48z(m+z)}$ when $z > m/2$ and by $\frac{(3m-2z)z}{12(m+z)}$ when $z \leq m/2$). We note that since the good is always sold, increasing all parameters by a constant increases both revenues by that same constant. Thus, the relationship also holds for $\underline{v}_1 > 0$.

Note that in the symmetric case $\underline{v}_1 = \underline{v}_2 = 0$, the condition of Proposition 6 implies $m = 0 < z = \bar{v} - \underline{v}$ and hence by the above formulas $R_{\text{FP}} = R_{\text{SP}} = z/3$ in accordance to the well known revenue equivalence results.

The following example helps illustrate the Proposition 6 by demonstrating that linearity is lost by stretching the upper range.

Example 4 $\underline{v}_1 = 0$, $\bar{v}_1 = 3$, $\underline{v}_2 = 3$, $\bar{v}_2 = 6$, $m = 2$. Here we obtain

$$v_1(b) = \frac{8(b - 1)}{(8 + b(b - 4))},$$
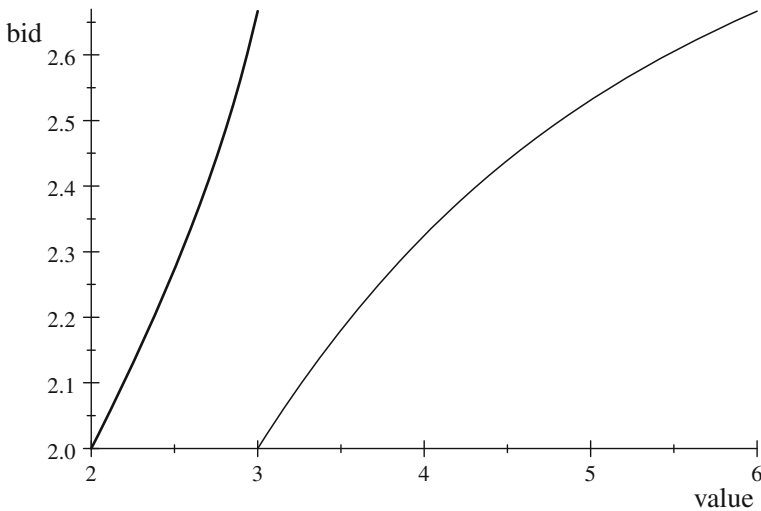
Fig. 4 Solution to Example 4, $\underline{v}_1 = 0$, $\underline{v}_2 = 3$, $m = 2$, $\overline{v}_1 = 3$, $\overline{v}_2 = 6$. The *thicker line* is $v_1(b)$

$$v_2(b) = 3 + \frac{10(b-2)}{(4+2b-b^2)}.$$

By inverting the functions, we get the following non-linear bid functions (see Fig. 4):

$$b_1(v) = \frac{2\left(2+v-\sqrt{4+2v-v^2}\right)}{v},$$

$$b_2(v) = \frac{v-8+\sqrt{5}\sqrt{8-4v+v^2}}{(v-3)}.$$

6 Concluding remarks

In this paper, we have analytically solved the first-price auction for two buyers with uniform distributions for any bounded supports, with or without a minimum bid. This is a fundamental case that would be useful to test conjectures, do comparative statics, or illustrate important features of auction design. A future direction of research would be to search for analytic solutions for other environments, such as extending our solution to N buyers. Another direction would be to find environments with simple equilibrium bid functions: the simplest being linear. There is a potential to expand the linear characterization in this paper in conjunction with the recent independently derived results of Cheng (2006) and Kirkegaard (2006). Cheng (2010) has a recent advance in that direction. Together these works should provide a useful set of examples for researchers and students. They may also suggest a set of parameters for additional experiments (see Güth et al. 2005) on asymmetric auctions.

Appendix A

A.1 Characterization of the equilibrium

Although Lebrun (2006) also characterizes the equilibria (and Griesmer et al. 1967 does so without a minimum bid), we include these results for both completeness (the proofs in Lebrun are only in the working paper version) and to aid the reader by using the notation of our paper.

Lemma 4 *The closure of the set of equilibrium bids in which buyer i has a positive probability of winning is a subinterval of $[\underline{b}_i, \overline{b}_i]$ and is the same for both buyers.*

Proof Note first that in equilibrium the set of winning bids for buyer i is an interval. Since the equilibrium consists of continuous bid functions defined on intervals, the range of the bid functions are also intervals. If b' and b'' are in the set of bids in which buyer i has a positive probability of winning, then so must any bid between them since the probability of winning is increasing in the bid. Denote the closure of the interval of equilibrium bids in which buyer i has a positive probability of winning as $[\underline{c}_i, \overline{c}_i]$. First, observe that due to independence of the value distributions, in equilibrium when a buyer bids b_a he wins with a (weakly) higher probability than when he bids $b_b \leq b_a$. This implies $\overline{c}_i = \overline{b}_i$. In addition, $\overline{b}_1 = \overline{b}_2$. Otherwise, if $\overline{b}_i > \overline{b}_j$, then there would be a small enough amount for buyer i (of value $\overline{v}_i$) to lower his bid from $\overline{b}_i$ without lowering his probability of winning. Since $\overline{c}_i = \overline{b}_i$, we have $\overline{c}_1 = \overline{c}_2$.

Finally, we show that $\underline{c}_1 = \underline{c}_2$. First note that for any bid above $\underline{c}_i$, buyer i has a positive probability of winning; hence, there is a positive probability that $b_j \leq \underline{c}_i$. Assume by contradiction that $\underline{c}_i < \underline{c}_j$. By definition, $b_j > \underline{c}_j$ with positive probability. It follows by continuity of buyer j's bid function that j bids b_j, where $\underline{c}_i < b_j < \underline{c}_j$, with positive probability. Consider a bid b'_j such that $\underline{c}_i < b'_j < \underline{c}_j$. By continuity of buyer i's bid function between $\underline{c}_i$ and $\overline{c}_i$, there is a positive measure of b_i for which $\underline{c}_i \leq b_i < b'_j$. In other words, buyer j has a positive probability of winning with bid b_j in contradiction to the definition of $\underline{c}_j$. Hence, $\underline{c}_1 = \underline{c}_2$.

In view of Lemma 4, denote by $[\underline{b}, \overline{b}]$ the region of equilibrium bids where if a buyer submits a bid, he has a positive probability of winning in equilibrium. From our assumption that in equilibrium a buyer with zero probability of winning bids his value, it follows that $b_i(v_i) = v_i$ for $v_i < \underline{b}$ and by continuity $b_i(\underline{b}) = \underline{b}$ (if $\underline{b} \geq \underline{v}_i$).

A.2 Proof of the boundary conditions

(B1) $v_1(\underline{b}) = \underline{b}$: as we noted above, $\underline{b}$ belongs to $[\underline{v}_1, \overline{v}_1]$. By continuity of the bid functions, we must have $v_1(\underline{b}) = \max\{\underline{v}_1, \underline{b}\}$ and $v_2(\underline{b}) = \max\{\underline{v}_2, \underline{b}\}$ (otherwise, there is a profitable deviation). Since there are no bids below $\underline{v}_1$, $\max\{\underline{v}_1, \underline{b}\} = \underline{b}$ and hence, $v_1(\underline{b}) = \underline{b}$.

(B2) $v_2(\underline{b}) = \max\{\underline{v}_2, m\}$: if $\underline{v}_2 \geq m$, then any buyer with $v_2 > \underline{v}_2 (\geq \underline{v}_1)$ can bid $\frac{\underline{v}_2 + v_2}{2}$ and thus win with positive probability with a bid strictly less than his

value. Hence, in equilibrium buyer 2 must have a positive probability of winning with these values, $v_2 \geq v_2(\underline{b})$ for all $v_2 > \underline{v}_2$. By continuity, $\underline{v}_2 \geq v_2(\underline{b})$. Since $v_2(\underline{b}) \in [\underline{v}_2, \overline{v}_2]$. Hence, we have $v_2(\underline{b}) = \underline{v}_2$.

If $\underline{v}_2 < m$, then any buyer with $v_2 > m > \underline{v}_2 (\geq \underline{v}_1)$ can bid $\frac{v_2+m}{2}$ and thus win with positive probability with a bid strictly less than his value. Hence, $v_2 \geq v_2(\underline{b})$ for all $v_2 > m$ and by continuity $v_2(\underline{b}) \leq m$. However, $v_2(\underline{b}) < m$ cannot hold, since then buyer 2 would sometimes be winning with a bid larger than his value. Thus, $v_2(\underline{b}) = m$. Therefore, $v_2(\underline{b}) = \max\{\underline{v}_2, m\}$.

(B3) $v_1(\overline{b}) = \overline{v}_1$ and $v_2(\overline{b}) = \overline{v}_2$: since the bid functions are monotonic, the highest bid of each buyer is reached for his highest value. It must be in the set of winning bids.

A.3 Proof of Lemma 2

We solve for $\underline{b}$ in terms $\underline{v}_1$ and $\underline{v}_2$. Buyer 2 with value v_2 solves the following maximization problem:

$$\max_b \left(\frac{v_1(b) - \underline{v}_1}{\overline{v}_1 - \underline{v}_1} \right) (v_2 - b).$$

Buyer 1 bids weakly below his value, $v_1(b) \geq b$. Buyer 2 with value $v_2(\underline{b})$ must not benefit from deviating from bidding $\underline{b}$. Also, remember by boundary condition B1, $v_1(\underline{b}) = \underline{b}$. Hence,

$$(\underline{b} - \underline{v}_1)(v_2(\underline{b}) - \underline{b}) \geq (v_1(b) - \underline{v}_1)(v_2(\underline{b}) - b) \geq (b - \underline{v}_1)(v_2(\underline{b}) - b) \quad \forall b \geq m.$$

Note the b for which the expression $(b - \underline{v}_1)(v_2(\underline{b}) - b)$ achieves its maximum is $b = \frac{\underline{v}_1+v_2(\underline{b})}{2}$. Hence, if $\frac{\underline{v}_1+v_2(\underline{b})}{2} \geq m$, we must have $\underline{b} = \frac{\underline{v}_1+v_2(\underline{b})}{2}$. If $\frac{\underline{v}_1+v_2(\underline{b})}{2} \leq m$, we must have $\underline{b} = m$ since $(b - \underline{v}_1)(v_2(\underline{b}) - b)$ is decreasing for all $b \geq m$ (if $\underline{b} > m$, buyer 2 could then gain choosing $b = m$).

Therefore, $\underline{b} = \max\{\frac{\underline{v}_1+v_2(\underline{b})}{2}, m\}$. By boundary condition B2, $v_2(\underline{b}) = \max\{\underline{v}_2, m\}$. Therefore, $\underline{b} = \max\{\frac{\underline{v}_1+v_2(\underline{b})}{2}, m\} = \max\{\frac{\underline{v}_1+\max\{\underline{v}_2,m\}}{2}, m\} = \max\{\frac{\underline{v}_1+\underline{v}_2}{2}, m\}$ (since $\underline{v}_1 \leq \underline{v}_2$).

Thus,

$$\underline{b} = \max \left\{ \frac{\underline{v}_1 + \underline{v}_2}{2}, m \right\}.$$

A.4 Proof of Proposition 2: solution with minimum bids

The solution that we presented with a binding minimum bid (and $m \neq \underline{v}_2$) is

$$v_1(b) = \underline{v}_1 + \frac{(m - \underline{v}_1)(m - \underline{v}_2)}{b - \underline{v}_2 - c_3(b - m)^\theta (b + m - \underline{v}_1 - \underline{v}_2)^{1-\theta}},$$

$$c_3 = \frac{(\bar{v}_2 - \underline{v}_2)}{(\bar{v}_1 - \underline{v}_1)} \frac{\left(\frac{\bar{v}_1 - m}{m - \underline{v}_1 + \bar{v}_2 - \underline{v}_2}\right)^{1-\theta}}{\left(\frac{a\bar{v}_2 - m}{m - \underline{v}_2 + \bar{v}_1 - \underline{v}_1}\right)^{\theta}}.$$

where $\theta \equiv \frac{m - \underline{v}_1}{(m - \underline{v}_1) + (m - \underline{v}_2)}$. To derive this solution we divide both sides of Eq. (15) by

$$(v_1(b) - \underline{v}_1)^2 (b - m)^{1+\theta} (b + m - \underline{v}_1 - \underline{v}_2)^{2-\theta}$$

to obtain

$$\frac{v_1'(b)}{(v_1(b) - \underline{v}_1)^2 (b - m)^{\theta} (b + m - \underline{v}_1 - \underline{v}_2)^{1-\theta}}$$
$$= \frac{(v_1(b) - b)}{(v_1(b) - \underline{v}_1)(b - m)^{1+\theta} (b + m - \underline{v}_1 - \underline{v}_2)^{2-\theta}}. \tag{29}$$

The RHS can be broken into two expressions:

$$\frac{1}{(b - m)^{1+\theta} (b + m - \underline{v}_1 - \underline{v}_2)^{2-\theta}}$$
$$+ \frac{(\underline{v}_1 - b)}{(v_1(b) - \underline{v}_1)(b - m)^{1+\theta} (b + m - \underline{v}_1 - \underline{v}_2)^{2-\theta}}.$$

It is straightforward to show that the derivative of $\frac{1}{(b-m)^{\theta}(b+m-\underline{v}_1-\underline{v}_2)^{1-\theta}}$ equals $\frac{(\underline{v}_1 - b)}{(b-m)^{1+\theta}(b+m-\underline{v}_1-\underline{v}_2)^{2-\theta}}$. Using this, observe that

$$\int \frac{1}{(b - m)^{1+\theta} (b + m - \underline{v}_1 - \underline{v}_2)^{2-\theta}} db$$
$$= \frac{1}{(b - m)^{\theta} (b + m - \underline{v}_1 - \underline{v}_2)^{1-\theta}} \cdot \frac{\underline{v}_2 - b}{(m - \underline{v}_1)(m - \underline{v}_2)} + C$$

and $\int [\frac{v_1'(b)}{(v_1(b)-\underline{v}_1)^2(b-m)^{\theta}(b+m-\underline{v}_1-\underline{v}_2)^{1-\theta}} - \frac{(\underline{v}_1-b)}{(v_1(b)-\underline{v}_1)(b-m)^{1+\theta}(b+m-\underline{v}_1-\underline{v}_2)^{2-\theta}}] db = -\frac{1}{(b-m)^{\theta}(b+m-\underline{v}_1-\underline{v}_2)^{1-\theta}} \cdot \frac{1}{v_1(b)-\underline{v}_1} + C$ (simply take the derivative of the RHS of each equation, of which both are of the form $u \cdot v + C$ where $u = \frac{1}{(b-m)^{\theta}(b+m-\underline{v}_1-\underline{v}_2)^{1-\theta}}$).

Hence, we can integrate (29). From this we can obtain $v_1(b)$ as in (16), and the expression for c_3 is obtained by the boundary condition B3.

A.5 Proof of Proposition 3: solution when $m = \underline{v}_2$

The differential equation for this case is obtained by substituting $m = \underline{v}_2$ in Eq. (15):

$$v_1'(b) \cdot (b - \underline{v}_2)(b - \underline{v}_1) = (v_1(b) - \underline{v}_1)(v_1(b) - b).$$

Dividing both sides by $(b - \underline{v}_2)^2(b - \underline{v}_1)(v_1(b) - \underline{v}_1)^2$ and rewriting yields

$$\frac{v_1'(b)}{(b - \underline{v}_2)(v_1(b) - \underline{v}_1)^2} - \frac{(v_1(b) - b)}{(b - \underline{v}_2)^2(b - \underline{v}_1)(v_1(b) - \underline{v}_1)} = 0.$$

By further rewriting, we have

$$\left(\frac{1}{\underline{v}_2 - \underline{v}_1}\right)^2 \left(\frac{1}{b - \underline{v}_2} - \frac{1}{b - \underline{v}_1}\right) + \left(\frac{1}{\underline{v}_2 - \underline{v}_1}\right)\left(-\frac{1}{(b - \underline{v}_2)^2}\right)$$
$$+ \left(\frac{v_1'(b)(b - \underline{v}_2) + (v_1(b) - \underline{v}_1)}{(v_1(b) - \underline{v}_1)^2(b - \underline{v}_2)^2}\right) = 0.$$

Now by integration, we derive the solution:

$$\left(\frac{1}{\underline{v}_2 - \underline{v}_1}\right)^2 \left(\log(b - \underline{v}_2) - \log(b - \underline{v}_1)\right) + \frac{1}{(\underline{v}_2 - \underline{v}_1)(b - \underline{v}_2)}$$
$$- \frac{1}{(v_1(b) - \underline{v}_1)(b - \underline{v}_2)} = C.$$

Rewriting this yields

$$\frac{1}{\underline{v}_2 - \underline{v}_1}\left[-\left(\frac{b - \underline{v}_2}{\underline{v}_2 - \underline{v}_1}\right)\log\left(\frac{b - \underline{v}_1}{b - \underline{v}_2}\right) + 1 - \frac{c_4}{(\underline{v}_2 - \underline{v}_1)}(b - \underline{v}_2)\right] = \frac{1}{(v_1(b) - \underline{v}_1)},$$

where $c_5 \equiv C \cdot (\underline{v}_2 - \underline{v}_1)^2$. Rearranging yields

$$v_1(b) = \underline{v}_1 + \frac{\underline{v}_2 - \underline{v}_1}{1 - \left(\frac{b - \underline{v}_2}{\underline{v}_2 - \underline{v}_1}\right)\left[c_5 + \log\left(\frac{b - \underline{v}_1}{b - \underline{v}_2}\right)\right]}.$$

Using boundary condition B3, $v_1(\bar{b}) = \bar{v}_1$, we have

$$\bar{v}_1 = \underline{v}_1 + \frac{\underline{v}_2 - \underline{v}_1}{1 - \left(\frac{\bar{b} - \underline{v}_2}{\underline{v}_2 - \underline{v}_1}\right)\left[c_5 + \log\left(\frac{\bar{b} - \underline{v}_1}{\bar{b} - \underline{v}_2}\right)\right]},$$

which implies

$$c_5 = \frac{(\bar{v}_1 - \underline{v}_2)(\underline{v}_2 - \underline{v}_1)}{(\bar{v}_1 - \underline{v}_1)(\bar{b} - \underline{v}_2)} - \log\left(\frac{\bar{b} - \underline{v}_1}{\bar{b} - \underline{v}_2}\right).$$

Substituting $\overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - \underline{v}_1 \cdot \underline{v}_2}{(\overline{v}_1 - \underline{v}_1) + (\overline{v}_2 - \underline{v}_2)}$ yields

$$c_5 = \frac{(\overline{v}_1 - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)(\underline{v}_2 - \underline{v}_1)}{(\overline{v}_1 - \underline{v}_1)(\overline{v}_2 - \underline{v}_2)} - \log\left(\frac{(\overline{v}_2 - \underline{v}_1)(\overline{v}_1 - \underline{v}_1)}{(\overline{v}_2 - \underline{v}_2)(\overline{v}_1 - \underline{v}_2)}\right).$$

Thus, we have $v_1(b)$ and c_5 equivalent to those in Eqs. (23) and (22).

A.6 Continuity at $m = (\underline{v}_1 + \underline{v}_2)/2$

Since for $m \leq (\underline{v}_1 + \underline{v}_2)/2$ the equilibrium does not depend on m, continuity is established by proving that the inverse bid function $v_1(b)$ given by (16) approaches that given by (8) as m approaches the critical value $(\underline{v}_1 + \underline{v}_2)/2$ from above. First, we verify that

$$\lim_{m \searrow (\underline{v}_1 + \underline{v}_2)/2} (b - m)^{\frac{m - \underline{v}_1}{(m - \underline{v}_1) + (m - \underline{v}_2)}} (b + m - \underline{v}_1 - \underline{v}_2)^{\frac{(m - \underline{v}_2)}{(m - \underline{v}_1) + (m - \underline{v}_2)}}$$
$$= \frac{1}{2} e^{-\frac{\underline{v}_2 - \underline{v}_1}{2b - \underline{v}_1 - \underline{v}_2}} (2b - \underline{v}_1 - \underline{v}_2),$$

$$\lim_{m \searrow (\underline{v}_1 + \underline{v}_2)/2} \left(\frac{(m - \underline{v}_2 + \overline{v}_1 - \underline{v}_1)(m - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)}{(\overline{v}_1 - m)(\overline{v}_2 - m)}\right)^{\frac{m - \underline{v}_1}{(m - \underline{v}_1) + (m - \underline{v}_2)}}$$
$$= e^{2\frac{(\overline{v}_1 - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)(\underline{v}_2 - \underline{v}_1)}{(2\overline{v}_1 - \underline{v}_2 - \underline{v}_1)(2\overline{v}_2 - \underline{v}_2 - \underline{v}_1)}}.$$

Using these in our solution for $v_1(b)$ and c_3 in Eqs. (16) and (17), we have

$$\lim_{m \searrow (\underline{v}_1 + \underline{v}_2)/2} v_1(b)$$
$$= \underline{v}_1 + \lim_{m \searrow (\underline{v}_1 + \underline{v}_2)/2} \frac{(m - \underline{v}_1)(m - \underline{v}_2)}{b - \underline{v}_2 + \frac{1}{2} e^{-\frac{\underline{v}_2 - \underline{v}_1}{2b - \underline{v}_1 - \underline{v}_2}} (2b - \underline{v}_1 - \underline{v}_2) \lim_{m \searrow (\underline{v}_1 + \underline{v}_2)/2} c_3}$$
$$= \underline{v}_1 + \frac{-(\underline{v}_2 - \underline{v}_1)^2/4}{b - \underline{v}_2 + \frac{1}{2} e^{-\frac{\underline{v}_2 - \underline{v}_1}{2b - \underline{v}_1 - \underline{v}_2}} (2b - \underline{v}_1 - \underline{v}_2) \lim_{m \searrow (\underline{v}_1 + \underline{v}_2)/2} c_3}, \tag{30}$$

$$\lim_{m \searrow (\underline{v}_1 + \underline{v}_2)/2} c_3 = -\frac{(2\overline{v}_1 - (\underline{v}_1 + \underline{v}_2))(\overline{v}_2 - \underline{v}_2) e^{2\frac{(\overline{v}_1 - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)(\underline{v}_2 - \underline{v}_1)}{(2\overline{v}_1 - \underline{v}_2 - \underline{v}_1)(2\overline{v}_2 - \underline{v}_2 - \underline{v}_1)}}}{(\overline{v}_1 - \underline{v}_1)(2\overline{v}_2 - (\underline{v}_1 + \underline{v}_2))}. \tag{31}$$

We now see that, indeed, this limit yields the equilibrium bid functions for the case of no minimum bid. Note that the range of bids is as follows:

$$\underline{b} = \frac{\underline{v}_1 + \underline{v}_2}{2}, \quad \overline{b} = \frac{\overline{v}_1 \cdot \overline{v}_2 - \frac{(\underline{v}_1 + \underline{v}_2)^2}{4}}{(\overline{v}_1 + \overline{v}_2) - (\underline{v}_1 + \underline{v}_2)}.$$

Notice that by (3) and (12) we have $\bar{b} - \underline{b} = \frac{1}{4}\frac{(2\bar{v}_1 - \underline{v}_2 - \underline{v}_1)(2\bar{v}_2 - \underline{v}_2 - \underline{v}_1)}{(\bar{v}_1 + \bar{v}_2) - (\underline{v}_1 + \underline{v}_2)}$ and that

$$\frac{(\underline{v}_2 - \underline{v}_1)^2}{\bar{v}_1 - \underline{v}_1} + 4(\bar{b} - \underline{v}_2) = \frac{(\bar{v}_2 - \underline{v}_2)(2\bar{v}_1 - (\underline{v}_1 + \underline{v}_2))}{(\bar{v}_1 - \underline{v}_1)(\bar{v}_1 - \underline{v}_1 + \bar{v}_2 - \underline{v}_2)}.$$

Using these two in Eqs. (30) and (31) yields the equilibrium bid function without a minimum bid; namely, it establishes the equality between (30), (31) and (8), (10), respectively.

A.7 Continuity when buyer 2's value is commonly known

Let us view this situation as a limiting case of our model where $[\underline{v}_1, \bar{v}_1] = [0, 1]$, $[\underline{v}_2, \bar{v}_2] = [\beta, \beta + \varepsilon]$, and $\varepsilon \to 0$. This may be viewed as a continuity property of the solution as $\bar{v}_2 \longrightarrow \underline{v}_2$. To see that, we first write the probability distribution of the bids of buyer 2, which is

$$P(b_2(V_2, \varepsilon) \leq b) = P(V_2 \leq v_2(b, \varepsilon)) = \frac{v_2(b, \varepsilon) - \beta}{\varepsilon}.$$

(We use V_2 for the random value of buyer 2, denote $b_i(v, \varepsilon)$ as the bid function for buyer i when the distribution is $[\beta, \beta + \varepsilon]$, and denote $v_i(b, \varepsilon)$ as the respective inverse bid function.)

Proposition 7 *The equilibrium in* Vickrey (1961) *is a limit of our solution in the following sense:*

(i) *The limit of buyer 1's bid function is that in Vickrey, namely,*

$$\lim_{\varepsilon \to 0} v_1(b, \varepsilon) = \frac{\beta^2}{4(\beta - b)}.$$

(ii) *The bid distribution of buyer 2 approaches the mixed bidding strategy in Vickrey, i.e.,*

$$\lim_{\varepsilon \to 0} \frac{v_2(b, \varepsilon) - \beta}{\varepsilon} = F(b),$$

where $F(b)$ is given by (25).

Proof First, we observe that for $[\underline{v}_1, \bar{v}_1] = [0, 1]$ and $\underline{v}_2 = \bar{v}_2 = \beta$ we obtain from our above equations for $\underline{b}$ and $\bar{b}$ ((3) and (12)) the correct range of bids: $\underline{b} = \frac{\beta}{2}$ and $\bar{b} = \beta - \frac{\beta^2}{4}$. Next, notice that $\bar{b} > \underline{b}$ whenever $\beta - \frac{\beta^2}{4} > \frac{\beta}{2}$ (i.e., $\beta < 2$). Assuming that this is indeed the case, we have a range of bids even when one buyer's value is known with near certainty. (This makes sense since it converges to a mixed-strategy

equilibrium.) Now, using the analytic solution for buyer 1's inverse bid function, (8) and (10), with the distributions of $[\underline{v}_1, \overline{v}_1] = [0, 1], [\underline{v}_2, \overline{v}_2] = [\beta, \beta + \varepsilon]$, we have

$$v_1(b, \varepsilon) = \frac{\beta^2}{(\beta - 2b)c_1(\varepsilon)e^{\frac{\beta}{\beta - 2b}} + 4(\beta - b)}$$

with $\overline{b} = \overline{b}(\varepsilon) = \frac{\beta + \varepsilon - \frac{\beta^2}{4}}{1 + \varepsilon}$ where

$$c_1(\varepsilon) = \frac{\beta^2 - 4(\beta - \overline{b})}{(\beta - 2\overline{b})} e^{-\frac{\beta}{\beta - 2\overline{b}}}$$

is the corresponding constant as a function of ε. We have

$$\lim_{\varepsilon \to 0} v_1(b, \varepsilon) = \frac{\beta^2}{(\beta - 2b) \lim_{\varepsilon \to 0} c_1(\varepsilon)e^{\frac{\beta}{\beta - 2b}} + 4(\beta - b)} = \frac{\beta^2}{4(\beta - b)},$$

since

$$\lim_{\varepsilon \to 0} c_1(\varepsilon) = \lim_{\varepsilon \to 0} \frac{\beta^2 - 4(\beta - \overline{b}(\varepsilon))}{(\beta - 2\overline{b}(\varepsilon))} e^{-\frac{\beta}{\beta - 2\overline{b}(\varepsilon)}} = 0.$$

Furthermore, using the analytic solution for buyer 2's inverse bid function, (9) and (11), we have

$$v_2(b, \varepsilon) = \frac{\beta^2/\varepsilon}{\left(\frac{4 - \beta + \frac{\beta}{\varepsilon}}{\frac{1}{2}\beta - 1}\right)(\beta - 2b) e^{-\frac{\beta}{\beta - 2b}} e^{-\frac{2}{2 - \beta}} - 4b}. \tag{32}$$

And finally it can be verified (by straightforward calculation using (32) and (25)) that indeed

$$\lim_{\varepsilon \to 0} \frac{v_2(b, \varepsilon) - \beta}{\varepsilon} = F(b).$$

A.8 Continuity when buyer 2's value is commonly known and there is a binding minimum bid

In this section, we check for continuity at the limit case when buyer 2's value is commonly known and the minimum bid is binding. We again use the normalization in Sect. A.7, that is, $[\underline{v}_1, \overline{v}_1] = [0, 1]$ and $\underline{v}_2 = \beta, \overline{v}_2 = \beta + \varepsilon$. From substituting these into Eqs. (16) and (17), it is clear that $c_3(\varepsilon) \to 0$, therefore

$$\lim_{\varepsilon \to 0} v_1(b, \varepsilon) = \frac{m(\beta - m)}{\beta - b}.$$

To find $v_2(b, \varepsilon)$, we again use (16) and (17) (but the roles of $\underline{v}_1$, $\overline{v}_1$ and $\underline{v}_2$, $\overline{v}_2$ reversed). Hence,

$$v_2(b, \varepsilon) = \beta + \frac{m(m - \beta)}{b + (b - m)^{\frac{m-\beta}{2m-\beta}} (b + m - \beta)^{\frac{m}{2m-\beta}} c_4(\varepsilon)},$$

where

$$c_4(\varepsilon) = -\frac{(\beta + \varepsilon - m)\left(\frac{(m+\varepsilon)(m-\beta+1)}{(1-m)(\beta-m)}\right)^{\frac{m-\beta}{2m-\beta}}}{\varepsilon(m - \beta + 1)}.$$

As in Sect. 3.4, buyer 2's strategy goes to a mixed strategy with cumulative distribution

$$\lim_{\varepsilon \to 0} \frac{v_2(b, \varepsilon) - \beta}{\varepsilon} = \frac{m}{\frac{\left(\frac{m(m-\beta+1)}{(1-m)(\beta-m)}\right)^{\frac{m-\beta}{2m-\beta}}}{m-\beta+1}(b - m)^{\frac{m-\beta}{2m-\beta}} (b + m - \beta)^{\frac{m}{2m-\beta}}}$$

$$= \frac{((1-m)(\beta - m))^{\frac{m-\beta}{2m-\beta}} (m(m - \beta + 1))^{\frac{m}{2m-\beta}}}{(b - m)^{\frac{m-\beta}{2m-\beta}} (b + m - \beta)^{\frac{m}{2m-\beta}}}.$$

This limit equals the equilibrium when $\varepsilon \to 0$.[17]

A.9 Continuity when $m \to \underline{v}_2$

Starting with Eqs. (16) and (17), denote

$$A(m) = (m - \underline{v}_1)(m - \underline{v}_2),$$

$$B(m) = (b + m - \underline{v}_1 - \underline{v}_2)^{\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}},$$

$$C(m) = -\frac{(\overline{v}_1 - m)(\overline{v}_2 - \underline{v}_2)}{(\overline{v}_1 - \underline{v}_1)(m - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)},$$

$$D(m) = \left(\frac{(b - m)(m - \underline{v}_2 + \overline{v}_1 - \underline{v}_1)(m - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)}{(\overline{v}_1 - m)(\overline{v}_2 - m)}\right)^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}.$$

Thus,

$$v_1(b) = \underline{v}_1 + \frac{A(m)}{b - \underline{v}_2 + B(m)C(m)D(m)}.$$

---

[17] Also when $m \to \beta (< 1)$, the solution approaches the equilibrium that buyer 2 stays out of the auction and buyer 1 wins the auction (for all values above m). Also, when $m \to \beta/2$, this goes to the solution in Sect. 3.4 (when the minimum bid is not binding).

Since $B(\underline{v}_2)C(\underline{v}_2)D(\underline{v}_2) = -(b - \underline{v}_2)$ as $m \to \underline{v}_2$, we get $\frac{A(m)}{b-\underline{v}_2+B(m)C(m)D(m)} \to \frac{0}{0}$, so we need to use L'Hopital's rule, which yields

$$\lim_{m \to \underline{v}_2} v_1(b)$$

$$= \underline{v}_1 + \frac{A'(m)}{B'(m)C(m)D(m) + B(m)C'(m)D(m) + B(m)C(m)D'(m)}\bigg|_{m=\underline{v}_2}.$$
(33)

Step 1. Finding $A'(\underline{v}_2)$.

$A'(m) = (m - \underline{v}_1) + (m - \underline{v}_2)$, which implies that

$$A'(\underline{v}_2) = \underline{v}_2 - \underline{v}_1.$$
(34)

Step 2. Finding $B'(\underline{v}_2)C(\underline{v}_2)D(\underline{v}_2)$.

$B(m) = (b + m - \underline{v}_1 - \underline{v}_2)^{\frac{(m-\underline{v}_2)}{(m-\underline{v}_1)+(m-\underline{v}_2)}} \equiv f(m)^{g(m)}$. By this definition, we have $f(\underline{v}_2) = b - \underline{v}_1, g(\underline{v}_2) = 0, f'(m) = 1$, and $g'(\underline{v}_2) = \frac{1}{\underline{v}_2-\underline{v}_1}$. Recall that $(f(m)^{g(m)})' = f(m)^{g(m)}[\log(f(m))g'(m) + g(m)\frac{f'(m)}{f(m)}]$; hence, $B'(\underline{v}_2) = \frac{\log(b-\underline{v}_1)}{\underline{v}_2-\underline{v}_1}$.

We also have $C(\underline{v}_2) = -\frac{(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)}{(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}$ and $D(\underline{v}_2) = \frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)}$.

Thus,

$$B'(\underline{v}_2)C(\underline{v}_2)D(\underline{v}_2) = -\frac{\log\left(b - \underline{v}_1\right)}{\underline{v}_2 - \underline{v}_1}\left(b - \underline{v}_2\right).$$
(35)

Step 3. Finding $B(\underline{v}_2)C'(\underline{v}_2)D(\underline{v}_2)$.

We have $C'(\underline{v}_2) = \frac{(\overline{v}_2-\underline{v}_2)(\overline{v}_1-\underline{v}_1+\overline{v}_2-\underline{v}_2)}{(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)^2}$ and $B(\underline{v}_2) = 1$. Thus,

$$B(\underline{v}_2)C'(\underline{v}_2)D(\underline{v}_2) = \frac{(\overline{v}_1 - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)}{(\overline{v}_2 - \underline{v}_1)}\left(\frac{b - \underline{v}_2}{\overline{v}_1 - \underline{v}_2}\right).$$
(36)

Step 4. Finding $B(\underline{v}_2)C(\underline{v}_2)D'(\underline{v}_2)$.

Similarly to Step 2, $D(m) = (\frac{(b-m)(m-\underline{v}_2+\overline{v}_1-\underline{v}_1)(m-\underline{v}_1+\overline{v}_2-\underline{v}_2)}{(\overline{v}_1-m)(\overline{v}_2-m)})^{\frac{m-\underline{v}_1}{(m-\underline{v}_1)+(m-\underline{v}_2)}}$ $\equiv f(m)^{g(m)}$. By simple substitution, we have $f(\underline{v}_2) = (\frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)})$, $g(\underline{v}_2) = 1$, and $g'(\underline{v}_2) = -\frac{1}{\underline{v}_2-\underline{v}_1}$.

Since $f(m) = \frac{(b-m)(m-\underline{v}_2+\overline{v}_1-\underline{v}_1)(m-\underline{v}_1+\overline{v}_2-\underline{v}_2)}{(\overline{v}_1-m)(\overline{v}_2-m)}$, we can take the log of both sides and then the derivative w.r.t. m. This evaluated at $m = \underline{v}_2$ yields

$$\frac{f'\left(\underline{v}_2\right)}{f\left(\underline{v}_2\right)} = \frac{1}{\overline{v}_1 - \underline{v}_1} + \frac{1}{\overline{v}_2 - \underline{v}_1} + \frac{1}{\overline{v}_1 - \underline{v}_2} + \frac{1}{\overline{v}_2 - \underline{v}_2} - \frac{1}{b - \underline{v}_2}.$$

Again, recall that $(f(m)^{g(m)})' = f(m)^{g(m)}[\log(f(m))g'(m) + g(m)\frac{f'(m)}{f(m)}]$; hence, we have

$$
D'(\underline{v}_2) = \left(\frac{(b - \underline{v}_2)(\overline{v}_1 - \underline{v}_1)(\overline{v}_2 - \underline{v}_1)}{(\overline{v}_1 - \underline{v}_2)(\overline{v}_2 - \underline{v}_2)} \right)
$$
$$
\times \left[\begin{array}{l} -\log\left(\frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)} \right) \frac{1}{\underline{v}_2-\underline{v}_1} + \frac{1}{\overline{v}_1-\underline{v}_1} + \\ \frac{1}{\overline{v}_2-\underline{v}_1} + \frac{1}{\overline{v}_1-\underline{v}_2} + \frac{1}{\overline{v}_2-\underline{v}_2} - \frac{1}{b-\underline{v}_2} \end{array} \right].
$$

This implies that

$$
B(\underline{v}_2)C(\underline{v}_2)D'(\underline{v}_2) = 1 + (b-\underline{v}_2)\left[\begin{array}{l} \log\left(\frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)} \right) \frac{1}{\underline{v}_2-\underline{v}_1} \\ -\frac{1}{\overline{v}_1-\underline{v}_1} - \frac{1}{\overline{v}_2-\underline{v}_1} - \frac{1}{\overline{v}_1-\underline{v}_2} - \frac{1}{\overline{v}_2-\underline{v}_2} \end{array} \right].
$$
(37)

Step 5. Finding $B'(\underline{v}_2)C(\underline{v}_2)D(\underline{v}_2) + B(\underline{v}_2)C'(\underline{v}_2)D(\underline{v}_2) + B(\underline{v}_2)C(\underline{v}_2)D'(\underline{v}_2)$.
By (35), (36), and (37), we now have

$$
B'(\underline{v}_2)C(\underline{v}_2)D(\underline{v}_2) + B(\underline{v}_2)C'(\underline{v}_2)D(\underline{v}_2) + B(\underline{v}_2)C(\underline{v}_2)D'(\underline{v}_2)
$$
$$
= -\frac{\log(b-\underline{v}_1)}{\underline{v}_2 - \underline{v}_1}(b-\underline{v}_2) + \frac{(\overline{v}_1 - \underline{v}_1 + \overline{v}_2 - \underline{v}_2)}{(\overline{v}_2 - \underline{v}_1)}\left(\frac{b-\underline{v}_2}{\overline{v}_1 - \underline{v}_2} \right)
$$
$$
+ 1 + (b-\underline{v}_2)\left[\log\left(\frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)} \right) \frac{1}{\underline{v}_2 - \underline{v}_1} \right.
$$
$$
\left. -\frac{1}{\overline{v}_1 - \underline{v}_1} - \frac{1}{\overline{v}_2 - \underline{v}_1} - \frac{1}{\overline{v}_1 - \underline{v}_2} - \frac{1}{\overline{v}_2 - \underline{v}_2} \right]
$$
$$
= 1 + (b-\underline{v}_2)\left[\log\left(\frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(b-\underline{v}_1)(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)} \right) \frac{1}{\underline{v}_2 - \underline{v}_1} \right.
$$
$$
\left. -\frac{1}{\overline{v}_1 - \underline{v}_1} - \frac{1}{\overline{v}_2 - \underline{v}_2} \right].
$$
(38)

Step 6. Finding $v_1(b)$.
By substituting of (34) and (38) into (33), we have

$$
\lim_{m \to \underline{v}_2} v_1(b)
$$
$$
= \underline{v}_1 + \frac{\underline{v}_2 - \underline{v}_1}{1 + (b-\underline{v}_2)\left[\log\left(\frac{(b-\underline{v}_2)(\overline{v}_1-\underline{v}_1)(\overline{v}_2-\underline{v}_1)}{(b-\underline{v}_1)(\overline{v}_1-\underline{v}_2)(\overline{v}_2-\underline{v}_2)} \right) \frac{1}{\underline{v}_2-\underline{v}_1} - \frac{1}{\overline{v}_1-\underline{v}_1} - \frac{1}{\overline{v}_2-\underline{v}_2} \right]}.
$$

This is equivalent to Eq. (22) after substituting the expression for c by Eq. (23).

A.10 Proof of Proposition 6: linear solutions

We know in the symmetric case that linear bid functions are possible for the uniform distribution. Here we ask what conditions are necessary for linear solutions to exist in general (for the uniform asymmetric case)?

Recall our two differential equations from the first-order conditions (1):

$$v_1'(b)(v_2(b) - b) = v_1(b) - \underline{v}_1,$$
$$v_2'(b)(v_1(b) - b) = v_2(b) - \underline{v}_2.$$

Assume that there is a linear solution for both inverse bid functions:

$$v_i(b) = \alpha_i b + \beta_i \quad \text{where } \alpha_i > 0.$$

This implies that

$$v_i'(b) = \alpha_i.$$

Substituting this into the above two differential equations yields

$$\alpha_1(\alpha_2 b + \beta_2 - b) = \alpha_1 b + \beta_1 - \underline{v}_1,$$
$$\alpha_2(\alpha_1 b + \beta_1 - b) = \alpha_2 b + \beta_2 - \underline{v}_2.$$

Since this is true for all b, the derivative of both sides must also be equal. Hence,

$$\alpha_1(\alpha_2 - 1) = \alpha_1, \quad \alpha_2(\alpha_1 - 1) = \alpha_2.$$

This implies that $\alpha_1 = \alpha_2 = 2$. Substituting this into the equations yields

$$2\beta_2 = \beta_1 - \underline{v}_1, \quad 2\beta_1 = \beta_2 - \underline{v}_2.$$

Combining these equations shows that

$$\beta_1 = -\frac{1}{3}\underline{v}_1 - \frac{2}{3}\underline{v}_2.$$

By boundary condition B1, $v_1(\underline{b}) = \underline{b}$, we have $\underline{b} = 2\underline{b} + \beta_1$. This implies that $\beta_1 = -\underline{b}$ and $\underline{b} = \frac{1}{3}\underline{v}_1 + \frac{2}{3}\underline{v}_2$. Since $\underline{b} > (\underline{v}_1 + \underline{v}_2)/2$, it must be, by (3), that there is a binding minimum bid $m = \underline{b}$.

Now rewriting, $m = \frac{1}{3}\underline{v}_1 + \frac{2}{3}\underline{v}_2$ yields $m - \underline{v}_1 = 2(\underline{v}_2 - m)$ (or $\underline{v}_2 = \frac{3}{2}m - \frac{1}{2}\underline{v}_1$). Finally, we use the upper boundary conditions in B3 to find that

$$\overline{v}_1 = 2\overline{b} - m,$$
$$\overline{v}_2 = 2\overline{b} - m/2 - \underline{v}_1/2.$$

Elimination of $\overline{b}$ implies that $\overline{v}_1 = \overline{v}_2 + \underline{v}_1/2 - m/2$ (or $\overline{v}_1 - m = \overline{v}_2 - \underline{v}_2$). Thus, if we define z such that $\overline{v}_1 = m + z$, we have $\overline{v}_2 = \frac{3}{2}m + z - \underline{v}_1/2$. Since $m = -\beta_1$ and $\alpha_i = 2$, we have $v_1(b) = 2b - m$. Since $2\beta_2 = \beta_1 - \underline{v}_1$, we have $v_2(b) = 2b - \frac{m}{2} - \frac{v_1}{2}$.

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