1. Hilbert spaces

1.1 Definitions

1.1.1 Vector spaces

Definition 1.1 — Vector space (מרחב וקטורי). A **vector space** over a field $\mathscr{F}$ is a set $\mathscr{V}$ that has the structure of an additive group. Moreover, a product $\mathscr{F} \times \mathscr{V} \to \mathscr{V}$, denoted $(\alpha, x) \mapsto \alpha x$, is defined, satisfying:

① Distributivity in $\mathscr{V}$: $\alpha(x+y) = \alpha x + \alpha y$.
② Distributivity in $\mathscr{F}$: $(\alpha+\beta)x = \alpha x + \beta x$.
③ Homogeneity in $\mathscr{F}$: $\alpha(\beta x) = (\alpha\beta)x$.
④ Scalar unit element: $1 \cdot x = x$.

The elements of $\mathscr{V}$ are called **vectors**; the elements of $\mathscr{F}$ are called **scalars**. Throughout this course the field $\mathscr{F}$ will be either the field of complex numbers $\mathbb{C}$ ($\mathscr{V}$ is a **complex vector space**) or the field of reals $\mathbb{R}$ ($\mathscr{V}$ is a **real vector space**).

Definition 1.2 Let $\mathscr{V}$ be a vector space. A (finite) set of vectors $\{x_1, \dots, x_n\} \subset \mathscr{V}$ is called **linearly independent** (בלתי תלויים ליניארית) if the identity

$$\sum_{k=1}^{n} \alpha_k x_k = 0$$

implies that $\alpha_k = 0$ for all k. Otherwise, this set of elements is said to be **linearly dependent**.

Definition 1.3 If a vector space $\mathscr{V}$ contains n linearly independent vectors and every $n+1$ vectors are linearly dependent, then we say that $\mathscr{V}$ has **dimension** n:

$$\dim \mathscr{V} = n.$$

If $\dim \mathscr{V} \neq n$ for every $n \in \mathbb{N}$ (for every n there exist n linearly independent vectors) then we say that $\mathscr{V}$ has **infinite dimension**.

Proposition 1.1 Let $\mathscr{V}$ be a vector space. Suppose that $\dim \mathscr{V} = n$ and let $(x_1, \ldots, x_n)$ be linearly independent (a **basis**). Then, every $y \in \mathscr{V}$ has a unique representation

$$y = \sum_{k=1}^{n} \alpha_k x_k.$$

Proof. Obvious. ■

Definition 1.4 Let $\mathscr{V}$ be a vector space. A subset $\mathscr{Y} \subset \mathscr{V}$ is called a **vector subspace** (תת מרחב וקטורי) (or a **linear subspace**) if it is a vector space with respect to the same addition and scalar multiplication operations (the vector space structure on $\mathscr{Y}$ is **inherited** from the vector space structure on $\mathscr{V}$).

Proposition 1.2 Let $\mathscr{V}$ be a vector space. A subset $\mathscr{Y} \subset \mathscr{V}$ is a vector subspace if and only if $0 \in \mathscr{Y}$ and for all $y_1, y_2 \in \mathscr{Y}$ and $\alpha_1, \alpha_2 \in \mathscr{F}$,

$$\alpha_1 y_1 + \alpha_2 y_2 \in \mathscr{Y},$$

i.e., the subset $\mathscr{Y}$ is closed under linear combinations.

Proof. Easy. ■

Comment 1.1 By definition, every linear subspace is closed under vector space operations (it is **algebraically closed**). This should not be confused with the topological notion of closedness, which is defined once we endow the vector space with a topology. A linear subspace may not be closed in the topological sense.

Definition 1.5 Let $\mathscr{V}$ and $\mathscr{Y}$ be vector spaces over the same field $\mathscr{F}$; let $\mathscr{D} \subseteq \mathscr{V}$ be a vector subspace. A mapping $T : \mathscr{D} \to \mathscr{Y}$ is said to be a **linear transformation** (העתקה ליניארית) if for all $x_1, x_2 \in \mathscr{D}$ and $\alpha_1 \alpha_2 \in \mathscr{F}$:

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2).$$

The set $\mathscr{D}$ is called the **domain** (תחום) of T. The set

$$\text{Image}(T) = \{T(x) : x \in \mathscr{D}\} \subseteq \mathscr{Y}$$

is called the **image** (תמונה) of T. If $\mathscr{D} = \mathscr{V} = \mathscr{Y}$ we call T a linear transformation on $\mathscr{V}$. If $\mathscr{Y} = \mathscr{F}$ we call T a **linear functional** (פונקציונל ליניארי).

Comment 1.2 Linear transformations preserve the vector space operations, and are therefore the natural isomorphisms in the category of vector spaces. This should be kept in mind, as the natural isomorphisms may change as we endow the vector space with additional structure[1].

Inverse transformation

If $T : \text{Domain}(T) \to \text{Image}(T)$ is one-to-one (injective) then we can define an **inverse transformation**

$$T^{-1} : \text{Image}(T) \to \text{Domain}(T),$$

such that

$$T^{-1}(Tx) = x \qquad \text{and} \qquad T(T^{-1}y) = y$$

for all $x \in \text{Domain}(T)$ and $y \in \text{Image}(T)$.

Notation 1.1 In these notes we will use $A \hookrightarrow B$ to denote injections, $A \twoheadrightarrow B$ to denote surjections, and $A \leftrightarrows B$ to denote bijections.

Proposition 1.3 Let $\mathscr{V}$ be a vector space and $\mathscr{D} \subset \mathscr{V}$ a linear subspace. Let $T : \mathscr{D} \to \mathscr{Y}$ be a linear transformation. Then, $\text{Image}(T)$ is a linear subspace of $\mathscr{Y}$.

Proof. Since $0 \in \mathscr{D}$,

$$\text{Image}(T) \ni T(0) = 0.$$

Let $x, y \in \text{Image}(T)$. By definition, there exist $u, v \in \mathscr{D}$ such that

$$x = T(u) \qquad \text{and} \qquad y = T(v).$$

By the linearity of T, for every $\alpha, \beta \in \mathscr{F}$:

$$\text{Image}(T) \ni T(\alpha u + \beta v) = \alpha x + \beta y.$$

Thus, $\text{Image}(T)$ is closed under linear combinations. ■

[1] A digression on **categories**: A category is an algebraic structure that comprises **objects** that are linked by **morphisms**. A category has two basic properties: the ability to compose the morphisms associatively and the existence of an identity morphism for each object.

A simple example is the category of **sets**, whose morphisms are functions. Another example is the category of **groups**, whose morphisms are homomorphisms. A third example is the category of **topological spaces**, whose morphisms are the continuous functions. As you can see, the chosen morphisms are not just arbitrary associative maps. They are maps that preserve a certain structure in each class of objects.

1.1.2 Normed spaces

A vector space is a set endowed with an algebraic structure. We now endow vector spaces with additional structures – all of them involving **topologies**. Thus, the vector space is endowed with a notion of convergence.

Definition 1.6 — Metric space. A **metric space** (מרחב מטרי) is a set $\mathscr{X}$, endowed with a function $d : \mathscr{X} \times \mathscr{X} \to \mathbb{R}$, such that

① Positivity: $d(x,y) \ge 0$ with equality iff $x = y$.
② Symmetry: $d(x,y) = d(y,x)$.
③ Triangle inequality: $d(x,y) \le d(x,z) + d(z,y)$.

Please note that a metric space does not need to be a vector space. On the other hand, a metric defines a **topology** on $\mathscr{X}$ generated by open balls,

$$\mathfrak{B}(x,r) = \{y \in \mathscr{X} \mid d(x,y) < r\}.$$

As topological spaces, metric spaces are paracompact (every open cover has an open refinement that is locally finite), Hausdorff spaces, and hence **normal** (given any disjoint closed sets E and F, there are open neighborhoods U of E and V of F that are also disjoint). Metric spaces are **first countable** (each point has a countable neighborhood base) since one can use balls with rational radius as a neighborhood base.

Definition 1.7 — Norm. A **norm** (נורמה) over a vector space $\mathscr{V}$ is a mapping $\|\cdot\| : \mathscr{V} \to \mathbb{R}$ such that

① *Positivity*: $\|x\| \ge 0$ with equality iff $x = 0$.
② *Homogeneity*: $\|\alpha x\| = |\alpha|\,\|x\|$.
③ *Triangle inequality*: $\|x+y\| \le \|x\| + \|y\|$.

A **normed space** (מרחב נורמי) is a pair $(\mathscr{V}, \|\cdot\|)$, where $\mathscr{V}$ is a vector space and $\|\cdot\|$ is a norm over $\mathscr{V}$.

A norm is a function that assigns a size to vectors. Any norm on a vector space induces a metric:

Proposition 1.4 Let $(\mathscr{V}, \|\cdot\|)$ be a normed space. Then

$$d(x,y) = \|x-y\|$$

is a metric on $\mathscr{V}$.

Proof. Obvious. ■

The converse is not necessarily true unless certain conditions hold:

Proposition 1.5 Let $\mathscr{V}$ be a vector space endowed with a metric d. If the following two conditions hold:

① *Translation invariance*: $d(x+z, y+z) = d(x,y)$
② *Homogeneity*: $d(\alpha x, \alpha y) = |\alpha| d(x,y)$,

then

$$\|x\| = d(x,0)$$

is a norm on $\mathscr{V}$.

Exercise 1.1 Prove Prop. 1.5. ■

1.1.3 Inner-product spaces

Vector spaces are a very useful construct (e.g., in physics). But to be even more useful, we often need to endow them with structure beyond the notion of a size.

Definition 1.8 — Inner product space. A complex vector field $\mathscr{V}$ is called an **inner-product space** (**מרחב מכפלה פנימית**) if there exists a product $(\cdot,\cdot) : \mathscr{V} \times \mathscr{V} \to \mathbb{C}$, satisfying:

① Symmetry: $(x,y) = \overline{(y,x)}$.
② Bilinearity: $(x+y,z) = (x,z)+(y,z)$.
③ Homogeneity: $(\alpha x, y) = \alpha(x,y)$.
④ Positivity: $(x,x) \geq 0$ with equality iff $x = 0$.

An inner-product space is also called a **pre-Hilbert space**.

Proposition 1.6 An inner-product $(\mathbb{H}, (\cdot,\cdot))$ space satisfies:

① $(x, y+z) = (x,y) + (x,z)$.

② $(x, \alpha y) = \overline{\alpha}(x,y)$.

Proof. Obvious. ∎

Proposition 1.7 — Cauchy-Schwarz inequality. Let $(\mathbb{H}, (\cdot,\cdot))$ be an inner-product space. Define $\|\cdot\| = (\cdot,\cdot)^{1/2}$. Then, for every $x, y \in \mathbb{H}$,

$$|(x,y)| \le \|x\| \, \|y\|.$$

Proof. There are many different proofs to this proposition[2]. For $x, y \in \mathbb{H}$ define $u = x/\|x\|$ and $v = y/\|y\|$. Using the positivity, symmetry, and bilinearity of the inner-product:

$$\begin{aligned} 0 &\le (u - (u,v)v, u - (u,v)v) \\ &= 1 + |(u,v)|^2 - |(u,v)|^2 - |(u,v)|^2 \\ &= 1 - |(u,v)|^2. \end{aligned}$$

That is,

$$\frac{|(x,y)|}{\|x\| \, \|y\|} \le 1.$$

Equality holds if and only if u and v are co-linear, i.e., if and only if x and y are co-linear. ∎

Corollary 1.8 — Triangle inequality. In every inner product space $\mathbb{H}$,

$$\|x+y\| \le \|x\| + \|y\|.$$

Proof. Applying the Cauchy-Schwarz inequality

$$\begin{aligned} \|x+y\|^2 = (x+y, x+y) &= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(x,y) \\ &\le \|x\|^2 + \|y\|^2 + 2|(x,y)| \\ &\le \|x\|^2 + \|y\|^2 + 2\|x\| \, \|y\| = (\|x\| + \|y\|)^2. \end{aligned}$$

∎

[2] There is a book called *The Cauchy-Schwarz Master Class* which presents more proofs that you want.

Corollary 1.9 An inner-product space is a normed space with respect to the norm:

$$\|x\| = (x,x)^{1/2}.$$

Proof. Obvious. ∎

Thus, every inner-product space is automatically a normed space and consequently a metric space. The (default) topology associated with an inner-product space is that induced by the metric (i..e, the open sets are generated by open metric balls).

Exercise 1.2 Show that the inner product $(\mathbb{H},(\cdot,\cdot))$ is continuous with respect to each of its arguments:

$$(\forall x,y \in \mathbb{H})(\forall \varepsilon > 0)(\exists \delta > 0) : (\forall z \in \mathbb{H} \mid \|z-x\| < \delta)(|(z,y)-(x,y)| < \varepsilon).$$

∎

Exercise 1.3 Let $(\mathbb{H},(\cdot,\cdot))$ be a complex inner-product space. Define

$$\langle x,y\rangle = \operatorname{Re}(x,y).$$

Show that $(\mathbb{H},\langle\cdot,\cdot\rangle)$ is a real inner-product space. ∎

Exercise 1.4 Prove that if a collection of non-zero vectors $\{x_1,\dots,x_n\}$ in an inner-product space are mutually orthogonal then they are linearly independent. ∎

Exercise 1.5 Prove that in an inner-product space $x = 0$ iff $(x,y) = 0$ for all y. ∎

Exercise 1.6 Let $(\mathbb{H},(\cdot,\cdot))$ be an inner-product space. Show that the following conditions are equivalent:

① $(x,y) = 0$.
② $\|x+\lambda y\| = \|x-\lambda y\|$ for all $\lambda \in \mathbb{C}$.
③ $\|x\| \le \|x+\lambda y\|$ for all $\lambda \in \mathbb{C}$.

Exercise 1.7 Consider the vector space $\mathscr{V} = C^1[0,1]$ (continuously-differentiable functions over the unit interval) and define the product $(\cdot,\cdot) : \mathscr{V} \times \mathscr{V} \to \mathbb{C}$:

$$(f,g) = \int_0^1 f(x)\overline{g(x)}\,dx.$$

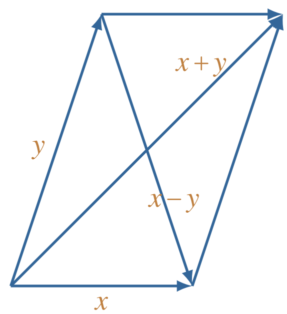
① Is $(\cdot,\cdot)$ an inner-product?

② Set $\mathscr{V}_0 = \{f \in \mathscr{V} \mid f(0) = 0\}$. Is $(\mathscr{V}_0, (\cdot,\cdot))$ an inner-product?

Proposition 1.10 — Parallelogram identity (שוויון המקבילית). In every inner-product space $(\mathbb{H}, (\cdot,\cdot))$:

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

(This equation is called the **parallelogram** identity because it asserts that in a parallelogram the sum of the squares of the sides equals to the sum of the squares of the diagonals.)



Proof. For every $x, y \in \mathbb{H}$:

$$\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \pm 2\,\mathrm{Re}(x,y).$$

The identity follows from adding both equations. ■

An inner product defines a norm. What about a converse? Suppose we are given the the norm induced by an inner product. Can we recover the inner product? The answer is positive, as shown by the following proposition:

Proposition 1.11 — Polarization identity (זהות הפולריזציה). In an inner-product space $(\mathbb{H}, (\cdot,\cdot))$,

$$(x,y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + \imath\|x+\imath y\|^2 - \imath\|x-\imath y\|^2\right).$$

Proof. It is easy to see that

$$\|x+y\|^2 - \|x-y\|^2 = 4\operatorname{Re}(x,y).$$

Setting $y \mapsto \imath y$,

$$\|x+\imath y\|^2 - \|x-\imath y\|^2 = -4\operatorname{Re}\imath(x,y) = 4\operatorname{Im}(x,y).$$

Multiplying the second equation by $\imath$ and adding it to the first equation we obtain the desired result. ∎

Comment 1.3 In a real inner-product space:

$$(x,y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right).$$

Definition 1.9 Let $(\mathbb{H}, (\cdot,\cdot))$ be an inner-product space. $x, y \in \mathbb{H}$ are said to be **orthogonal** (**ניצבים**) if $(x,y) = 0$; we denote $x \perp y$.

Proposition 1.12 Orthogonal vectors in an inner-product space satisfy **Pytagoras' law**:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Proof. Obvious. ∎

Exercise 1.8 Show that a norm $\|\cdot\|$ over a real vector space $\mathscr{V}$ is induced from an inner-product over $\mathscr{V}$ if and only if the parallelogram law holds. Hint: set

$$(x,y) = \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right),$$

and show that it is an inner product and that the induced norm is indeed $\|\cdot\|$. ▪

Exercise 1.9 Show that the ℓ^p spaces (the spaces of sequences with the appropriate norms) can be turned into an inner-product space (i.e., the norm can be induced from an inner-product) only for $p = 2$. ■

1.1.4 Hilbert spaces

Definition 1.10 — Hilbert space. A complete inner-product space is called a **Hilbert space**. (Recall: a space is complete (שלם) if every Cauchy sequence converges.)

Comment 1.4 An inner-product space $(\mathbb{H}, (\cdot,\cdot))$ is a Hilbert space if it is complete with respect to the metric

$$d(x,y) = (x-y, x-y)^{1/2}.$$

Completeness is a property of metric spaces. A sequence $(x_n) \subset \mathbb{H}$ is a Cauchy sequence if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for every $m,n > N$:

$$\|x_n - x_m\| < \varepsilon.$$

Exercise 1.10 Let $\mathbb{H}_1, \dots, \mathbb{H}_n$ be a finite collection of inner-product spaces. Define the space

$$\mathbb{H} = \mathbb{H}_1 \times \cdots \times \mathbb{H}_n,$$

along with coordinate-wise vector space operations. Define a product $(\cdot,\cdot)_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$:

$$((x_1, \dots, x_n), (y_1, \dots, y_n))_{\mathbb{H}} = \sum_{k=1}^{n} (x_k, y_k)_{\mathbb{H}_k}.$$

① Show that $(\cdot,\cdot)_{\mathbb{H}}$ is an inner-product on $\mathbb{H}$.

② Show that convergence in $\mathbb{H}$ is equivalent to component-wise convergence in each of the $\mathbb{H}_k$.

③ Show that $\mathbb{H}$ is complete if and only if all the $\mathbb{H}_k$ are complete.

■

We mentioned the fact that an inner-product space is also called a pre-Hilbert space. The reason for this nomenclature is the following theorem: any inner-product space can be completed *canonically* into a Hilbert space. This completion is analogous to the completion of the field of rationals into the field of reals.

Theorem 1.13 — Completion. Let $(\mathscr{G}, (\cdot,\cdot)_{\mathscr{G}})$ be an inner-product space. Then, there exists a Hilbert space $(\mathscr{H}, (\cdot,\cdot)_{\mathscr{H}})$, such that:

① There exists a linear injection $T : \mathscr{G} \hookrightarrow \mathscr{H}$, that preserves the inner-product, $(x,y)_{\mathscr{G}} = (Tx, Ty)_{\mathscr{H}}$ for all $x, y \in \mathscr{G}$ (i.e., elements in $\mathscr{G}$ can be identified with elements in $\mathscr{H}$).

② Image(T) is dense in $\mathscr{H}$ (i.e., $\mathscr{G}$ is identified with "almost all of" $\mathscr{H}$).

Moreover, the inclusion of $\mathscr{G}$ in $\mathscr{H}$ is unique: For any linear inner-product preserving injection $T_1 : \mathscr{G} \hookrightarrow \mathscr{H}_1$ where $\mathscr{H}_1$ is a Hilbert space and Image(T_1) is dense in $\mathscr{H}_1$, there is a linear isomorphism $S : \mathscr{H} \leftrightarrows \mathscr{H}_1$, such that $T_1 = S \circ T$ (i.e., $\mathscr{H}$ and $\mathscr{H}_1$ are isomorphic in the category of inner-product spaces). In other words, the completion $\mathscr{G}$ is unique modulo isomorphisms.

Proof. We start by defining the space $\mathscr{H}$. Consider the set of Cauchy sequences (x_n) in $\mathscr{G}$. Two Cauchy sequences (x_n) and (y_n) are defined to be equivalent (denoted $(x_n) \sim (y_n)$) if

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

It is easy to see that this establishes an equivalence relation among all Cauchy sequences in $\mathscr{G}$. We denote the equivalence class of a Cauchy sequence (x_n) by $[x_n]$ and define $\mathscr{H}$ as the set of equivalence classes.

We endow $\mathscr{H}$ with a vector space structure by defining

$$\alpha[x_n] + \beta[y_n] = [\alpha x_n + \beta z_n].$$

(It is easy to see that this definition is independent of representing elements.)

Let (x_n) and (y_n) be Cauchy sequence in $\mathscr{G}$. Consider the series

$$\lim_{n\to\infty} (x_n, y_n)_{\mathscr{G}}.$$

This limit exists because

$$\begin{aligned} |(x_n,y_n)_{\mathscr{G}} - (x_m,y_m)_{\mathscr{G}}| &= |(x_n,y_n)_{\mathscr{G}} - (x_n,y_m)_{\mathscr{G}} + (x_n,y_m)_{\mathscr{G}} - (x_m,y_m)_{\mathscr{G}}| \\ \text{(triangle ineq.)} &\le |(x_n, y_n - y_m)_{\mathscr{G}}| + |(x_n - x_m, y_m)_{\mathscr{G}}| \\ \text{(Cauchy-Schwarz)} &\le \|x_n\|_{\mathscr{G}} \, \|y_n - y_m\|_{\mathscr{G}} + \|x_n - x_m\| \, \|y_m\|_{\mathscr{G}}, \end{aligned}$$

Since Cauchy sequences are bounded (easy!), there exists an $M > 0$ such that for all m, n,

$$|(x_n,y_n)_{\mathscr{G}} - (x_m,y_m)_{\mathscr{G}}| \le M \left(\|y_n - y_m\|_{\mathscr{G}} + \|x_n - x_m\|_{\mathscr{G}} \right).$$

It follows that $(x_n, y_n)_{\mathscr{G}}$ is a Cauchy sequence in $\mathscr{F}$, hence converges (because $\mathscr{F}$ is complete).

Moreover, if $(u_n) \sim (x_n)$ and $(v_n) \sim (y_n)$ then,

$$\begin{aligned}|(x_n,y_n)_{\mathscr{G}} - (u_n,v_n)_{\mathscr{G}}| &= |(x_n,y_n)_{\mathscr{G}} - (x_n,v_n)_{\mathscr{G}} + (x_n,v_n)_{\mathscr{G}} - (u_n,v_n)_{\mathscr{G}}| \\ &\le |(x_n,y_n-v_n)_{\mathscr{G}}| + |(x_n-u_n,v_n)_{\mathscr{G}}| \\ &\le \|x_n\|_{\mathscr{G}}\,\|y_n-v_n\|_{\mathscr{G}} + \|x_n-v_n\|_{\mathscr{G}}\,\|v_n\|_{\mathscr{G}},\end{aligned}$$

from which follows that

$$\lim_{n\to\infty}(x_n,y_n)_{\mathscr{G}} = \lim_{n\to\infty}(u_n,v_n)_{\mathscr{G}}.$$

Thus, we can define unambiguously a product $(\cdot,\cdot)_{\mathscr{H}} : \mathscr{H} \to \mathscr{F}$;

$$([x_n],[y_n])_{\mathscr{H}} = \lim_{n\to\infty}(x_n,y_n)_{\mathscr{G}}.$$

It remains to show that $(\cdot,\cdot)_{\mathscr{H}}$ is indeed an inner product (do it).

The next step is to define the inclusion $T : \mathscr{G} \hookrightarrow \mathscr{H}$. For $x \in \mathscr{G}$ let

$$Tx = [(x,x,\dots)],$$

namely, it maps every vector in $\mathscr{G}$ into the equivalence class of a constant sequence. By the definition of the linear structure on $\mathscr{H}$, T is linear. It is preserves the inner-product as

$$(Tx,Ty)_{\mathscr{H}} = \lim_{n\to\infty}((Tx)_n,(Ty)_n)_{\mathscr{G}} = \lim_{n\to\infty}(x,y)_{\mathscr{G}} = (x,y)_{\mathscr{G}}.$$

The next step is to show that $\text{Image}(T)$ is dense in $\mathscr{H}$. Let $h \in \mathscr{H}$ and let (x_n) be a representative of h. Since (x_n) is a Cauchy sequence in $\mathscr{G}$,

$$\lim_{n\to\infty}\|Tx_n - h\|_{\mathscr{H}} = \lim_{n\to\infty}\lim_{k\to\infty}\|x_n - x_k\|_{\mathscr{G}} = 0.$$

which proves that $Tx_n \to h$, and therefore $\text{Image}(T)$ is dense in $\mathscr{H}$.

The next step is to show that $\mathscr{H}$ is complete. Let (h_n) be a Cauchy sequence in $\mathscr{H}$. For every n let $(x_{n,k})$ be a Cauchy sequence in $\mathscr{G}$ in the equivalence class of h_n. Since $\text{Image}(T)$ is dense in $\mathscr{H}$, there exists for every n a $y_n \in \mathscr{G}$, such that

$$\|Ty_n - h_n\|_{\mathscr{H}} = \lim_{k\to\infty}\|y_n - x_{n,k}\|_{\mathscr{G}} \le \frac{1}{n}.$$

It follows that

$$\begin{aligned}\|y_n - y_m\|_{\mathscr{G}} &= \|Ty_n - Ty_m\|_{\mathscr{H}} \\ &\le \|Ty_n - h_n\|_{\mathscr{H}} + \|h_n - h_m\|_{\mathscr{H}} + \|h_m - Ty_m\|_{\mathscr{H}} \\ &\le \|h_n - h_m\|_{\mathscr{H}} + \frac{1}{n} + \frac{1}{m},\end{aligned}$$

i.e., (y_n) is a Cauchy sequence in $\mathscr{G}$ and therefore $h = [y_n]$ is an element of $\mathscr{H}$.

We will show that

$$\lim_{n\to\infty} \|h_n - h\|_{\mathscr{H}} = 0,$$

which will prove that any Cauchy sequence in $\mathscr{H}$ converges.

By definition,

$$\lim_{n\to\infty} \|h_n - h\|_{\mathscr{H}} = \lim_{n\to\infty} \lim_{k\to\infty} \|x_{n,k} - y_k\|_{\mathscr{G}}.$$

Now

$$\|x_{n,k} - y_k\|_{\mathscr{G}} \le \|x_{n,k} - y_n\|_{\mathscr{G}} + \|y_n - y_k\|_{\mathscr{G}},$$

and

$$\lim_{n\to\infty} \lim_{k\to\infty} \|x_{n,k} - y_n\|_{\mathscr{G}} \le \lim_{n\to\infty} \frac{1}{n} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \lim_{k\to\infty} \|y_n - y_k\|_{\mathscr{G}} = 0.$$

The last step is to show the uniqueness of the completion modulo isomorphisms. Let $h \in \mathscr{H}$. Since Image(T) is dense in $\mathscr{H}$, there exists a sequence $(y_n) \subset \mathscr{G}$, such that

$$\lim_{n\to\infty} \|Ty_n - h\|_{\mathscr{H}} = 0.$$

It follows that (Ty_n) is a Cauchy sequence in $\mathscr{H}$, and because T preserves the inner-product, (y_n) is a Cauchy sequence in $\mathscr{G}$. It follows that $(T_1 y_n)$ is a Cauchy sequence in $\mathscr{H}_1$, and because the latter is complete $(T_1 y_n)$ has a limit in $\mathscr{H}_1$. This limit is independent of the choice of the sequence (y_n), hence it is a function of h, which we denote by

$$S(h) = \lim_{n\to\infty} T_1 Y_n.$$

We leave it as an exercise to show that S satisfies the required properties. ∎

Exercise 1.11 Complete the missing details in the above proof. ■

1.1.5 Examples of Hilbert spaces

1. The space $\mathbb{R}^n$ is a real vector space. The mapping

$$(x,y) \mapsto \sum_{i=1}^{n} x_i y_i$$

is an inner product. The induced metric

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

is called the **Euclidean metric**. It is known that $\mathbb{R}^n$ is complete with respect to this metric, hence it is a Hilbert space (in fact, any finite-dimensional normed space is complete, so that the notion of completeness is only of interest in infinite-dimensional spaces).

2. The space $\mathbb{C}^n$ is a complex vector space. The mapping

$$(x,y) \mapsto \sum_{i=1}^{n} x_i \overline{y_i}$$

as an inner product. $\mathbb{C}^n$ endowed with this metric is a Hilbert space.

3. Consider the space of square summable sequences:

$$\ell^2 = \left\{ x \in \mathbb{C}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}.$$

It is a complex vector space with respect to pointwise operations. We define

$$(x,y) = \sum_{i=1}^{\infty} x_i \overline{y_i}.$$

This series converges absolutely as for every finite n, the Cauchy-Schwarz inequality implies:

$$\sum_{i=1}^{n} |x_i \overline{y_i}| \le \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |y_i|^2 \right)^{1/2},$$

and the right hand side is uniformly bounded. It can be shown that ℓ^2 is complete. Moreover, it is easy to show that the subset of rational-valued sequences that have a finite number of non-zero terms is dense in ℓ^2, i.e., ℓ^2 is a **separable** (ספרילי) Hilbert space (has a countable dense subset).

Exercise 1.12 Prove ("by hand") that ℓ^2 is complete. ■

4. Let Ω be a bounded set in $\mathbb{R}^n$ and let $C(\overline{\Omega})$ be the set of continuous complex-valued functions on its closure[3]. This space is made into a complex vector space by pointwise addition and scalar multiplication:

$$(f+g)(x) = f(x) + g(x) \qquad \text{and} \qquad (\alpha f)(x) = \alpha f(x).$$

We define on $C(\overline{\Omega})$ an inner product

$$(f,g) = \int_{\Omega} f(x)\overline{g}(x)\, dx,$$

with the corresponding norm:

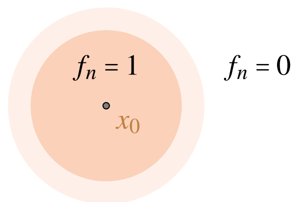
$$\|f\| = \left(\int_{\Omega} |f(x)|^2 \, dx \right)^{1/2}.$$

[3]The fact that the domain is compact is crucial; we shall see later in this course that such a structure cannot be applied for continuous functions over open domains.

This space is not complete and hence not a Hilbert space. To show that, let $x_0 \in \Omega$ and $r > 0$ be such that $B(x_0, 2r) \subset \Omega$. Define the sequence of functions,

$$f_n(x) = \begin{cases} 1 & |x - x_0| \le r \\ 1 + n(r - |x - x_0|) & r \le |x - x_0| \le r + 1/n \\ 0 & |x - x_0| > 1 + 1/n, \end{cases}$$

which are defined for sufficiently large n.



The functions f_n are continuous and converge pointwise to the discontinuous function

$$f(x) = \begin{cases} 1 & |x - x_0| \le r \\ 0 & |x - x_0| > r. \end{cases}$$

The sequence (f_n) is a Cauchy sequence as for $n > m$,

$$\|f_n - f_m\| \le (|B(x_0, r + 1/m) \setminus B(x_0, r)|)^{1/2},$$

which tends to zero as $m, n \to \infty$. Suppose that the space was complete. It would imply the existence of a function $g \in C(\overline{\Omega})$, such that

$$\lim_{n\to\infty} \|f_n - g\| = \lim_{n\to\infty} \left(\int_\Omega |f_n(x) - g(x)|^2 \right)^{1/2} = 0.$$

By Lebesgue's bounded convergence (משפט ההתכנסות החסומה) theorem

$$\lim_{n\to\infty} \|f_n - g\| = \lim_{n\to\infty} \left(\int_\Omega |f(x) - g(x)|^2 \right)^{1/2} = 0,$$

i.e., $g = f$ a.e., which is a contradiction.
The completion of $C(\overline{\Omega})$ with respect to this metric is isomorphic to the Hilbert space $L^2(\overline{\Omega})$ of square integrable functions.

Comment 1.5 The construction provided by the completion theorem is not convenient to work with. We prefer to work with functions rather than with equivalence classes of Cauchy sequences of functions.

TA material 1.1 — Hilbert-Schmidt matrices. Let $\mathscr{M}$ be a collection of all infinite matrices over $\mathbb{C}$ that only have a finite number of non-zero elements. For $A \in \mathscr{M}$ we denote by $n(A)$ the smallest number for which $A_{ij} = 0$ for all $i, j > n(A)$. (i)

Show that $\mathscr{M}$ is a vector space over $\mathbb{C}$ with respect to matrix addition and scalar multiplication. (ii) Define

$$(A,B) = \text{Tr}(AB^*)$$

and show that it is an inner-product. (iii) Show that $\mathscr{M}$ is not complete. (iv) Show that it is possible to identify the completion $\mathscr{H}$ of $\mathscr{M}$ with the set

$$\mathscr{H} = \left\{ A = (a_{ij})_{i,j=1}^{\infty} \mid \sum_{i,j=1}^{\infty} |a_{ij}|^2 < \infty \right\},$$

along with the inner-product

$$(A,B) = \sum_{i,j=1}^{\infty} A_{ij}\overline{B_{ij}}.$$

This space is known as the space of **Hilbert-Schmidt matrices**.

Exercise 1.13 Prove that every finite-dimensional inner product space is complete (and hence a Hilbert space). ■

TA material 1.2 — Sobolev spaces. Endow the space $C^k(\mathbb{R})$ with the inner-product

$$(f,g) = \sum_{i \le k} \int_{\mathbb{R}} \frac{d^i f}{dx_i} \frac{d^i g}{dx_i}\, dx/$$

Its completion is denoted $H^k(\mathbb{R})$ (or $W^{k,2}(\mathbb{R})$). Define the notion of a weak derivative and show that if it exists, then it is unique. Show how to identify $H^k(\mathbb{R})$ with the space of functions that have square-integrable k weak derivatives.

1.2 Convexity and projection

Orthogonality is one of the central concepts in the theory of Hilbert spaces. Another concept, intimately related to orthogonality, is **orthogonal projection** (הטלה ניצבת). Before getting to projections we need to develop the notion of a convex set. Convexity, is a purely algebraic concept, but as we will see, it interacts with the topology induced by the inner-product.

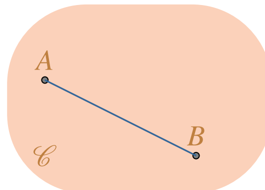
1.2.1 Convexity

Definition 1.11 — Convex set. Let $\mathscr{V}$ be a vector space. A subset $\mathscr{C} \subset \mathscr{V}$ is called **convex** (קמור) if

$$\forall x,y \in \mathscr{C} \quad \text{and} \quad \forall 0 \le t \le 1 \qquad tx + (1-t)y \in \mathscr{C}.$$

(The segment that connects any two points in $\mathscr{C}$ is in $\mathscr{C}$). Differently stated, for all $t \in [0,1]$:

$$t\mathscr{C} + (1-t)\mathscr{C} \subset \mathscr{C}.$$



Lemma 1.14 For any collection of sets $\{\mathscr{C}_\alpha\}$ and $\mathscr{D}_\alpha$ and every $t \in \mathbb{R}$:

$$t \bigcap_{\alpha\in A} \mathscr{C}_\alpha = \bigcap_{\alpha\in A} t\mathscr{C}_\alpha,$$

and

$$\bigcap_{\alpha\in A} \mathscr{C}_\alpha + \bigcap_{\alpha\in A} \mathscr{D}_\alpha \subset \bigcap_{\alpha\in A} (\mathscr{C}_\alpha + \mathscr{D}_\alpha).$$

Proof. First,

$$t \bigcap_{\alpha\in A} \mathscr{C}_\alpha = \{tx \mid x \in \mathscr{C}_\alpha \ \forall\alpha\} = \bigcap_{\alpha\in A} t\mathscr{C}_\alpha.$$

Second, if

$$x \in \bigcap_{\alpha\in A} \mathscr{C}_\alpha + \bigcap_{\alpha\in A} \mathscr{D}_\alpha,$$

then there is a $c \in \mathscr{C}_\alpha$ for all α and a $d \in \mathscr{D}_\beta$ for all β, such that $x = c + d$. Now $c + d = \mathscr{C}_\alpha + \mathscr{D}_\alpha$ for all α, hence

$$x \in \bigcap_{\alpha\in A} (\mathscr{C}_\alpha + \mathscr{D}_\alpha).$$

■

Proposition 1.15 — Convexity is closed under intersections. Let $\mathscr{V}$ be a vector space. Let $\{\mathscr{C}_\alpha \subset \mathscr{V} \mid \alpha \in A\}$ be a collection of convex sets (not necessarily countable). Then

$$\mathscr{C} = \bigcap_{\alpha\in A} \mathscr{C}_\alpha$$

is convex.

Proof. An “obscure" proof relies on Lemma 1.14. For all $t \in [0,1]$:

$$\begin{aligned} t\mathscr{C} + (1-t)\mathscr{C} &= t \bigcap_{\alpha \in A} \mathscr{C}_\alpha + (1-t) \bigcap_{\alpha \in A} \mathscr{C}_\alpha \\ &= \bigcap_{\alpha \in A} t\mathscr{C}_\alpha + \bigcap_{\alpha \in A} (1-t)\mathscr{C}_\alpha \\ &\subset \bigcap_{\alpha \in A} \left(t\mathscr{C}_\alpha + (1-t)\mathscr{C}_\alpha \right) \\ &\subset \bigcap_{\alpha \in A} \mathscr{C}_\alpha = \mathscr{C}. \end{aligned}$$

Now for a more transparent proof: let $x, y \in \mathscr{C}$. By definition:

$$(\forall \alpha \in A)(x, y \in \mathscr{C}_\alpha).$$

Since all the $\mathscr{C}_\alpha$ are convex:

$$(\forall \alpha \in A)(\forall 0 \le t \le 1)\left(tx + (1-t)y \in \mathscr{C}_\alpha\right).$$

Interchanging the order of the quantifiers,

$$(\forall 0 \le t \le 1)(\forall \alpha \in A)\left(tx + (1-t)y \in \mathscr{C}_\alpha\right),$$

which implies that

$$(\forall 0 \le t \le 1)(tx + (1-t)y \in \mathscr{C}).$$

■

Proposition 1.16 — Convex sets are closed under convex linear combinations. Let $\mathscr{V}$ be a vector space. Let $\mathscr{C} \subset \mathscr{V}$ be a convex set. Then for every $(x_1, \dots, x_n) \subset \mathscr{C}$ and every non-negative $(t_1, \dots, t_n)$ real numbers that sum up to 1,

$$\sum_{i=1}^{n} t_i x_i \in \mathscr{C}. \tag{1.1}$$

Proof. Equation (1.1) holds for $n = 2$ by the very definition of convexity. Suppose (1.1) were true for $n = k$. Given

$$(x_1, \dots, x_{k+1}) \subset \mathscr{C} \qquad \text{and} \qquad (t_1, \dots, t_{k+1}) \ge 0, \quad \sum_{i=1}^{k+1} t_i = 1,$$

define $t = \sum_{i=1}^{k} t_i$. Then,

$$\sum_{i=1}^{k+1} t_i x_i = \sum_{i=1}^{k} t_i x_i + t_{k+1} x_{k+1} = \underbrace{t \underbrace{\sum_{i=1}^{k} \frac{t_i}{t} x_i}_{\in \mathscr{C}} + (1-t) x_{k+1}}_{\in \mathscr{C}}$$

■

Proposition 1.17 Let $(\mathscr{X}, \|\cdot\|)$ be a normed space and $\mathscr{C} \subset \mathscr{X}$ a convex subset. Then,

① The closure $\overline{\mathscr{C}}$ is convex.
② The interior $\mathscr{C}^\circ$ is convex.

Comment 1.6 Interior and closure are topological concepts, whereas convexity is a vector space concept. The connection between the two stems from the fact that a normed space has both a topology and a vector space structure.

Proof. ① Let $x, y \in \overline{\mathscr{C}}$. For every $\varepsilon > 0$ there are points $x_\varepsilon, y_\varepsilon \in \mathscr{C}$ with

$$\|x - x_\varepsilon\| < \varepsilon \qquad \text{and} \qquad \|y - y_\varepsilon\| < \varepsilon.$$

Let $0 \le t \le 1$. Then, $tx_\varepsilon + (1-t)y_\varepsilon \in \mathscr{C}$ and

$$\|(tx + (1-t)y) - (tx_\varepsilon + (1-t)y_\varepsilon)\| \le t\|x - x_\varepsilon\| + (1-t)\|y - y_\varepsilon\| < \varepsilon,$$

which implies that $tx + (1-t)y \in \overline{\mathscr{C}}$, hence $\overline{\mathscr{C}}$ is convex.

② Let $x, y \in \mathscr{C}^\circ$. By definition of the interior there exists an $r > 0$ such that

$$B(x,r) \subset \mathscr{C} \qquad \text{and} \qquad B(y,r) \subset \mathscr{C}.$$

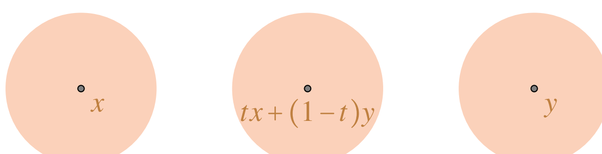
Since $\mathscr{C}$ is convex,

$$\forall t \in [0,1] \qquad t\,B(x,r) + (1-t)B(y,r) \subset \mathscr{C},$$

but

$$B(tx + (1-t)y, r) \subset t\,B(x,r) + (1-t)B(y,r),$$

which proves that $tx + (1-t)y \in \mathscr{C}^\circ$. Hence, $\mathscr{C}^\circ$ is convex.



■

■ Examples 1.1

- Every open **ball** $\mathfrak{B}(a,r)$ in a normed vector space is convex, for if $x, y \in \mathfrak{B}(a,r)$, then for all $0 \le t \le 1$:

$$\|tx + (1-t)y - a\| = \|t(x-a) + (1-t)(y-a)\| \le t\,\|x-a\| + (1-t)\,\|y-a\| < r.$$

- Every **linear subspace** of a vector space is convex, because it is closed under any linear combinations and in particular, convex ones. For example, let $\mathscr{V} = L^2[0,1]$ and let $\mathscr{C}$ be the subset of polynomials. $\mathscr{C}$ is a linear subspace of $\mathscr{V}$, hence it is convex.
- Let $\Omega \subset \mathbb{R}^n$ be a domain and consider the Hilbert space $L^2(\Omega)$. The subset of functions that are **non-negative** (up to a set of measure zero) is convex (but it is not a linear subspace).

■

Exercise 1.14 Let $\mathscr{V}$ be a vector space and $C \subset \mathscr{V}$. The **convex hull** (קמור) of C is defined by

$$\operatorname{Conv}(C) = \{x \in \mathscr{V} \mid x \text{ is a convex combinations of elements in } C\}.$$

Show that $\operatorname{Conv}(C)$ is the smallest convex set that contains C. ■

Exercise 1.15

① Prove Carathéodory's theorem: let $A \subset \mathbb{R}^n$ and let $x \in \operatorname{Conv}(A)$. Then x is a convex combination of $n+1$ points in A or less. (Hint: suppose that x is a convex combination of $x_1, \dots, x_p \in A$, where $p > n+1$. Use the fact that $\{x_i - x_1\}_{i=2}^{p}$ are linearly dependent to show that x can be written as a convex sum of $p-1$ points).

② Show that Carathéodory's theorem may fail if the dimension of the vector space is infinite.

■

1.2.2 Orthogonal projection

Definition 1.12 Let $(\mathbb{H}, (\cdot,\cdot))$ be an inner-product space, and let $\mathcal{S} \subset \mathbb{H}$ (it can be any subset; not necessarily a vector subspace). We denote by $\mathcal{S}^\perp$ the set of vectors that are perpendicular to all the elements in $\mathcal{S}$,

$$\mathcal{S}^\perp = \{x \in \mathbb{H} \mid (x,y) = 0 \quad \forall y \in \mathcal{S}\}.$$

Proposition 1.18 Let $(\mathbb{H},(\cdot,\cdot))$ be an inner-product space. Let $\mathcal{S}\subset\mathbb{H}$. The set $\mathcal{S}^\perp$ is a closed linear subspace of $\mathbb{H}$, and

$$\mathcal{S}\cap\mathcal{S}^\perp\subset\{0\},$$

Proof. We start by showing that $\mathcal{S}^\perp$ is a linear subspace. Let $x,y\in\mathcal{S}^\perp$, i.e.,

$$\forall z\in\mathcal{S}\qquad (x,z)=(y,z)=0.$$

For all $\alpha,\beta\in\mathscr{F}$,

$$\forall z\in\mathcal{S}\qquad (\alpha x+\beta y,z)=\alpha(x,z)+\beta(y,z)=0,$$

which implies that $\alpha x+\beta y\in\mathcal{S}^\perp$, i.e., $\mathcal{S}^\perp$ is a linear subspace.

We next show that $\mathcal{S}^\perp$ is closed. Let (x_n) be a sequence in $\mathcal{S}^\perp$ that converges to $x\in\mathbb{H}$. By the continuity of the inner product,

$$\forall z\in\mathcal{S}\qquad (x,z)=\lim_{n\to\infty}(x_n,z)=0,$$

i.e., $x\in\mathcal{S}^\perp$.

Suppose $x\in\mathcal{S}\cap\mathcal{S}^\perp$. As an element in $\mathcal{S}^\perp$, x is orthogonal to all the elements in $\mathcal{S}$, and in particular to itself, hence $(x,x)=0$, which by the defining property of the inner-product implies that $x=0$. ■

Exercise 1.16 Show that

$$\mathcal{S}^\perp=\bigcap_{x\in\mathcal{S}}\{x\}^\perp.$$

■

Exercise 1.17

① Show that if M and N are closed subspaces of a Hilbert space $\mathscr{H}$, and N is finite dimensional, then $M+N$ is a closed subspace (hint: induction on the dimension of N).

② Show that $M+N$ may not be closed if N is infinite dimensional.

■

The following theorem states that given a closed convex set $\mathscr{C}$ in a Hilbert space $\mathscr{H}$, every point in $\mathscr{H}$ has a unique point in $\mathscr{C}$ that is the closest to it among all points in $\mathscr{C}$:

Theorem 1.19 Let $(\mathscr{H},(\cdot,\cdot))$ be a Hilbert space and $\mathscr{C}\subset\mathscr{H}$ closed and convex. Then,

$$\forall x\in\mathscr{H} \qquad \exists! y\in\mathscr{C} \qquad \text{such that} \qquad \|x-y\| = d(x,\mathscr{C}),$$

where

$$d(x,\mathscr{C}) = \inf_{y\in\mathscr{C}} \|x-y\|.$$

The mapping $x\mapsto y$ is called the **projection** (הטלה) of x onto the set $\mathscr{C}$ and it is denoted by $\mathbb{P}_{\mathscr{C}}$.

Comment 1.7 Note the conditions of this theorem. The space must be complete and the subset must be convex and closed. We will see how these conditions are needed in the proof. A very important point is that the space must be an inner-product space. Projections do not generally exist in (complete) normed spaces.

Proof. We start by showing the **existence** of a distance minimizer. By the definition of the infimum, there exists a sequence $(y_n)\subset\mathscr{C}$ satisfying,

$$\lim_{n\to\infty} d(x,y_n) = d(x,\mathscr{C}).$$

Since $\mathscr{C}$ is convex, $\frac{1}{2}(y_n+y_m)\in\mathscr{C}$ for all m,n, and therefore,

$$\|\tfrac{1}{2}(y_n+y_m)-x\| \ge d(x,\mathscr{C}).$$

By the parallelogram identity (which is where the inner-product property enters), $\|a-b\|^2 = 2(\|a\|^2+2\|b\|^2) - \|a+b\|^2$, and so

$$\begin{aligned}0 \le \|y_n-y_m\|^2 &= \|(y_n-x)-(y_m-x)\|^2\\ &= 2\|y_n-x\|^2+2\|y_m-x\|^2 - \|y_n+y_m-2x\|^2\\ &\le 2\|y_n-x\|^2+2\|y_m-x\|^2 - 2d(x,\mathscr{C}) \overset{m,n\to\infty}{\longrightarrow} 0.\end{aligned}$$

It follows that (y_n) is a Cauchy sequence and hence converges to a limit y (which is where completeness is essential). Since $\mathscr{C}$ is closed, $y\in\mathscr{C}$. Finally, by the continuity of the norm,

$$\|x-y\| = \lim_{n\to\infty}\|x-y_n\| = d(x,\mathscr{C}),$$

which completes the existence proof of a distance minimizer.

Next, we show the **uniqueness** of the distance minimizer. Suppose that $y,z\in\mathscr{C}$ both satisfy

$$\|y-x\| = \|z-x\| = d(x,\mathscr{C}).$$

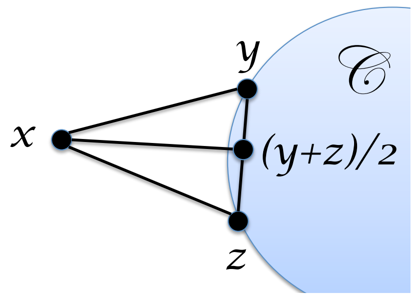
By the parallelogram identity,

$$\|y+z-2x\|^2 + \|y-z\|^2 = 2\|y-x\|^2 + 2\|z-x\|^2,$$

i.e.,

$$\left\|\tfrac{y+z}{2} - x\right\|^2 = d^2(x,\mathscr{C}) - \tfrac{1}{4}\|y-z\|.$$

If $y \neq z$ then $(y+z)/2$, which belongs to $\mathscr{C}$ is closer to x than the distance of x from $\mathscr{C}$, which is a contradiction.



■

TA material 1.3 — Projections in Banach spaces. The existence of a unique projection does not hold in general in complete normed spaces (i.e., Banach spaces). A distance minimizer does exist in finite-dimensional normed spaces, but it may not be unique)=. In infinite-dimensional Banach spaces distance minimizers may fail to exist.

TA material 1.4 — Conditional expectations. The following is an important application of orthogonal projections. Let $(\Omega,\mathscr{F},P)$ be a probability space, and let $\mathscr{A} \subset \mathscr{F}$ be a sub-σ-algebra. Let $X:\Omega \to \mathbb{C}$ be a random variable (i.e., a measurable function) satisfying $\|X\|_1 < \infty$. The random variable $Y:\Omega \to \mathbb{C}$ is called the **conditional expectation** of X with respect to the σ-algebra $\mathscr{A}$ if (i) Y is $\mathscr{A}$-measurable, and (ii) for every $A \in \mathscr{A}$,

$$\int_A Y\,dP = \int_A X\,dP.$$

Prove that the conditional expectation exists and is unique in $L^1(\Omega,\mathscr{F},P)$. (Note that L^1 is not a Hilbert space, so that the construction has to start with the subspace $L^2(\Omega,\mathscr{F},P)$, and end up with a density argument.)

Proposition 1.20 Let $\mathscr{H}$ be a Hilbert space. Let $\mathscr{C}$ be a closed convex set. The mapping $\mathbb{P}_{\mathscr{C}}:\mathscr{H} \to \mathscr{C}$ is idempotent, $\mathbb{P}_{\mathscr{C}} \circ \mathbb{P}_{\mathscr{C}} = \mathbb{P}_{\mathscr{C}}$.

Proof. Obvious, since $\mathbb{P}_{\mathscr{C}} = \mathrm{Id}$ on $\mathscr{C}$. ■

Exercise 1.18 Let $(\mathscr{H},(\cdot,\cdot))$ be a Hilbert space and let $\mathbb{P}_A$ and $\mathbb{P}_B$ be orthogonal projections on closed subspaces A and B.

① Show that *if* $\mathbb{P}_A\mathbb{P}_B$ is an orthogonal projection then it projects on $A \cap B$.
② Show that $\mathbb{P}_A\mathbb{P}_B$ is an orthogonal projection if and only if $\mathbb{P}_A\mathbb{P}_B = \mathbb{P}_B\mathbb{P}_A$.
③ Show that *if* $\mathbb{P}_A\mathbb{P}_B$ is an orthogonal projection then $\mathbb{P}_A + \mathbb{P}_B - \mathbb{P}_A\mathbb{P}_B$ is an orthogonal projection on $A + B$.
④ Find an example in which $\mathbb{P}_A\mathbb{P}_B \neq \mathbb{P}_B\mathbb{P}_A$.

■

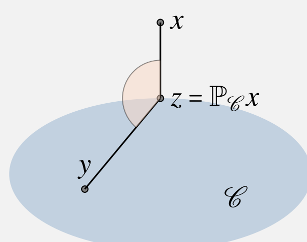
The next proposition has a geometric interpretation: the segment connecting a point $x \notin \mathscr{C}$ with its projection $\mathbb{P}_{\mathscr{C}}x$ makes an obtuse angle with any segment connecting $\mathbb{P}_{\mathscr{C}}x$ with another point in $\mathscr{C}$. The proposition states that this is in fact a characterization of the projection.

Proposition 1.21 Let $\mathscr{C}$ be a closed convex set in a Hilbert space $(\mathscr{H},(\cdot,\cdot))$. Then for every $x \in \mathscr{H}$,

$$z = \mathbb{P}_{\mathscr{C}}x$$

if and only if

$$z \in \mathscr{C} \qquad \text{and} \qquad \forall y \in \mathscr{C} \quad \operatorname{Re}(x - z, y - z) \leq 0.$$



Proof. Suppose first that $z = \mathbb{P}_{\mathscr{C}}x$. By definition $z \in \mathscr{C}$. Let $y \in \mathscr{C}$. Since $\mathscr{C}$ is convex then $ty + (1-t)z \ni \mathscr{C}$ for all $t \in [0,1]$, and since z is the unique distance minimizer from x in $\mathscr{C}$:

$$\begin{aligned} 0 &> \|x - z\|^2 - \|x - (ty + (1-t)z)\|^2 \\ &= \|x - z\|^2 - \|(x - z) - t(y - z)\|^2 \\ &= -t^2\|y - z\|^2 + 2t\operatorname{Re}(x - z, y - z). \end{aligned}$$

Thus, for all $0 < t \leq 1$,

$$\operatorname{Re}(x - z, y - z) < \tfrac{1}{2}t\|y - z\|^2.$$

Letting $t \to 0$ we get that $\operatorname{Re}(x-z, y-z) \le 0$.

Conversely, suppose that $z \in \mathscr{C}$ and that for every $y \in \mathscr{C}$,

$$\operatorname{Re}(x-z, y-z) \le 0.$$

For every $y \in \mathscr{C}$,

$$\begin{aligned} \|x-y\|^2 - \|x-z\|^2 &= \|(x-z)+(z-y)\|^2 - \|x-z\|^2 \\ &= \|y-z\|^2 - 2\operatorname{Re}(x-z, y-z) \ge 0, \end{aligned}$$

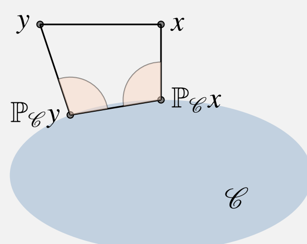
which implies that z is the distance minimizer, hence $z = \mathbb{P}_{\mathscr{C}} x$. ■

Corollary 1.22 — Projections are distance reducing. Let $\mathscr{C}$ be a closed convex set in a Hilbert space $(\mathscr{H}, (\cdot,\cdot))$. Then for all $x, y \in \mathscr{H}$,

$$\operatorname{Re}(\mathbb{P}_{\mathscr{C}} x - \mathbb{P}_{\mathscr{C}} y, x-y) \ge \|\mathbb{P}_{\mathscr{C}} x - \mathbb{P}_{\mathscr{C}} y\|^2$$

and

$$\|\mathbb{P}_{\mathscr{C}} x - \mathbb{P}_{\mathscr{C}} y\|^2 \le \|x-y\|^2.$$



Proof. By Proposition 1.21, with $\mathbb{P}_{\mathscr{C}} y$ as an arbitrary point in $\mathscr{C}$,

$$\operatorname{Re}(x - \mathbb{P}_{\mathscr{C}} x, \mathbb{P}_{\mathscr{C}} y - \mathbb{P}_{\mathscr{C}} x) \le 0.$$

Similarly, with $\mathbb{P}_{\mathscr{C}} x$ as an arbitrary point in $\mathscr{C}$

$$\operatorname{Re}(y - \mathbb{P}_{\mathscr{C}} y, \mathbb{P}_{\mathscr{C}} x - \mathbb{P}_{\mathscr{C}} y) \le 0.$$

Adding up both inequalities:

$$\operatorname{Re}((x-y) - (\mathbb{P}_{\mathscr{C}} x - \mathbb{P}_{\mathscr{C}} y), \mathbb{P}_{\mathscr{C}} x - \mathbb{P}_{\mathscr{C}} y) \ge 0,$$

which proves the first assertion.

Next, using the first assertion and the Cauchy-Schwarz inequality,

$$\begin{aligned} \|\mathbb{P}_{\mathscr{C}} x - \mathbb{P}_{\mathscr{C}} y\|^2 &\le \operatorname{Re}(\mathbb{P}_{\mathscr{C}} x - \mathbb{P}_{\mathscr{C}} y, x-y) \\ &\le |(\mathbb{P}_{\mathscr{C}} x - \mathbb{P}_{\mathscr{C}} y, x-y)| \\ &\le \|\mathbb{P}_{\mathscr{C}} x - \mathbb{P}_{\mathscr{C}} y\| \, \|x-y\|, \end{aligned}$$

and it remains to divide by $\|\mathbb{P}_{\mathscr{C}} x - \mathbb{P}_{\mathscr{C}} y\|$. ■

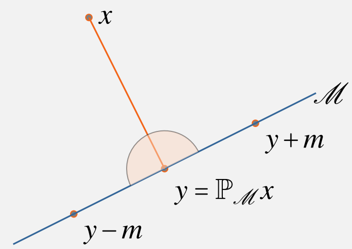
The next corollary characterizes the projection in the case of a closed linear subspace (which is a particular case of a closed convex set).

Corollary 1.23 Let $\mathscr{M}$ by a closed linear subspace of a Hilbert space $(\mathscr{H}, (\cdot,\cdot))$ Then,

$$y = \mathbb{P}_{\mathscr{M}} x$$

if and only if

$$y \in \mathscr{M} \quad \text{and} \quad x - y \in \mathscr{M}^{\perp}.$$



Proof. Let $y \in \mathscr{M}$ and suppose that $x - y \in \mathscr{M}^{\perp}$. Then, for all $m \in \mathscr{M}$:

$$(x-y, \underbrace{m-y}_{\in \mathscr{M}}) = 0 \le 0,$$

hence $y = \mathbb{P}_{\mathscr{M}} x$ by Proposition 1.21.

Conversely, suppose that $y = \mathbb{P}_{\mathscr{M}} x$ and let $m \in \mathscr{M}$. By Proposition 1.21,

$$\operatorname{Re}(y-x, y-m) \le 0.$$

We may replace m by $y - m \in \mathscr{M}$, hence for all $m \in \mathscr{M}$:

$$\operatorname{Re}(y-x, m) \le 0.$$

Since we may replace m by $(-m)$, it follows that for all $m \in \mathscr{M}$:

$$\operatorname{Re}(y-x, m) = 0.$$

Replacing m by $\imath m$ we obtain $\operatorname{Im}(y-x, m) = 0$. ■

Finite dimensional case

This last characterization of the projection provides a constructive way to calculate the projection when $\mathscr{M}$ is a finite-dimensional subspace (hence a closed subspace). Let $n = \dim \mathscr{M}$ and let

$$(e_1, \dots, e_n)$$

be a set of linearly independent vectors in $\mathscr{M}$ (i.e., a basis for $\mathscr{M}$). Let $x \in \mathscr{H}$. Then, $x - \mathbb{P}_{\mathscr{M}}x$ is orthogonal to each of the basis vectors:

$$(x - \mathbb{P}_{\mathscr{M}}x, e_\ell) = 0 \qquad \text{for } \ell = 1, \dots, n.$$

Expanding $\mathbb{P}_{\mathscr{M}}x$ with respect to the given basis:

$$\mathbb{P}_{\mathscr{M}}x = \sum_{k=1}^{n} \alpha_k e_k,$$

we obtain,

$$(x, e_\ell) = \sum_{k=1}^{n} \alpha_k (e_k, e_\ell) = 0 \qquad \text{for } \ell = 1, \dots, n.$$

The matrix G whose entries are $G_{ij} = (e_i, e_j)$ is known as the **Gram matrix**. Because the e_i's are linearly independent, this matrix is non-singular, and

$$\alpha_k = \sum_{\ell=1}^{n} G^{-1}_{k\ell}(x, e_\ell),$$

i.e., we have an explicit expression for the projection of any vector:

$$\mathbb{P}_{\mathscr{M}}x = \sum_{k=1}^{n} \sum_{\ell=1}^{n} G^{-1}_{k\ell}(x, e_\ell) e_k.$$

Exercise 1.19 Let $\mathscr{H} = L^2(\mathbb{R})$ and set

$$\mathscr{M} = \{ f \in \mathscr{H} \mid f(t) = f(-t) \text{ a.e.} \}.$$

① Show that $\mathscr{M}$ is a closed subspace.
② Express the projection $\mathbb{P}_{\mathscr{M}}$ explicitly.
③ Find $\mathscr{M}^\perp$.

■

Exercise 1.20 What is the orthogonal complement of the following sets of $L^2[0,1]$?

① The set of polynomials.
② The set of polynomials in x^2.
③ The set of polynomials with $a_0 = 0$.
④ The set of polynomials with coefficients summing up to zero.

Theorem 1.24 — Projection theorem (משפט ההטלה). Let $\mathscr{M}$ be a closed linear subspace of a Hilbert space $(\mathscr{H},(\cdot,\cdot))$. Then every vector $x\in\mathscr{H}$ has a unique decomposition

$$x = m+n \qquad m\in\mathscr{M},\, n\in\mathscr{M}^{\perp}.$$

Furthermore, $m=\mathbb{P}_{\mathscr{M}}x$. In other words,

$$\mathscr{H}=\mathscr{M}\oplus\mathscr{M}^{\perp}.$$

Proof. Let $x\in\mathscr{H}$. By Corollary 1.23

$$x-\mathbb{P}_{\mathscr{M}}x\in\mathscr{M}^{\perp},$$

hence

$$x=\mathbb{P}_{\mathscr{M}}x+(x-\mathbb{P}_{\mathscr{M}}x)$$

satisfies the required properties of the decomposition.

Next, we show that the decomposition is unique. Assume

$$x=m_1+n_1=m_2+n_2,$$

where $m_1,m_2\in\mathscr{M}$ and $n_1,n_2\in\mathscr{M}^{\perp}$. Then,

$$\mathscr{M}\ni m_1-m_2=n_2-n_1\in\mathscr{M}^{\perp}.$$

Uniqueness follows from the fact that $\mathscr{M}\cap\mathscr{M}^{\perp}=\{0\}$.

■

TA material 1.5 Show that the projection theorem does not holds when the conditions are not satisfied. Take for example $\mathscr{H}=\ell^2$, with the linear subspace

$$\mathscr{M}=\{(a_n)\in\ell^2 \mid \exists N:\forall n>N\ a_n=0\}.$$

This linear subspace it not closed, and its orthogonal complement is $\{0\}$, i.e.,

$$\mathscr{M}\oplus\mathscr{M}^{\perp}=\mathscr{M}\neq\mathscr{H}.$$

Corollary 1.25 For every linear subspace $\mathscr{M}$ of a Hilbert space $\mathscr{H}$,

$$(\mathscr{M}^{\perp})^{\perp}=\overline{\mathscr{M}}.$$

Proof. Let $m \in \mathscr{M}$. By the definition of $\mathscr{M}^\perp$:

$$\forall n \in \mathscr{M}^\perp \qquad (m,n) = 0,$$

which implies that $m \in (\mathscr{M}^\perp)^\perp$, i.e.,

$$\mathscr{M} \subseteq (\mathscr{M}^\perp)^\perp.$$

By Proposition 1.18 any orthogonal complement is closed. If a set is contained in a closed set, so is its closure (prove it!),

$$\overline{\mathscr{M}} \subseteq (\mathscr{M}^\perp)^\perp.$$

Let $x \in (\mathscr{M}^\perp)^\perp$. Since $\overline{\mathscr{M}}$ is a closed linear subspace, there exists a (unique) decomposition

$$x = m + n, \qquad m \in \overline{\mathscr{M}} \quad n \in (\overline{\mathscr{M}})^\perp.$$

Taking an inner product with n, using the fact that $m \perp n$:

$$\|n\|^2 = (x,n).$$

Since $\mathscr{M} \subset \overline{\mathscr{M}}$, then

$$(\mathscr{M})^\perp \supset (\overline{\mathscr{M}})^\perp$$

(the smaller the set, the larger its orthogonal complement). Thus

$$n \in (\mathscr{M})^\perp,$$

and therefore $n \perp x$. It follows that $n = 0$, which means that $x \in \overline{\mathscr{M}}$, i.e.,

$$(\mathscr{M}^\perp)^\perp \subseteq \overline{\mathscr{M}}.$$

This completes the proof. ∎

Corollary 1.26 Let $\mathscr{M}$ be a closed linear subspace of a Hilbert space $(\mathscr{H}, (\cdot,\cdot))$. Then, every $x \in \mathscr{H}$ has a decomposition

$$x = \mathbb{P}_{\mathscr{M}} x + \mathbb{P}_{\mathscr{M}^\perp} x,$$

and

$$\|x\|^2 = \|\mathbb{P}_{\mathscr{M}} x\|^2 + \|\mathbb{P}_{\mathscr{M}^\perp} x\|^2.$$

Proof. As a consequence of the projection theorem, using the fact that both $\mathscr{M}$ and $\mathscr{M}^\perp$ are closed and the fact that $(\mathscr{M}^\perp)^\perp = \mathscr{M}$:

$$x = \underbrace{\mathbb{P}_{\mathscr{M}} x}_{\in \mathscr{M}} + \underbrace{(x - \mathbb{P}_{\mathscr{M}} x)}_{\in \mathscr{M}^\perp}$$

$$x = \underbrace{(x - \mathbb{P}_{\mathscr{M}^\perp} x)}_{\in \mathscr{M}} + \underbrace{\mathbb{P}_{\mathscr{M}^\perp} x}_{\in \mathscr{M}^\perp}.$$

By the uniqueness of the decomposition, both decompositions are identical, which proves the first part. The second identity follows from Pythagoras' law. ■

Corollary 1.27 — A projection is linear, norm-reducing and idempotent. Let $\mathscr{M}$ be a closed linear subspace of a Hilbert space $(\mathscr{H},(\cdot,\cdot))$. Then the projection $\mathbb{P}_{\mathscr{M}}$ is a linear operator satisfying $\mathbb{P}_{\mathscr{M}}^2 = \mathbb{P}_{\mathscr{M}}$, and

$$\forall x \in \mathscr{H} \qquad \|\mathbb{P}_{\mathscr{M}} x\| \le \|x\|.$$

Proof. ① We have already seen that $\mathbb{P}_{\mathscr{M}}$ is idempotent.

② The norm reducing property follows from

$$\|x\|^2 = \|\mathbb{P}_{\mathscr{M}} x\|^2 + \|\mathbb{P}_{\mathscr{M}^\perp} x\|^2 \ge \|\mathbb{P}_{\mathscr{M}} x\|^2.$$

③ It remains to show that $\mathbb{P}_{\mathscr{M}}$ is linear. Let $x,y \in \mathscr{H}$. It follows from Corollary 1.26 that

$$\begin{aligned} x &= \mathbb{P}_{\mathscr{M}} x + \mathbb{P}_{\mathscr{M}^\perp} x \\ y &= \mathbb{P}_{\mathscr{M}} y + \mathbb{P}_{\mathscr{M}^\perp} y, \end{aligned}$$

hence

$$x + y = \underbrace{(\mathbb{P}_{\mathscr{M}} x + \mathbb{P}_{\mathscr{M}} y)}_{\in \mathscr{M}} + \underbrace{(\mathbb{P}_{\mathscr{M}^\perp} x + \mathbb{P}_{\mathscr{M}^\perp} y)}_{\in \mathscr{M}^\perp}.$$

On the other hand, it also follows from Corollary 1.26 that

$$x + y = \underbrace{\mathbb{P}_{\mathscr{M}}(x+y)}_{\in \mathscr{M}} + \underbrace{\mathbb{P}_{\mathscr{M}^\perp}(x+y)}_{\in \mathscr{M}^\perp}.$$

By the uniqueness of the decomposition,

$$\mathbb{P}_{\mathscr{M}}(x+y) = \mathbb{P}_{\mathscr{M}} x + \mathbb{P}_{\mathscr{M}} y.$$

Similarly,

$$\alpha x = \alpha \mathbb{P}_{\mathscr{M}} x + \alpha \mathbb{P}_{\mathscr{M}^\perp} x,$$

but also

$$\alpha x = \mathbb{P}_{\mathscr{M}}(\alpha x) + \mathbb{P}_{\mathscr{M}^\perp}(\alpha x),$$

and from the uniqueness of the decomposition,

$$\mathbb{P}_{\mathscr{M}}(\alpha x) = \alpha \mathbb{P}_{\mathscr{M}} x.$$

■

The next theorem shows that the last corollary is in fact a characterization of projections: there is a one-to-one correspondence between closed subspaces of $\mathscr{H}$ and orthogonal projections.

Theorem 1.28 — Every linear, norm-reducing, idempotent operator is a projection. Let $(\mathscr{H},(\cdot,\cdot))$ be a Hilbert space and let $\mathbb{P}:\mathscr{H}\to\mathscr{H}$ be a linear, norm reducing, idempotent operator. Then $\mathbb{P}$ is a projection on a closed linear subspace of $\mathscr{H}$.

Proof. The first step is to identify the closed subspace of $\mathscr{H}$ that $\mathbb{P}$ projects onto. Define

$$\mathscr{M} = \{x\in\mathscr{H} \mid \mathbb{P}x = x\}$$
$$\mathscr{N} = \{y\in\mathscr{H} : \mid \mathbb{P}y = 0\}.$$

Both $\mathscr{M}$ and $\mathscr{N}$ are **linear subspaces** of $\mathscr{H}$. If $x\in\mathscr{M}\cap\mathscr{N}$ then $x=\mathbb{P}x=0$, namely,

$$\mathscr{M}\cap\mathscr{N} = \{0\},$$

Both $\mathscr{M}$ and $\mathscr{N}$ are **closed** because for every $x,y\in\mathscr{H}$:

$$\|\mathbb{P}x-\mathbb{P}y\| = \|\mathbb{P}(x-y)\| \le \|x-y\|,$$

from which follows that if $x_n\in\mathscr{M}$ is a sequence with limit $x\in\mathscr{H}$, then

$$\|x-\mathbb{P}x\| = \lim_{n\to\infty}\|x_n-\mathbb{P}x\| = \lim_{n\to\infty}\|\mathbb{P}x_n-\mathbb{P}x\| \le \lim_{n\to\infty}\|x_n-x\| = 0,$$

i.e., $\mathbb{P}x=x$, hence $x\in\mathscr{M}$. By a similar argument ee show that $\mathscr{N}$ is closed.

Let $x\in\mathscr{H}$. We write

$$x = \mathbb{P}x + (\mathrm{Id}-\mathbb{P})x.$$

By the idempotence of $\mathbb{P}$,

$$\mathbb{P}x\in\mathscr{M} \qquad \text{and} \qquad (\mathrm{Id}-\mathbb{P})x\in\mathscr{N}.$$

To prove that $\mathbb{P}=\mathbb{P}_{\mathscr{M}}$ it remains to show that $\mathscr{N}=\mathscr{M}^\perp$ (because of the uniqueness of the decomposition). Let

$$x\in\mathscr{N}^\perp \qquad \text{and} \qquad y = \mathbb{P}x - x.$$

Obviously, $y\in\mathscr{N}$, hence $(x,y)=0$, and

$$\|x\|^2 \ge \|\mathbb{P}x\|^2 = \|x+y\|^2 = \|x\|^2+\|y\|^2 \ge \|x\|^2,$$

i.e., $y=0$, i.e., $x=\mathbb{P}x$, i.e., $x\in\mathscr{M}$. We have just shown that

$$\mathscr{N}^\perp\subseteq\mathscr{M}.$$

Take now $x \in \mathscr{M}$. By the projection theorem, there exists a unique decomposition,

$$\underbrace{x}_{\in \mathscr{M}} = \underbrace{u}_{\in \mathscr{N}} + \underbrace{v}_{\in \mathscr{N}^\perp}.$$

But since $\mathscr{N}^\perp \subseteq \mathscr{M}$, it follows that u is both in $\mathscr{M}$ and in $\mathscr{N}$, i.e., it is zero and $x \in \mathscr{N}^\perp$, namely

$$\mathscr{M} \subseteq \mathscr{N}^\perp.$$

Thus $\mathscr{N}^\perp = \mathscr{M}$ and further $\mathscr{N} = (\mathscr{N}^\perp)^\perp = \mathscr{M}^\perp$. This concludes the proof. ■

1.3 Linear functionals

Among all linear maps between normed spaces stand out the linear maps into the field of scalars. The study of linear functionals is a central theme in functional analysis.

1.3.1 Boundedness and continuity

Definition 1.13 Let $(\mathscr{X}_1, \|\cdot\|_1)$ and $(\mathscr{X}_2, \|\cdot\|_2)$ be normed spaces and let $\mathscr{D} \subseteq \mathscr{X}_1$ be a linear subspace. A linear transformation $T : \mathscr{D} \to \mathscr{X}_2$ is said to be **continuous** if

$$\forall x \in \mathscr{D} \qquad \lim_{y\to x} \|Ty - Tx\|_2 \to 0.$$

It is said to be **bounded** if there exists a constant $C > 0$ such that

$$\forall x \in \mathscr{D} \qquad \|Tx\|_2 \le C\,\|x\|_1.$$

If T is bounded, then the lowest bound C is called the **norm** of T:

$$\|T\| = \sup_{0\ne x\in\mathscr{D}} \frac{\|Tx\|_2}{\|x\|_1} = \sup_{0\ne x\in\mathscr{D}} \left\| T\frac{x}{\|x\|_1} \right\|_2 = \sup_{\|x\|_1=1} \|Tx\|_2.$$

(Recall that if $\mathscr{X}' = \mathbb{R}$ then we call T a **linear functional**.)

Comments 1.1

① We are dealing here with normed spaces; no inner-product is needed.

② As for now, we call $\|T\|$ a norm, but we need to show that it is indeed a norm on a vector space.

③ If T is bounded then

$$\forall x \in \mathscr{D} \qquad \|Tx\| \le \|T\|\,\|x\|.$$

④ All linear operators between finite-dimensional normed spaces are bounded. This notion is therefore only relevant to infinite-dimensional cases.

Proposition 1.29 — Boundedness and continuity are equivalent. Let $(\mathscr{X}_1, \|\cdot\|_1)$ and $(\mathscr{X}_2, \|\cdot\|_2)$ be normed spaces. A linear operator $T : \mathscr{D} \subseteq \mathscr{X}_1 \to \mathscr{X}_2$ is bounded if and only if it is continuous.

Proof. ① Suppose T is bounded. Then,

$$\|Tx - Ty\|_2 = \|T(x-y)\|_2 \le \|T\| \, \|x-y\|_1,$$

and $y \to x$ implies $Ty \to Tx$.

① Suppose that T is continuous at $0 \in \mathscr{D}$. Then there exists a $\delta > 0$ such that

$$\forall y \in \mathfrak{B}(0,\delta) \qquad \|Ty - 0\|_2 \le 1.$$

Using the homogeneity of the norm and the linearity of T:

$$\forall x \in \mathscr{D} \qquad \|Tx\|_2 = \frac{2}{\delta}\|x\|_1 \left\| T\left(\frac{\delta}{2}\frac{x}{\|x\|_1}\right)\right\|_2 \le \frac{2}{\delta}\|x\|_1.$$

which implies that T is bounded, $\|T\| \le 2/\delta$. ■

Comment 1.8 We have only used the continuity of T at zero. This means that if T is continuous at zero, then it is bounded, and hence continuous everywhere.

■ Examples 1.2

1. An orthogonal projection in a Hilbert space is bounded, since

$$\|\mathbb{P}_{\mathscr{M}} x\| \le \|x\|,$$

i.e., $\|\mathbb{P}_{\mathscr{M}}\| \le 1$. Since $\mathbb{P}_{\mathscr{M}} x = x$ for $x \in \mathscr{M}$, it follows that

$$\|\mathbb{P}_{\mathscr{M}}\| = \sup_{\|x\|=1} \|\mathbb{P}_{\mathscr{M}} x\| \ge \sup_{x\in\mathscr{M}\ \|x\|=1} \|\mathbb{P}_{\mathscr{M}} x\| = \sup_{x\in\mathscr{M}\ \|x\|=1} \|x\| = 1,$$

$\|\mathbb{P}_{\mathscr{M}}\| = 1$.

2. Let $\mathscr{H} = L^2[0,1]$ and let $\mathscr{D} = C[0,1] \subset L^2[0,1]$. Define the linear functional $T : \mathscr{D} \to \mathbb{R}$,

$$Tf = f(0).$$

This operator is unbounded (and hence not-continuous), for take the sequence of functions,

$$f_n(x) = \begin{cases} 1 - nx & 0 \le x \le \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}.$$

Then $\|f_n\| \to 0$, whereas $\|Tf_n\| = |f_n(0)| = 1$, i.e.,

$$\lim_{n\to\infty} \frac{\|Tf_n\|}{\|f_n\|} = \infty.$$

3. Consider the Hilbert space $\mathscr{H} = L^2[0,1]$ with $\mathscr{D}$ the subspace of differentiable functions with derivatives in $\mathscr{H}$. Define the linear operator $T : \mathscr{D} \to \mathscr{H}$,

$$(Tf)(x) = f'(x).$$

This operator is **unbounded**. Take for example the sequence of functions,

$$f_n(x) = \left(\frac{1}{\sqrt{2\pi n}} e^{-nx^2/2} \right)^{1/2}.$$

Then $\|f_n\| = 1$ and $\lim_{n\to\infty} \|Tf_n\| = \infty$.

4. Important example! Let $\mathscr{H}$ be a Hilbert space. Set $y \in \mathscr{H}$ and define the functional:

$$T_y = (\cdot, y).$$

This functional is linear, and it is bounded as

$$\|y\| = \frac{|(y,y)|}{\|y\|} \le \sup_{x\neq 0} \frac{|(x,y)|}{\|x\|} \le \sup_{x\neq 0} \frac{\|y\|\,\|x\|}{\|x\|} = \|y\|,$$

hence

$$\|T\| = \sup_{x\neq 0} \frac{|T_y(x)|}{\|x\|} = \|y\|.$$

In other words, to every $y \in \mathscr{H}$ corresponds a bounded linear functional T_y.

■

1.3.2 Extension of bounded linear functionals

Lemma 1.30 Given a bounded linear functional T defined on a dense linear subspace $\mathscr{D}$ of a Hilbert space $(\mathscr{H}, (\cdot,\cdot))$, it has a unique extension $\bar{T}$ over $\mathscr{H}$. Moreover, $\|\bar{T}\| = \|T\|$.

Proof. We start by defining $\bar{T}$. For $x \in \mathscr{H}$, take a sequence $(x_n) \subset \mathscr{D}$ that converges to x. Consider the sequence Tx_n. Since T is linear and bounded,

$$|Tx_n - Tx_m| \le |T(x_n - x_m)| \le \|T\|\,\|x_n - x_m\|,$$

which implies that (Tx_n) is a scalar-valued Cauchy sequence. The limit does not depend on the chosen sequence: if $(y_n) \subset \mathscr{D}$ converges to x as well, then

$$\lim_{n\to\infty} |Tx_n - Ty_n| \le \lim_{n\to\infty} \|T\|\,\|x_n - y_n\| = 0.$$

Thus, we can define unambiguously

$$\bar{T}x = \lim_{n\to\infty} Tx_n.$$

For $x \in \mathscr{D}$ we can take the constant sequence $x_n = x$, hence

$$\bar{T}x = \lim_{n\to\infty} Tx_n = Tx,$$

which shows that $\bar{T}$ is indeed an extension of T: $\bar{T}|_{\mathscr{D}} = T$.

Next, we show that $\bar{T}$ is linear. Let $x, y \in \overline{\mathscr{D}}$. Let $x_n, y_n \in \mathscr{D}$ converge to x, y, respectively, then:

$$\bar{T}(\alpha x + \beta y) = \lim_{n\to\infty} T(\alpha x_n + \beta y_n) = \alpha \lim_{n\to\infty} Tx_n + \beta \lim_{n\to\infty} Ty_n = \alpha \bar{T}x + \beta \bar{T}y.$$

It remains to calculate the norm of $\bar{T}$:

$$\|\bar{T}\| = \sup_{x\neq 0} \frac{\|\bar{T}x\|}{\|x\|} = \sup_{x\neq 0} \frac{\lim_{n\to\infty}\|Tx_n\|}{\|x\|} \le \sup_{x\neq 0} \frac{\|T\|\lim_{n\to\infty}\|x_n\|}{\|x\|} = \|T\|,$$

and since $\|\bar{T}\|$ is an extension of T: $\|\bar{T}\| = \|T\|$. ■

The following theorem is an instance of the **Hahn-Banach** theorem, which we will meet when we study Banach spaces:

Theorem 1.31 — Extension theorem. Given a bounded linear functional T defined on a linear subspace $\mathscr{D}$ of a Hilbert space $(\mathscr{H}, (\cdot,\cdot))$, it can be extended into a linear functional over all $\mathscr{H}$, without changing its norm. That is, there exists a linear functional $\bar{T}$ on $\mathscr{H}$, such that $\bar{T}|_{\mathscr{D}} = T$ and $\|\bar{T}\| = \|T\|$.

Proof. By the previous lemma we may assume without loss of generality that $\mathscr{D}$ is a closed linear subspace of $\mathscr{H}$. We define

$$\bar{T} = T \circ \mathbb{P}_{\mathscr{D}}.$$

Since $\bar{T}$ is a composition of two linear operators, it is linear. Also, $\bar{T}|_{\mathscr{D}} = T$. Finally,

$$\|\bar{T}\| = \|T \circ \mathbb{P}_{\mathscr{D}}\| \le \|T\|\,\|\mathbb{P}_{\mathscr{D}}\| = \|T\|.$$

Since $\bar{T}$ is an extension of T it follows that $\|\bar{T}\| = \|T\|$. ■

TA material 1.6 — Hamel basis.

Definition 1.14 Let $\mathscr{V}$ be a vector space. A set of vectors $\{v_\alpha \mid \alpha \in A\} \subset \mathscr{V}$ is called a **Hamel basis** (or algebraic basis) if every $v \in \mathscr{V}$ has a unique representation as a linear combination of a finite number of vectors from $\{v_\alpha \mid \alpha \in A\}$.

Proposition 1.32 Every vector space $\mathscr{V}$ has a Hamel basis.

Exercise 1.21 Prove it. Hint: use the axiom of choice. ▪

Proposition 1.33 Let $(\mathscr{X}, \|\cdot\|)$ be an infinite-dimensional normed space. Then, there exists an unbounded linear functional on $\mathscr{X}$.

Proof. Let $\{x_\alpha \mid \alpha \in A\}$ be a Hamel basis. Since $\mathscr{X}$ is infinite-dimensional there is a sequence (x_{α_n}) such that

$$\{x_{\alpha_n} \mid n \in \mathbb{N}\}$$

is linearly independent.

For $x = \sum_{\alpha\in A} t_\alpha x_\alpha$ define

$$\Phi(x) = \sum_{n=1}^{\infty} n\|x_{\alpha_n}\| t_{\alpha_n}.$$

It is easy to see that this is a linear functional. However, it is not continuous. Define

$$y_n = \frac{x_{\alpha_n}}{n\|x_{\alpha_n}\|},$$

Then, $y_n \to 0$ by $\Phi(y_n) = 1$ for all n. ■

1.3.3 The Riesz representation theorem

The next (very important) theorem asserts that all bounded linear functionals on a Hilbert space can be represented as an inner-product with a fixed element of $\mathscr{H}$:

Theorem 1.34 — Riesz representation theorem, 1907. Let $\mathscr{H}$ be a Hilbert space and T a bounded linear functional on $\mathscr{H}$. Then,

$$(\exists! y_T \in \mathscr{H}) : T = (\cdot, y_T),$$

and moreover $\|T\| = \|y_T\|$.

Comment 1.9 The representation theorem was proved by Frigyes Riesz (1880–1956), a Hungarian mathematician, and brother of the mathematician Marcel Riesz (1886–1969).

Proof. We start by proving the uniqueness of the representation. If y and z satisfy $T = (\cdot, y) = (\cdot, z)$, then

$$\|y-z\|^2 = (y-z, y-z) = (y-z, y) - (y-z, z) = T(y-z) - T(y-z) = 0,$$

which implies that $y = z$.

Next, we show the existence of y_T. Since T is linear it follows that $\ker T$ is a linear subspace of $\mathscr{H}$. Since T is moreover continuous it follows that $\ker T$ is closed, as $(x_n) \subset \ker T$ with limit $x \in \mathscr{H}$, implies that

$$Tx = \lim_{n\to\infty} Tx_n = 0,$$

i.e., $x \in \ker T$.

If $\ker T = \mathscr{H}$, then

$$\forall x \in \mathscr{H} \qquad Tx = 0 = (x, 0),$$

and the theorem is proved with $y_T = 0$.

If $\ker T \neq \mathscr{H}$, then we will show that $\dim(\ker T)^\perp = 1$. Let $y_1, y_2 \in (\ker T)^\perp$, and set

$$y = T(y_2)y_1 - T(y_1)y_2 \in (\ker T)^\perp.$$

By the linearity of T, $T(y) = 0$, i.e., $y \in \ker T$, but since $\ker T \cap (\ker T)^\perp = \{0\}$, it follows that

$$T(y_2)y_1 = T(y_1)y_2,$$

i.e., every two vectors $(\ker T)^\perp$ are co-linear.

Take $y_0 \in (\ker T)^\perp$ with $\|y_0\| = 1$. Then,

$$(\ker T)^\perp = \operatorname{Span}\{y_0\}.$$

Then, set

$$y_T = \overline{T(y_0)}\, y_0.$$

By the projection theorem, for every $x \in \mathscr{H}$,

$$x = \underbrace{(x, y_0)y_0}_{\in(\ker T)^\perp} + \underbrace{[x - (x, y_0)y_0]}_{\in\ker T},$$

and applying T,

$$T(x) = (x, y_0)T(y_0) = (x, y_T).$$

Finally, we have already seen that $\|T\| = \|y_T\|$. ■

The space dual to a Hilbert space

Consider the set of all bounded linear functionals on a Hilbert space. These form a vector space by the pointwise operations,

$$(\alpha T + \beta S)(x) = \alpha T(x) + \beta S(x).$$

We denote this vector space by $\mathscr{H}^*$; it is called the space **dual** (מרחב דואלי) to $\mathscr{H}$. The Riesz representation theorem states that there is a bijection $\mathscr{H}^* \leftrightarrows \mathscr{H}$, $T \mapsto y_T$. $\mathscr{H}^*$ is made into a Hilbert space by defining the inner-product

$$(T,S) = (y_T, y_S),$$

and the corresponding norm over $\mathscr{H}^*$ is

$$\|T\| = \|y_T\|$$

coincides with the previously-defined “norm" (we never showed it was indeed a norm)[4].

The following theorem (the **Radon-Nikodym theorem** restricted to finite measure spaces) is an application of the Riesz representation theorem:

Theorem 1.35 — Radon-Nikodym. Let $(\Omega, \mathscr{B}, \mu)$ be a finite measure space. If ν is a finite measure on $(\Omega, \mathscr{B})$ that is absolutely continuous with respect to μ (i.e., every zero set of μ is also a zero set of ν), then there exists a non-negative function $f \in L^1(\Omega)$, such that

$$\forall B \in \mathscr{B} \qquad \nu(B) = \int_B f \, d\mu.$$

(The function f is called the **density** of ν with respect to μ.)

Proof. Let $\lambda = \mu + \nu$. Every zero set of μ is also a zero set of λ. Define the functional $F : L^2(\lambda) \to \mathbb{C}$:

$$F(g) = \int_\Omega g \, d\nu.$$

F is a **linear** functional; it is **bounded** as

$$|F(g)| \le \int_\Omega |g| \, d\nu \le \left(\int_\Omega |g|^2 \, d\nu \right)^{1/2} \left(\int_\Omega d\nu \right)^{1/2} \le \|g\|_{L^2(\lambda)} \, (\nu(\Omega))^{1/2}.$$

(This is where the finiteness of the measure is crucial; otherwise an L^2 functions is not necessarily in L^1.) By the Riesz representation theorem there is a unique function $h \in L^2(\lambda)$, such that

$$\int_\Omega g \, d\nu = \int_\Omega h g \, d\lambda. \tag{1.2}$$

We now show that h satisfies the following properties:

[4] Our definition of a genuine norm over $\mathscr{H}^*$ is somewhat awkward. When we get to Banach spaces it will be made clear that we do not need inner-products and representation theorems in order to show that the operator norm is indeed a norm.

1. $h \geq 0$ **λ-a.e**: Because the measures are finite, indicator functions are integrable. Then, setting $g = I_B$,
$$0 \leq \nu(B) = \int_B h d\lambda,$$
which implies that $h \geq 0$ λ-a.e.
2. $h < 1$ **λ-a.e**: Setting,
$$B = \{x \in \Omega \mid h(x) \geq 1\},$$
we get
$$\nu(B) = \int_B h d\lambda \geq \lambda(B) = \nu(B) + \mu(B),$$
which implies that $\mu(B) = \nu(B) = \lambda(B) = 0$, i.e., $h < 1$ λ-a.e.

Since $0 \leq h < 1$, we may represent h as
$$h = \frac{f}{1+f},$$
where f is λ-a.e. non-negative.

Back to Eq. (1.2),
$$\int_\Omega g\, d\nu = \int_\Omega \left(\frac{f}{1+f}\right) g\, d\nu + \int_\Omega \left(\frac{f}{1+f}\right) g\, d\mu,$$
hence
$$\int_\Omega \left(\frac{1}{1+f}\right) g\, d\nu = \int_\Omega \left(\frac{f}{1+f}\right) g\, d\mu.$$

Let now $k : \Omega \to \mathbb{R}$ be non-negative and bounded. Define
$$B_m = \{x \in \Omega : \; k(x)(1 + f(x)) \leq m\},$$
and $g = k(1+f) I_{B_m}$. Then,
$$\int_{B_m} k\, d\nu = \int_{B_m} k f\, d\mu,$$
Since k is non-negative we can let $m \to \infty$ and get for all measurable and bounded k:[5]
$$\int_\Omega k\, d\nu = \int_\Omega k f\, d\mu.$$
Setting $k \equiv 1$ we get that f is in $L^1(\mu)$. Setting $k = I_B$ we get
$$\int_B d\nu = \int_B f\, d\mu,$$
which completes the proof. ■

[5] Here we apply Lebesgue's Monotone Convergence Theorem, whereby the sequence of integrands is monotone and has a pointwise limit.

1.3.4 Bilinear and quadratic forms

Definition 1.15 Let $(\mathscr{X}, \|\cdot\|)$ be a normed space. A function $B : \mathscr{X} \times \mathscr{X} \to \mathbb{C}$ is called a **bilinear form** (תבנית בי ליניארית) if it is linear in its first entry and anti-linear in its second entry, namely,

$$B(x, \alpha y + \beta z) = \overline{\alpha} B(x,y) + \overline{\beta} B(x,z).$$

It is said to be **bounded** if there exists a constant $C \geq 0$ such that

$$\forall x, y \in \mathscr{X} \qquad |B(x,y)| \leq C \|x\| \, \|y\|.$$

The smallest such constant C is called the **norm** of B, i.e.,

$$\|B\| = \sup_{x,y \neq 0} \frac{|B(x,y)|}{\|x\| \, \|y\|} \qquad \text{or} \qquad \|B\| = \sup_{\|x\|=\|y\|=1} |B(x,y)|.$$

Definition 1.16 Let $(\mathscr{X}, \|\cdot\|)$ be a normed space. A mapping $Q : \mathscr{X} \to \mathbb{C}$ is called a **quadratic form** (תבנית ריבועית) if $Q(x) = B(x,x)$, where B is a bilinear form. It is said to be bounded if there exists a constant C such that

$$\forall x \in \mathscr{X} \qquad |Q(x)| \leq C \|x\|^2.$$

The smallest such C is called the **norm** of Q, i.e.,

$$\|Q\| = \sup_{x \neq 0} \frac{|Q(x)|}{\|x\|^2}, \qquad \text{or} \qquad \|Q\| = \sup_{\|x\|=1} |Q(x)|.$$

Proposition 1.36 — Polarization identity. Let $(\mathscr{X}, \|\cdot\|)$ be a normed space. For every bilinear form $B(x,y)$ with $Q(x) = B(x,x)$,

$$B(x,y) = \frac{1}{4} \{Q(x+y) - Q(x-y) + \imath [Q(x+\imath y) - Q(x - \imath y)]\}.$$

Proof. Just follow the same steps as for the relation between the inner-product and the norm (which is a particular case). By the properties of B,

$$Q(x \pm y) = B(x \pm y, x \pm y) = Q(x) + Q(y) \pm B(x,y) \pm B(y,x).$$

Hence

$$Q(x+y) - Q(x-y) = 2\,(B(x,y) + B(y,x)).$$

Letting $y \mapsto \imath y$,

$$Q(x + \imath y) - Q(x - \imath y) = 2\imath\,(-B(x,y) + B(y,x)).$$

Multiplying the second equation by $\imath$ and adding to the first we recover the desired result. ■

Proposition 1.37 Let $(\mathscr{X}, \|\cdot\|)$ be a complex normed space. Let $Q(x) = B(x,x)$ be a quadratic form over $\mathscr{X}$. Then Q is bounded if and only if B is bounded, in which case

$$\|Q\| \le \|B\| \le 2\|Q\|.$$

If, moreover, $|B(x,y)| = |B(y,x)|$ for all $x,y \in \mathscr{X}$, then $\|Q\| = \|B\|$.

Proof. First, suppose that B is bounded. Then,

$$|Q(x)| = |B(x,x)| \le \|B\|\,\|x\|^2,$$

i.e., Q is bounded and $\|Q\| \le \|B\|$.

Second, suppose that Q is bounded. Taking the polarization identity,

$$B(x,y) = \frac{1}{4}\left\{Q(x+y) - Q(x-y) + \imath\left[Q(x+\imath y) - Q(x-\imath y)\right]\right\},$$

and using the definition of $\|Q\|$:

$$|B(x,y)| \le \frac{1}{4}\|Q\|\left(\|x+y\|^2 + \|x-y\|^2 + \|x+\imath y\|^2 + \|x-\imath y\|^2\right).$$

Using (twice) the parallelogram law:

$$|B(x,y)| \le \|Q\|\left(\|x\|^2 + \|y\|^2\right),$$

hence

$$\|B\| = \sup_{\|x\|=\|y\|=1} |B(x,y)| \le 2\|Q\|.$$

Remains the last part of the theorem. Using again the polarization identity, and noting that

$$Q(y+\imath x) = Q(\imath(x-\imath y)) = Q(x-\imath y)$$
$$Q(y-\imath x) = Q(-\imath(x+\imath y)) = Q(x+\imath y)$$

we get

$$\begin{aligned} B(x,y) + B(y,x) &= \frac{1}{4}\left\{Q(x+y) - Q(x-y) + \imath\left[Q(x+\imath y) - Q(x-\imath y)\right]\right\} \\ &+ \frac{1}{4}\left\{Q(x+y) - Q(y-x) + \imath\left[Q(y+\imath x) - Q(y-\imath x)\right]\right\} \\ &= \frac{1}{2}\left\{Q(x+y) - Q(x-y)\right\}. \end{aligned}$$

Thus, using once again the parallelogram law,

$$|B(x,y)+B(y,x)| \le \frac{1}{2}\|Q\|\left(\|x+y\|^2+\|x-y\|^2\right) = \|Q\|\left(\|x\|^2+\|y\|^2\right), \tag{1.3}$$

If $|B(x,y)| = |B(y,x)|$, then there exists a phase $\alpha = \alpha(x,y)$ such that

$$B(y,x) = e^{\imath\alpha(x,y)}B(x,y).$$

Note that

$$\begin{aligned} B(y,e^{\imath\beta}x) &= e^{\imath\alpha(x,y)-\imath\beta}B(x,y) = e^{\imath\alpha(x,y)-2\imath\beta}B(e^{\imath\beta}x,y) \\ B(e^{\imath\beta}y,x) &= e^{\imath\alpha(x,y)+\imath\beta}B(x,y) = e^{\imath\alpha(x,y)+2\imath\beta}B(x,e^{\imath\beta}y), \end{aligned}$$

that is

$$\begin{aligned} \alpha(e^{\imath\beta}x,y) &= \alpha(x,y)-2\beta \\ \alpha(x,e^{\imath\beta}y) &= \alpha(x,y)+2\beta. \end{aligned}$$

For all $\|x\| = \|y\| = 1$,

$$|1+e^{\imath\alpha(x,y)}||B(x,y)| \le 2\|Q\|.$$

And letting $x \mapsto e^{\imath\beta}x$, we get for all β:

$$|1+e^{\imath(\alpha(x,y)-2\beta)}||B(x,y)| \le 2\|Q\|.$$

In particular, for $\beta = \frac{1}{2}\alpha(x,y)$:

$$|B(x,y)| \le \|Q\|,$$

i.e., $\|B\| \le \|Q\|$. ■

■ **Example 1.1** Consider the real normed space $\mathscr{X} = \mathbb{R}^2$ endowed with the Euclidean inner-product. Let

$$B(x,y) = x_1y_2 - x_2y_1.$$

Clearly $\|B\| > 0$, however

$$Q(x) = B(x,x) = 0,$$

hence $\|Q\| = 0$, in contradiction to $\|B\| \le 2\|Q\|$. What going on? The above proof was based on the assumption that $\mathscr{F} = \mathbb{C}$. The proposition does not hold when $\mathscr{F} = \mathbb{R}$. ■

■ **Example 1.2** Let $(\mathbb{H},(\cdot,\cdot))$ be an inner-product space, and let T be a bounded linear operator on $\mathbb{H}$. Set

$$B(x,y) = (Tx,y).$$

B is a bilinear form. By the Cauchy-Schwarz inequality,

$$\|T\| = \sup_{\|x\|=1}\|Tx\| = \sup_{\|x\|=1}\frac{B(x,Tx)}{\|Tx\|} \le \sup_{\|x\|=1}\sup_{\|y\|=1}|B(x,y)| \le \sup_{\|x\|=1}\sup_{\|y\|=1}|(Tx,y)| \le \|T\|.$$

It follows that

$$\|B\| = \sup_{\|x\|=1}\sup_{\|y\|=1}|B(x,y)| = \|T\|.$$

■

The following theorem states that, in fact, every bounded bilinear form on a Hilbert space can be represented by a bounded linear operator (this theorem is very similar to the Riesz representation theorem). Please note that like for the Riesz representation theorem, completeness is essential.

Theorem 1.38 To every bounded bilinear form B over a Hilbert space $(\mathscr{H},(\cdot,\cdot))$ corresponds a unique bounded linear operator on $\mathscr{H}$, such that

$$\forall x,y\in\mathscr{H} \qquad B(x,y)=(Tx,y).$$

Furthermore, $\|B\|=\|T\|$.

Comment 1.10 The Riesz representation theorem asserts that

$$\mathscr{H}\cong\mathscr{H}^*.$$

This theorem asserts that

$$\mathscr{H}^*\otimes\mathscr{H}^*\cong\mathscr{H}^*\otimes\mathscr{H}.$$

Proof. We start by constructing T. Fix $x\in\mathscr{H}$ and define the functional:

$$F_x=\overline{B(x,\cdot)}.$$

F_x is linear and bounded, as

$$|F_x(y)|\le\|B\|\|x\|\|y\|,$$

i.e., $\|F_x\|\le\|B\|\|x\|$. It follows from the Riesz representation theorem that there exists a unique $z_x\in\mathscr{H}$, such that

$$F_x=(\cdot,z_x)=\overline{B(x,\cdot)}.$$

Denote the mapping $x\mapsto z_x$ by T, thus for every $x\in\mathscr{H}$:

$$(\cdot,Tx)=\overline{B(x,\cdot)}$$

i.e., for every $x,y\in\mathscr{H}$:

$$B(x,y)=(Tx,y).$$

Next, we show that T is **linear**. By definition of T and by the bilinearity of both B and the inner-product:

$$\begin{aligned}(T(\alpha_1x_1+\alpha_2x_2),y)&=B(\alpha_1x_1+\alpha_2x_2,y)\\&=\alpha_1B(x_1,y)+\alpha_2B(x_2,y)\\&=\alpha_1(Tx_1,y)+\alpha_2(Tx_2,y)\\&=(\alpha_1Tx_1+\alpha_2Tx_2,y).\end{aligned}$$

Since this holds for all $y \in \mathscr{H}$ it follows that

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T x_1 + \alpha_2 T x_2.$$

T is **bounded**, as we have already proved that $\|T\| = \|B\|$.

It remains to prove that T is **unique**. Suppose T and S are both bounded linear operators satisfying

$$(\forall x, y \in \mathscr{H}) \quad B(x,y) = (Tx, y) = (Sx, y).$$

Then,

$$(\forall x, y \in \mathscr{H}) \quad ((T - S)x, y) = 0,$$

hence

$$(\forall x \in \mathscr{H}) \qquad \|(T - S)x\|^2 = 0,$$

and hence $T = S$. ∎

1.3.5 The Lax-Milgram theorem

Definition 1.17 Let $(\mathscr{X}, \|\cdot\|)$ be a normed space. A bilinear form B on $\mathscr{X}$ is called **coercive** (חסומה מלרע) if there exists a constant $\delta > 0$ such that

$$\forall x \in \mathscr{H} \qquad |B(x,x)| \geq \delta \|x\|^2.$$

The following theorem is a central pillar in the theory of partial differential equations:

Theorem 1.39 — Lax-Milgram, 1954. Let B be a bounded and coercive bilinear form on a Hilbert space $\mathscr{H}$. Then, there exists a unique bounded linear operator S on $\mathscr{H}$ such that

$$\forall x, y \in \mathscr{H} \qquad (x, y) = B(Sx, y).$$

Furthermore, S^{-1} exists, it is bounded, and

$$\|S\| \leq \frac{1}{\delta} \qquad \text{and} \qquad \|S^{-1}\| = \|B\|,$$

where δ is the coercivity parameter.

Comment 1.11 This theorem is named after Peter Lax (1926–) and Arthur Milgram (1912–1961).

Proof. By Theorem 1.38 there is a unique bounded linear operator T such that

$$\forall x, y \in \mathscr{H} \qquad B(x, y) = (Tx, y),$$

and $\|T\| = \|B\|$. If we succeed to show that T is invertible with bounded inverse $S = T^{-1}$ and $\|S\| \le 1/\delta$ then we are done, as

$$(x,y) = (TSx,y) = B(Sx,y),$$

and we know that $\|T\| = \|B\|$.

Setting $x = y$ and using the coercivity of B and the Cauchy-Schwarz inequality, we get

$$\delta\|x\|^2 \le |B(x,x)| = |(Tx,x)| \le \|Tx\|\,\|x\|,$$

i.e.,

$$\forall x \in \mathscr{H} \qquad \|Tx\| \ge \delta\|x\|. \tag{1.4}$$

It follows that $\ker T = \{0\}$, hence T **is injective**.

We next show that $\text{Image}(T)$ is a **closed linear subspace**. If $\text{Image}(T) \ni x_n = Tu_n \to x \in \mathscr{H}$, then

$$\|u_n - u_m\| \le \frac{1}{\delta}\|Tu_n - Tu_m\| = \frac{1}{\delta}\|x_n - x_m\|,$$

which implies that (u_n) is a Cauchy sequence with limit which we denote by u. By the continuity of T, we have $x = Tu$.

Next, we show that T is **surjective**. Let $z \in (\text{Image}\,T)^\perp$, then

$$\delta\,\|z\|^2 \le B(z,z) = (Tz,z) = 0,$$

which implies that $z = 0$, i.e., $(\text{Image}(T))^\perp = \{0\}$, and since $\text{Image}(T)$ is closed it is equal to $\mathscr{H}$. We conclude that T is a **bijection**. Hence, T^{-1} exists. It is linear because the inverse of a linear operator is always linear. It is bounded because by Eq. (1.4)

$$\|T^{-1}x\| \le \frac{1}{\delta}\|T(T^{-1}x)\| \le \frac{1}{\delta}\|x\|,$$

from which we conclude that

$$\|T^{-1}\| \le \frac{1}{\delta}.$$

This completes the proof. ∎

■ **Example 1.3** We will study a typical (relatively simple) application of the Lax-Milgram theorem. Let $k \in C[0,1]$ satisfy

$$\forall x \in [0,1] \qquad 0 < c_1 \le k(x) \le c_2,$$

and consider the boundary value problem

$$\frac{d}{dx}\left(k\frac{du}{dx}\right) = f \qquad u(0) = u(1) = 0, \tag{1.5}$$

for some $f \in C[0,1]$. Does a solution exist?[6]

We will slightly modify the problem. Let $v \in C^1[0,1]$ satisfy $v(0) = v(1) = 0$ (we denote this space by $C_0^1[0,1]$). Multiplying the equation by v, integrating over $[0,1]$ and integrating by parts, we get

$$\forall v \in C_0^1[0,1] \qquad -\int_0^1 k u' v' \, dx = \int_0^1 f v \, dx. \tag{1.6}$$

Equation (1.6) is called the **weak formulation** of (1.5). Does a solution exist to the weak formulation?

First, endow the space $C_0^1[0,1]$ with the inner-product

$$(u,v) = \int_0^1 uv \, dx + \int_0^1 u'v' \, dx.$$

In fact, this space is not complete. Its completion is known as the **Sobolev space** $W_0^{1,2}[0,1] = H_0^1[0,1]$. So. Eq. 1.6 really takes the following form: find $u \in H_0^1[0,1]$ such that

$$\forall v \in H_0^1[0,1] \qquad -\int_0^1 k Du Dv \, dx = \int_0^1 f v \, dx,$$

where D is the **weak derivative**.

Define the bilinear form

$$B(u,v) = \int_0^1 k u' v' \, dx.$$

B is bounded, for by the Cauchy-Schwarz inequality:

$$\begin{aligned} |B(u,v)| &\le c_2 \int_0^1 |u'||v'| \, dx \\ &\le c_2 \left(\int_0^1 (u')^2 \, dx \right)^{1/2} \left(\int_0^1 (v')^2 \, dx \right)^{1/2} \\ &\le c_2 \|u\| \|v\|. \end{aligned}$$

Coercivity is more tricky. First,

$$|B(u,u)| = \int_0^1 k (u')^2 \, dx \ge c_1 \int_0^1 (u')^2 \, dx.$$

This is not good enough because the norm also has a part that depends on the integral of u^2.

To solve this difficulty we will derive a very important inequality–the **Poincaré inequality**. Since $u(0) = 0$,

$$u(x) = \int_0^x u'(t) \, dt,$$

[6] Recall the existence and uniqueness theorem applies to **initial value problems** but not to **boundary value problems**.

hence

$$u^2(x) = \left(\int_0^x u'(t)\,dt\right)^2 \le \left(\int_0^x (u'(t))^2\,dt\right)\left(\int_0^x dt\right) \le \int_0^1 (u')^2\,dt.$$

Integrating over $[0,1]$,

$$\int_0^1 u^2\,dx \le \int_0^1 (u')^2\,dx.$$

Thus,

$$\int_0^1 (u')^2\,dx \ge \frac{1}{2}\int_0^1 u^2\,dx + \frac{1}{2}\int_0^1 (u')^2\,dx = \frac{1}{2}\|u\|^2,$$

which implies that B is coercive with

$$|B(u,u)| \ge \frac{c_1}{2}\|u\|^2.$$

Next, we consider the mapping

$$v \mapsto -\int_0^1 f v\,dx,$$

which is a bounded linear functional. By the Riesz representation theorem, there exists a vector $g \in H_0^1[0,1]$ such that

$$(g,v) = -\int_0^1 f v\,dx.$$

By the Lax-Milgram theorem, there exists a bounded linear operator S on $H_0^1[0,1]$, such that

$$\forall v \in H_0^1[0,1] \qquad B(Sg,v) = (g,v),$$

i.e.,

$$\forall v \in H_0^1[0,1] \qquad \int_0^1 k(Sg)'v'\,dx = -\int_0^1 f v\,dx,$$

i.e., Sg satisfies the weak equation.[7] ∎

Theorem 1.40 — Töplitz-Haussdorff. Let Q be a quadratic form on an inner-product space. Then the **numerical range** (**הטווח הנומרי**),

$$W(Q) = \{Q(x) \mid \|x\| = 1\} = \left\{\frac{Q(x)}{\|x\|^2} \mid \|x\| \ne 0\right\},$$

is a convex subset of the complex plane.

[7] Note that this is an existence proof; we haven't solved the equation.

Comment 1.12 In the finite-dimensional case, a bilinear form is represented by a matrix Q, and the corresponding numerical range is

$$W(Q) = \left\{ \frac{x^\dagger Q x}{x^\dagger x} \mid \|x\| \neq 0 \right\}.$$

It can be shown, for example, that all the eigenvalues of Q are within its numerical range.

Proof. We need to show that if α and β belong to the numerical range of Q, then so does any $t\alpha + (1-t)\beta$ for even $0 \le t \le 1$. In other words, we need to show that:

$$(\exists x, y \neq 0): \; Q(x) = \alpha \|x\|^2 \quad \text{and} \quad Q(y) = \beta \|y\|^2$$

implies

$$(\forall 0 \le t \le 1)(\exists z \neq 0): \; Q(z) = (t\alpha + (1-t)\beta)\|z\|^2.$$

We can formulate it differently: we need to show that

$$(\exists x, y \neq 0): \; \frac{\frac{Q(x) - \alpha\|x\|^2}{\beta - \alpha}}{\|x\|^2} = 0 \quad \text{and} \quad \frac{\frac{Q(y) - \alpha\|y\|^2}{\beta - \alpha}}{\|y\|^2} = 1,$$

implies

$$(\forall 0 \le t \le 1)(\exists z \neq 0): \; \frac{\frac{Q(z) - \alpha\|z\|^2}{\beta - \alpha}}{\|z\|^2} = 1 - t.$$

Thus it suffices to show that if 0 and 1 are in the numerical range of a quadratic form Q, so is any number on the unit segment.

Suppose then that

$$Q(x) = 0 \quad \text{and} \quad Q(y) = 1, \qquad \|x\| = \|y\| = 1.$$

Let B be the bilinear form that defines Q, and consider

$$B(x,y) + B(y,x).$$

We may assume that it is real-valued for we can always replace y by $e^{\iota s} y$.

Now,

$$Q((1-t)x + ty) = (1-t)^2 \underbrace{Q(x)}_{0} + t^2 \underbrace{Q(y)}_{1} + t(1-t)\,(B(x,y) + B(y,x)) \in \mathbb{R},$$

and

$$F(t) = \frac{Q((1-t)x + ty)}{\|(1-t)x + ty\|^2} = \frac{t^2 + t(1-t)\,(B(x,y) + B(y,x))}{\|(1-t)x + ty\|^2}.$$

This is a continuous function equal to zero at zero and to one at one, hence it assumes all intermediate values. ∎

1.4 Orthonormal systems

Definition 1.18 Let A be some index set (not necessarily countable), and let

$$\{u_\alpha \mid \alpha \in A\}$$

be a set of vectors in a inner-product space $(\mathbb{H}, (\cdot,\cdot))$. The set $\{u_\alpha\}$ is called an **orthonormal system** (**מערכת אורתונורמלית**) if

$$\forall \alpha, \beta \in A \qquad (u_\alpha, u_\beta) = \delta_{\alpha\beta}.$$

Definition 1.19 Let $\{u_\alpha \mid \alpha \in A\}$ be an orthonormal system and let $x \in \mathbb{H}$. The set of scalars

$$\{\hat{x}(\alpha) = (x, u_\alpha) \mid \alpha \in A)$$

are called the **Fourier components** (**רכיבי פורייה**) of x with respect to the orthonormal system.

Theorem 1.41 — Gram-Schmidt orthonormalization. Let (x_n) be either a finite or a countable sequence of linearly independent vectors in an inner-product space $\mathbb{H}$. Then it is possible to construct an orthonormal sequence (y_n) that has the same cardinality as the sequence (x_n), such that

$$\forall n \in \mathbb{N} \qquad \operatorname{Span}\{y_k \mid 1 \le k \le n\} = \operatorname{Span}\{x_k \mid 1 \le k \le n\}.$$

Proof. You learned it in linear algebra for spaces of finite dimension. The same recursive construction holds for a countable sequence. ■

Proposition 1.42 Let $(u_1, \dots, u_n)$ be vectors in an inner-product space $(\mathbb{H}, (\cdot,\cdot))$. Then,

$$\forall x \in \mathbb{H} \qquad \sum_{i=1}^{n} |(x, u_i)|^2 \le M \|x\|^2,$$

where

$$M = \max_i \sum_{j=1}^{n} |(u_i, u_j)|.$$

Proof. For every set of scalars $(c_1, \dots, c_n)$:

$$0 \le \left\| x - \sum_{i=1}^{n} c_i u_i \right\|^2 = \|x\|^2 - 2\sum_{i=1}^{n} \operatorname{Re}[\bar{c}_i (x, u_i)] + \sum_{i,j=1}^{n} c_i \bar{c}_j (u_i, u_j).$$

Using the inequality $2|a||b| \le |a|^2 + |b|^2$:

$$\begin{aligned} 0 &\le \|x\|^2 - 2\sum_{i=1}^n \operatorname{Re}\bar{c}_i(x,u_i) + \frac{1}{2}\sum_{i,j=1}^n \left(|c_i|^2 + |c_j|^2\right)|(u_i,u_j)| \\ &= \|x\|^2 - 2\sum_{i=1}^n \operatorname{Re}\bar{c}_i(x,u_i) + \sum_{i=1}^n |c_i|^2 \sum_{j=1}^n |(u_i,u_j)| \\ &\le \|x\|^2 - 2\sum_{i=1}^n \operatorname{Re}\bar{c}_i(x,u_i) + M\sum_{i=1}^n |c_i|^2. \end{aligned}$$

Choose $c_i = (x,u_i)/M$, then

$$0 \le \|x\|^2 - \frac{2}{M}\sum_{i=1}^n |(x,u_i)|^2 + \frac{1}{M}\sum_{i=1}^n |(x,u_i)|^2,$$

which yields the desired result. ■

Corollary 1.43 — Bessel inequality. Let $\{u_\alpha\}$ be an orthonormal system in an inner-product space $(\mathbb{H},(\cdot,\cdot))$. For every countable subset $\{u_{\alpha_k}\}_{k=1}^\infty$,

$$\forall x \in \mathbb{H} \qquad \sum_{k=1}^\infty |\hat{x}(\alpha_k)|^2 = \sum_{k=1}^\infty |(x,u_{\alpha_k})|^2 \le \|x\|^2.$$

Proof. Immediate from the previous proposition. ■

Corollary 1.44 Let $\{u_\alpha \mid \alpha \in A\}$ be an orthonormal system in an inner-product space $(\mathbb{H},(\cdot,\cdot))$. For every $x \in \mathbb{H}$ there is at most a countable set of non-vanishing Fourier components.

Proof. Fix $x \in \mathbb{H}$, and consider the sets

$$B_k = \{\alpha \in A \mid |\hat{x}(\alpha)|^2 \ge 1/k\}.$$

From Bessel's inequality,

$$\frac{1}{k}|B_k| \le \sum_{\alpha \in B_k} |\hat{x}(\alpha)|^2 \le \|x\|^2,$$

which implies that each set B_k is finite. Hence

$$\{\alpha \in A \mid |(\hat{x}(\alpha)| > 0\} = \bigcup_{k=1}^\infty B_k$$

is at most countable. ■

Proposition 1.45 — Riesz-Fischer. Let (u_n) be an orthonormal sequence in a Hilbert space $(\mathscr{H},(\cdot,\cdot))$. Let (c_n) be a sequence of scalars. Then,

$$s_n = \sum_{k=1}^{n} c_k u_k$$

converges as $n \to \infty$ if and only if

$$\sum_{k=1}^{\infty} |c_k|^2 < \infty.$$

Proof. Look at the difference between s_n and s_m:

$$\|s_n - s_m\|^2 = \| \sum_{k=m+1}^{n} c_k u_k \|^2 = \sum_{k=m+1}^{n} |c_k|^2,$$

where we used the orthonormality of the (u_k). Thus the series of $|c_k|^2$ is a Cauchy sequence if and only if (s_n) is a Cauchy sequence. Note that the completeness of $\mathscr{H}$ is crucial. ■

Fourier coefficients as optimizers

Let $(\mathscr{H},(\cdot,\cdot))$ be a Hilbert space. Let $(u_1,\dots,u_n)$ be a finite sequence of orthonormal vectors and let

$$\mathscr{M} = \operatorname{Span}\{u_1,\dots,u_n\},$$

which is a closed subspace of $\mathscr{H}$. For every $x \in \mathscr{H}$:

$$\mathbb{P}_{\mathscr{M}} x = \sum_{k=1}^{n} (x,u_k) u_k.$$

Why? Because $x - \mathbb{P}_{\mathscr{M}} x \in \mathscr{M}^{\perp}$ as for every u_j:

$$(x - \mathbb{P}_{\mathscr{M}} x, u_j) = (x,u_j) - (x,u_j) = 0.$$

By the definition of the projection as a distance minimizer, it follows that the Fourier coefficients $\hat{x}(k)$ satisfy:

$$\left\| x - \sum_{k=1}^{n} \hat{x}(k) u_j \right\| \le \left\| x - \sum_{k=1}^{n} \lambda_k u_j \right\|,$$

for any set of scalars $(\lambda_1,\dots,\lambda_n)$.

Definition 1.20 An orthonormal system

$$\{u_\alpha \mid \alpha \in A\}$$

is said to be **complete** if it is not contained (in the strict sense) in any other orthonormal system. That is, it is complete if the only vector orthogonal to all $\{u_\alpha\}$ is zero:

$$(\operatorname{Span}\{u_\alpha \mid \alpha \in A\})^\perp = \{0\}.$$

A complete orthonormal system is also called an **orthonormal basis**.

Proposition 1.46 Every separable Hilbert space $(\mathscr{H}, (\cdot,\cdot))$ contains a countable complete orthonormal system.

Proof. Recall that $\mathscr{H}$ is separable if it contains a countable dense subset. Now, Let (z_n) be a dense countable set. In particular:

$$\overline{\operatorname{Span}\{z_n \mid n \in \mathbb{N}\}} = \mathscr{H}.$$

We can construct inductively a subset (x_n) of independent vectors such that

$$\forall n \quad \exists N < N \quad \text{such that} \quad \operatorname{Span}\{x_k \mid 1 \le k \le N\} = \operatorname{Span}\{z_k \mid 1 \le k \le n\},$$

that is,

$$\overline{\operatorname{Span}\{x_n \mid n \in \mathbb{N}\}} = \mathscr{H}.$$

By applying Gram-Schmidt orthonormalization we obtain an orthonormal system (u_n) such that

$$\overline{\operatorname{Span}\{u_n \mid n \in \mathbb{N}\}} = \mathscr{H}.$$

We will show that this orthonormal system is complete. Suppose that v were orthogonal to all (u_n). Since every $x \in \mathscr{H}$ is a limit

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} c_{n,k} u_k,$$

it follows by the continuity of the inner-product that

$$(x, v) = \lim_{n\to\infty} \sum_{k=1}^{n} c_{n,k} (u_k, v) = 0,$$

i.e., v is orthogonal to all vectors in $\mathscr{H}$, hence it is zero, from which follows that the (u_n) form a complete orthonormal system.

■

Theorem 1.47 Let $(\mathscr{H}, (\cdot,\cdot))$ be a (non-trivial) Hilbert space. Then, it contains a complete orthonormal system. Moreover, every orthonormal system in $\mathscr{H}$ is contained in a complete orthonormal system.

Proof. The proof replies on the axiom of choice. Let P denotes the set of all orthonormal systems in $\mathscr{H}$. P is not empty because every normalized vector in $\mathscr{H}$ constitutes an orthonormal system.

Let $S \in P$ and consider

$$P_S = \{T \in P \mid T \supseteq S\}$$

be the collection of all the orthonormal systems that contain S. P_S is a partially ordered set with respect to set inclusion. Let $P_S' \subset P_S$ be fully ordered (a chain). Then,

$$T_0 = \bigcup_{T \in P_S'} T$$

is an element of P_S which is an upper bound to all the elements in P_S'. It follows from Zorn's lemma that P_S contains at least one maximal element[8], which we denote by S_0. S_0 is an orthonormal system that contains S. Since it is maximal, it is by definition complete. ■

The importance of complete orthonormal systems

We have seen that if $\{u_\alpha \mid \alpha \in A\}$ is an orthonormal set, then for every $x \in \mathscr{H}$ there is at most a countable number of Fourier components $\{\hat{x}(\alpha) \mid \alpha \in A\}$ that are not zero.

Let $x \in \mathscr{H}$ be given, and let (α_n) be the indexes for which $\hat{x}$ does not vanish. It follows from Bessel's inequality that

$$\sum_{k=1}^{\infty} |\hat{x}(\alpha_k)|^2 \le \|x\|.$$

It follows then from the Riesz-Fischer Theorem that

$$\sum_{k=1}^{\infty} \hat{x}(\alpha_k)\, u_{\alpha_k} \qquad \text{exists.}$$

As for all indexes not in (α_n) the Fourier coefficients vanish, there is no harm in writing for all $x \in \mathscr{H}$:

$$\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 \le \|x\|.$$

and

$$\sum_{\alpha \in A} \hat{x}(\alpha)\, u_\alpha \qquad \text{exists.}$$

[8]Zorn's lemma: suppose a partially ordered set P has the property that every totally ordered subset has an upper bound in P. Then the set P contains at least one maximal element.

Theorem 1.48 — Characterization of completeness. Let $\{u_\alpha \mid \alpha \in A\}$ be an orthonormal system in $\mathscr{H}$. All the following conditions are equivalent:

① $\{u_\alpha \mid \alpha \in A\}$ is complete.
② For all $x \in \mathscr{H}$: $\sum_{\alpha\in A} \hat{x}(\alpha)\, u_\alpha = x$.
③ **Generalized Parseval identity**. For all $x, y \in \mathscr{H}$: $(x, y) = \sum_{\alpha\in A} \hat{x}(\alpha)\overline{\hat{y}(\alpha)}$.
④ **Parseval identity**. For all $x \in \mathscr{H}$: $\|x\|^2 = \sum_{\alpha\in A} |\hat{x}(\alpha)|^2$.

Proof. Suppose that ① holds, i.e., the orthonormal system, is complete. Given x let (α_n) be a sequence of indexes that contains all indexes for which the Fourier coefficients of x do not vanish. For every index α_n,

$$\left(x - \sum_{k=1}^{\infty} \hat{x}(\alpha_k) u_{\alpha_k}, u_{\alpha_n}\right) = 0.$$

In fact, for all $\alpha \in A$

$$\left(x - \sum_{k=1}^{\infty} \hat{x}(\alpha_k) u_{\alpha_k}, u_{\alpha}\right) = 0.$$

If follows that $x - \sum_{k=1}^{\infty} \hat{x}(\alpha_k) u_{\alpha_k}$ is orthogonal all vectors $\{u_\alpha\}$ but since we assumed that the orthonormal system is complete, it follows that it is zero, i.e.,

$$x = \sum_{k=1}^{\infty} \hat{x}(\alpha_k) u_{\alpha_k},$$

and once again we may extend the sum over all $\alpha \in A$.

Suppose that ② holds:

$$\forall x \in \mathscr{H} \qquad \sum_{\alpha\in A} \hat{x}(\alpha)\, u_\alpha = x.$$

Given $x, y \in \mathscr{H}$ let (α_n) be a sequence of indexes that contains all the indexes for which at least one of the Fourier components of either x and y does not vanish. By the continuity of the inner-product:

$$(x, y) = \left(\sum_{k=1}^{\infty} \hat{x}(\alpha_k) u_{\alpha_k}, \sum_{k=1}^{\infty} \hat{y}(\alpha_k) u_{\alpha_k}\right) = \sum_{k=1}^{\infty} \hat{x}(\alpha_k)\overline{\hat{y}(\alpha_k)}.$$

Suppose that ③ holds. Setting $x = y$ we obtain the Parseval identity.

Suppose that ④ holds. Let $x \in \mathscr{H}$ be orthogonal to all the $\{u_\alpha\}$, then

$$\forall \alpha \in A \qquad \hat{x}(\alpha) = (x, u_\alpha) = 0.$$

It follows from the Parseval identity that $x = 0$, i.e., the orthonormal system is complete.

∎

■ **Example 1.4** Consider the Hilbert space ℓ^2 and the sequence of vectors

$$u_n = (0,0,\dots,0,1,0,\dots).$$

Clearly, the (u_n) form an orthonormal set. Let $x \in \ell^2$. $x \perp u_n$ implies that the n-th entry of x is zero, hence if x is orthogonal to all (u_n) then it is zero. This implies that the orthonormal system (u_n) is complete. ■

Theorem 1.49 All the complete orthonormal systems in a Hilbert space $\mathscr{H}$ have the same cardinality. Thus we can unambiguously define the **dimension** of a Hilbert space as the cardinality of any complete orthonormal system.

Proof. Let

$$\{u_\alpha \mid \alpha \in A\} \qquad \text{and} \qquad \{v_\beta \mid \beta \in B\}$$

be two complete orthonormal system.

Suppose first that A is a finite set, $|A| = n$. The vectors $(u_1,\dots,u_n P$ are a basis in $\mathscr{H}$, i.e., $\dim \mathscr{H} = n$. Since the vectors $\{v_\beta\}$ are orthonormal, they are linearly independent, and hence

$$|B| \le |A|.$$

By symmetry, $|A| = |B|$.

Suppose now that $|A|$ is an infinite set. To every $\alpha \in A$ we attach a set

$$F_\alpha = \{\beta \in B \mid (u_\alpha, v_\beta) \ne 0\},$$

which we know to be either finite or countable. By the completeness of the system $\{u_\alpha\}$, if $\beta \in B$, then there exists an $\alpha \in A$ such that $(u_\alpha, v_\beta) \ne 0$, i.e., there exits an α such that $\beta \in F_\alpha$, from which we conclude that

$$B \subseteq \bigcup_{\alpha \in A} F_\alpha,$$

and hence,

$$|B| \le \aleph_0 |A| = |A|,$$

and by symmetry, $|A| = |B|$. ■

Theorem 1.50 Two Hilbert space are isomorphic if and only if they have the same dimension.

Proof. Let $(\mathscr{H}, (\cdot,\cdot)_{\mathscr{H}})$ and $(\mathscr{G}, (\cdot,\cdot)_{\mathscr{G}})$ be two Hilbert spaces of equal dimension and let

$$\{u_\alpha \mid \alpha \in A\} \subset \mathscr{H} \qquad \text{and} \qquad \{v_\beta \mid \beta \in B\} \subset \mathscr{G}$$

be complete orthonormal systems. Since they have the same cardinality, we can use the same index set, and create a one-to-one correspondence,

$$T : u_\alpha \mapsto v_\alpha.$$

We extend this correspondence on all $\mathscr{H}$ and $\mathscr{G}$

$$T : \sum_{n=1}^{\infty} c_n u_{\alpha_n} \mapsto \sum_{n=1}^{\infty} c_n v_{\alpha_n}.$$

(Note that both sides are well defined if and only if $\sum_n |c_n|^2 < \infty$.) It is easy to see this this correspondence preserves the linear structure, and by the generalized Parseval identity also the inner product, as for $x = \sum_{n=1}^{\infty} c_n u_{\alpha_n}$ and $y = \sum_{n=1}^{\infty} d_n u_{\alpha_n}$

$$(x,y)_{\mathscr{H}} = \sum_{n=1}^{\infty} x(\alpha_n)\overline{y(\alpha_n)} = \sum_{n=1}^{\infty} c_n \overline{d_n} = (Tx, Ty)_{\mathscr{G}}.$$

Thus $\mathscr{H}$ and $\mathscr{G}$ are isomorphic.

Conversely, if $\mathscr{H}$ and $\mathscr{G}$ are isomorphic, then we can map any complete orthonormal system in $\mathscr{H}$ into a complete orthonormal system in $\mathscr{G}$, hence both have the same dimension. ■

Corollary 1.51 All separable Hilbert spaces are isomorphic and in particular isomorphic to ℓ^2.

Proof. We saw that all separable Hilbert spaces have dimension $\aleph_0$. ■

■ **Example 1.5** The **Haar functions** are a sequence of functions in $L^2[0,1]$ defined as follows:

$$\phi_0(t) = 1$$
$$\phi_1(t) = 1_{[0,1/2)} - 1_{[1/2,1]}$$
$$\phi_2(t) = \sqrt{2}(1_{[0,1/4)} - 1_{[1/4,1/2)}) \qquad \phi_3(t) = \sqrt{2}(1_{[1/2,3/4)} - 1_{[3/4,1]}),$$

and in general,

$$\phi_{2^n+k}(t) = 2^{n/2}\left(1_{[2^{-n}k,2^{-n}(k+1/2))} - 1_{[2^{-n}(k+1/2),2^{-n}(k+1))}\right), \qquad n \in \mathbb{N},\ k = 0,\cdots 2^n - 1.$$

The Haar functions form an orthonormal system: because take ϕ_j—on the interval on which it is non-zero any ϕ_i with $i < j$ is constant. Also, the span of all (ϕ_n) is the same as the span of all step functions with dyadic intervals. It is known that this span is dense in $L^2[0,1]$, hence the Haar functions form an orthonormal basis in $L^2[0,1]$.

If therefore follows that for every $f \in L^2[0,1]$:

$$f = \sum_{n=0}^{\infty} (f, \phi_n)\phi_n.$$

The limit is in $L^2[0,1]$. The question is whether the sum also converges pointwise (a.e.).

Such questions are usually quite hard. For the specific choice of the Haar basis it is relatively easy, due to the “good" ordering of those functions.

The first observation is that

$$\text{Span}\{\phi_i \mid 0 \le i \le 2^n - 1\} = \{\text{step functions over dyadic intervals of length } 2^{-n}\} \equiv \Pi_n,$$

i.e., functions of the form

$$\sum_{k=0}^{2^n-1} c_k \psi_{n,k},$$

where

$$\psi_{n,k} = 2^{n/2}\, 1_{[2^{-n}k, 2^{-n}(k+1))}.$$

Thus, the linear operator S_n

$$S_n f = \sum_{k=0}^{2^n-1} (f, \phi_k)\phi_k,$$

returns the orthogonal projection of f over the space of step functions Π_n, and by the uniqueness of this projection,

$$S_n f = \sum_{k=0}^{2^n-1} (f, \psi_{n,k})\psi_{n,k}.$$

Note that,

$$S_n f = \sum_{k=0}^{2^n-1} \left(2^n \int_{2^{-n}k}^{2^{-n}(k+1)} f(t)\, dt \right) 1_{[2^{-n}k, 2^{-n}(k+1))}.$$

It follows that $(S_n f)(x)$ is equal to the average of f in an interval of size 2^{-n} around x. It is known that as $n \to \infty$ this average converges to $f(x)$ a.e.

■

1.5 Weak convergence

Definition 1.21 Let (x_n) be a sequence of vectors in an inner-product space $(\mathbb{H},(\cdot,\cdot))$. We say that this sequence **weakly converges** (**מתכנסת חלש**) to a vector $x \in \mathbb{H}$ if

$$\forall y \in \mathbb{H} \qquad \lim_{n\to\infty}(x_n, y) = (x, y).$$

x is called the **weak limit** (**גבול חלש**) of (x_n), and we write

$$x_n \rightharpoonup x.$$

Definition 1.22 A sequence (x_n) of vectors in an inner-product space $(\mathbb{H},(\cdot,\cdot))$ is said to be **weakly Cauchy** if

$$\forall y \in \mathbb{H} \qquad (x_n, y) \text{ is a Cauchy sequence.}$$

Comments 1.2

① **Uniqueness of weak limit**: If a sequence has a weak limit then the weak limit is unique, for if x and z are both weak limits of (x_n) then

$$\forall y \in \mathscr{H} \qquad \lim_{n\to\infty}(x_n, y) = (x, y) = (z, y),$$

from which follows that $x = z$.

② Since there is a one-to-one correspondence between $(\cdot, y)$ and bounded linear functionals we can reformulate the definition of weak convergence as follows: a sequence (x_n) of vectors in a Hilbert space is said to weakly converge to x if

$$\forall f \in \mathscr{H}^* \qquad \lim_{n\to\infty} f(x_n) = f(x).$$

Proposition 1.52 — Strong convergence implies weak convergence. Let $(\mathbb{H},(\cdot,\cdot))$ be an inner-product space. If $x_n \to x$ then $x_n \rightharpoonup x$.

Proof. Let $x_n \to x$. For all $f \in \mathbb{H}^*$,

$$\lim_{n\to\infty}|f(x_n) - f(x)| \le \lim_{n\to\infty}\|f\|\,\|x_n - x\| = 0,$$

i.e., $x_n \rightharpoonup x$. ∎

Proposition 1.53 In a finite-dimensional Hilbert space strong and weak convergence coincide.

Proof. It only remains to prove that weak convergence implies strong convergence. Let $\{u_k \mid 1 \le k \le N\}$ be an orthonormal basis for $\mathscr{H}$. Let x_n we a weakly converging sequence with limit x. Expanding we get,

$$x_n = \sum_{k=0}^{N} (x_n, u_k) u_k \qquad \text{and} \qquad x = \sum_{k=0}^{N} (x, u_k) u_k.$$

Now,

$$\|x_n - x\| = \left\| \sum_{k=0}^{N} (x_n - x, u_k) u_k \right\| \le \sum_{k=0}^{N} \|(x_n - x, u_k) u_k\| = \sum_{k=0}^{N} |(x_n - x, u_k)|.$$

Since the right hand side tends to zero as $n \to \infty$ it follows that $x_n \to x$. ■

What about the general case? Does weak convergence imply strong convergence. The following proposition shows that the answer is no:

Proposition 1.54 Every infinite orthonormal sequence in an inner-product space $(\mathbb{H}, (\cdot,\cdot))$ weakly converges to zero.

Comment 1.13 Obviously, an infinite orthonormal sequence does not (strongly) converge to zero as the corresponding sequence of norms is constant and equal to one, and the norm is continuous with respect to (strong) convergence.

Proof. This is an immediate consequence of the Bessel inequality: if (u_n) is an orthonormal sequence, then for every $x \in \mathbb{H}$

$$\sum_{n=1}^{\infty} |(x, u_n)|^2 \le \|x\|^2,$$

from which follows that

$$\forall x \in \mathbb{H} \qquad \lim_{n \to \infty} |(u_n, x)| = 0,$$

i.e., $u_n \rightharpoonup 0$. ■

■ **Example 1.6 — Riemann-Lebesgue lemma.** Let $\mathscr{H} = L^2[0, 2\pi]$, and consider the vectors

$$u_n(x) = \frac{1}{\sqrt{2\pi}} e^{\imath n x}.$$

It is easy to check that they constitute an orthonormal system, hence

$$\forall f \in \mathscr{H} \qquad \lim_{n \to \infty} \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{\imath n x}\, dx = 0.$$

■

Proposition 1.55 — The norm is weakly lower-semicontinuous. Let $(\mathbb{H},(\cdot,\cdot))$ be an inner-product space. If $x_n \rightharpoonup x$ then

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|.$$

Proof. By definition of weak convergence:

$$\|x\|^2 = \lim_{n\to\infty} (x_n, x) = \lim_{n\to\infty} |(x_n, x)| \le \liminf_{n\to\infty} \|x_n\| \, \|x\|.$$

■

The next proposition shows that for a weakly-converging sequence to strongly converge, it is sufficient for the norm to converge as well:

Proposition 1.56 — Weak + convergence of norm = strong. Let $x_n \rightharpoonup x$. Then $x_n \to x$ if and only if

$$\lim_{n\to\infty} \|x_n\| = \|x\|.$$

Proof. Suppose $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. Then,

$$\|x_n - x\|^2 = \underbrace{\|x_n\|^2}_{\to\|x\|^2} + \|x\|^2 - \underbrace{(x, x_n)}_{\to\|x\|^2} - \underbrace{(x_n, x)}_{\to\|x\|^2} \to 0.$$

The other direction has already been proved. ■

So far we haven't made any use of the weak Cauchy property. Here is comes:

Proposition 1.57 A weak Cauchy sequence in a Hilbert space is bounded.

Comment 1.14 This is a generalization of the statement that a strong Cauchy sequence is bounded. Obviously, the current statement is stronger (and harder to prove).

Comment 1.15 Baire's category theorem: a set A is called **nowhere dense** (קבוצה דלילה) if its closure has an empty interior. Equivalently, if there is no open set B such that $A \cap B$ is dense in B. (A line is nowhere dense in $\mathbb{R}^2$. The rational are not nowhere dense in $\mathbb{R}$). A set A that is a countable union of nowhere dense sets is called of the **first category**; otherwise it is called of the **second category**.

Baire's theorem states that in a complete metric space a set of first category has an empty interior. Equivalently, a set that has a non-empty interior is not of the first category. It follows that if $\mathscr{X}$ is a complete metric space and

$$\mathscr{X} = \bigcup_n A_n,$$

then not all the A_n are nowhere dense. That is, there exists an m such that $(\overline{A_m})^\circ$ is not empty. If the A_n happen to be closed, then one of them must contain an open ball.

Proof. Recall that a sequence is weakly Cauchy if for every $y \in \mathscr{H}$, (x_n, y) is a Cauchy sequence, i.e., converges (but not necessarily to the same (x, y)). For every $n \in \mathbb{N}$ define the set

$$V_n = \{y \in \mathscr{H} \mid \forall k \in \mathbb{N},\ (x_k, y)| \le n\}.$$

These sets are increasing $V_1 \subseteq V_2 \subseteq \dots$. They are closed (by the continuity of the inner product). Since for every $y \in \mathscr{H}$ the sequence $|(x_k, y)|$ is bounded,

$$\forall y \in \mathscr{H} \quad \exists n \in \mathbb{N} \quad \text{such that} \quad y \in V_n,$$

i.e.,

$$\bigcup_{n=1}^{\infty} V_n = \mathscr{H}.$$

By Baire's Category theorem there exists an $m \in \mathbb{N}$ for which V_m contains a ball, say, $B(y_0, \rho)$. That is,

$$\forall y \in B(y_0, \rho) \qquad \forall k \in \mathbb{N} \qquad |(x_k, y)| \le m.$$

It follows that for every $k \in \mathbb{N}$,

$$\|x_k\| = \left(x_k, \frac{x_k}{\|x_k\|}\right) = \frac{2}{\rho}\left|\left(x_k, \frac{\rho}{2}\frac{x_k}{\|x_k\|}\right)\right| = \frac{2}{\rho}\left|\left(x_k, y_0 + \frac{\rho}{2}\frac{x_k}{\|x_k\|}\right) - (x_k, y_0)\right| \le \frac{4m}{\rho}.$$

■

Why are weak Cauchy sequences important? The following theorem provides the answer.

Theorem 1.58 — Hilbert spaces are weakly complete. Every weak Cauchy sequence in a Hilbert space weakly converges.

Proof. Let (x_n) be a weak Cauchy sequence. For every $y \in \mathscr{H}$, the sequence of scalars (x_n, y) converges, Define the functional

$$F(y) = \lim_{n\to\infty} (y, x_n).$$

It is a linear functional, and it is bounded, as

$$|F(y)| \le \left(\limsup_{n\to\infty} \|x_n\|\right) \|y\|.$$

By the Riesz representation theorem there exists an x, such that $F(y) = (y,x)$, i.e.,

$$\forall y \in \mathscr{H} \qquad (y,x) = \lim_{n\to\infty} (y,x_n),$$

which completes the proof. ■

Theorem 1.59 — The unit ball is weakly sequentially compact. Every bounded sequence in a Hilbert space $(\mathscr{H},(\cdot,\cdot))$ has a weakly converging subsequence. (Equivalently, the unit ball in a Hilbert space is weakly compact.)

Comment 1.16 Note that it is not true for strong convergence! Take for example $\mathscr{H} = \ell^2$ and $x_n = e_n$ (the n-th unit vector). This is a bounded sequence that does not have any (strongly) converging subsequence. On the other hand, we saw that it has a weak limit (zero).

Proof. We first prove the theorem for the case where $\mathscr{H}$ is separable. Let (x_n) be a bounded sequence. Let (y_n) be a dense sequence. Consider the sequence (y_1,x_n). Since it is bounded there exists a subsequence $(x_n^{(1)})$ of (x_n) such that $(y_1,x_n^{(1)})$ converges. Similarly, there exists a sub-subsequence $(x_n^{(2)})$ such that $(y_2,x_n^{(2)})$ converges (and also $(y_1,x_n^{(2)})$ converges). We proceed inductively to construct the subsequence $(x_n^{(k)})$ for which all the $(y_\ell,x_n^{(k)})$ for $\ell \le k$ converges. Consider the diagonal sequence $(x_n^{(n)})$, which is a subsequence of (x_n). For every k, $(x_n^{(n)})$ has a tail that is a subsequence of $(x_n^{(k)})$, from which follows that for every k,

$$\ell_k \equiv \lim_{n\to\infty} (y_k,x_n^{(n)}) \quad \text{exists.}$$

Next, we show that,

$$\forall y \in \mathscr{H} \qquad \lim_{n\to\infty} (y,x_n^{(n)}) \quad \text{exists,}$$

from which follows that $x_n^{(n)}$ is a weak Cauchy sequence, and by the previous theorem weakly converges.

Let $y \in \mathscr{H}$ and let $y_m \to y$ be a sequence in the dense countable set. Set,

$$\ell_m = \lim_{n\to\infty} (y_m,x_n^{(n)}).$$

(ℓ_m) is a Cauchy sequence, as

$$|\ell_m - \ell_k| = \left|\lim_{n\to\infty} (y_m - y_k,x_n^{(n)})\right| \le \|y_m - y_k\| \limsup_{n\to\infty} \|x_n^{(n)}\|.$$

Let ℓ be its limit, then

$$|(y,x_n^{(n)})-\ell| \le |(y-y_m,x_n^{(n)})|+|(y_m,x_n^{(n)})-\ell| \le \|y-y_m\|\,\|x_n^{(n)}\|+|(y_m,x_n^{(n)})-\ell|,$$

and it remains to take sequentially $\limsup_n$ and $m\to\infty$.

Next, consider the case where $\mathscr{H}$ is not separable. Denote

$$\mathscr{H}_0 = \overline{\operatorname{Span}\{x_n \mid n\in\mathbb{N}\}}.$$

$\mathscr{H}_0$ is a closed separable subspace of $\mathscr{H}$. Hence there exists a subsequence (x_{n_k}) of (x_n) that weakly converges in $\mathscr{H}_0$, namely, there exists an $x\in\mathscr{H}_0$, such that

$$\forall y\in\mathscr{H}_0 \qquad \lim_{n\to\infty}(y,x_n) = (y,x).$$

Take any $y\in\mathscr{H}$. From the projection theorem,

$$y = \mathbb{P}_{\mathscr{H}_0}y + \mathbb{P}_{\mathscr{H}_0^\perp}y,$$

hence

$$\lim_{n\to\infty}(y,x_n) = \lim_{n\to\infty}(\mathbb{P}_{\mathscr{H}_0}y,x_n) = (\mathbb{P}_{\mathscr{H}_0}y,x) = (\mathbb{P}_{\mathscr{H}_0}y+\mathbb{P}_{\mathscr{H}_0^\perp}y,x) = (y,x),$$

which completes the proof. ■

Weak convergence does not imply strong convergence, and does not even implies a strongly convergent subsequence. The following theorem establishes another relation between weak and strong convergence.

Theorem 1.60 — Banach-Saks. Let $(\mathscr{H},(\cdot,\cdot))$ be a Hilbert space. If $x_n \rightharpoonup x$ then there exists a subsequence (x_{n_k}) such that the sequence of running averages

$$S_k = \frac{1}{k}\sum_{j=1}^{k} x_{n_j}$$

strongly converges to x.

Proof. Without loss of generality we may assume that $x=0$, otherwise consider the sequence (x_n-x). As (x_n) weakly converges it is bounded; denote $M=\limsup_n\|x_n\|$. We construct the subsequence (x_{n_k}) as follows. Because $x_n\rightharpoonup 0$, we can choose x_{n_k} such that

$$\forall j<k \qquad |(x_{n_k},x_{n_j})| \le \frac{1}{k}.$$

Then,

$$\left\| \frac{1}{k} \sum_{j=1}^{k} x_{n_j} \right\|^2 = \frac{1}{k^2} \left(\sum_{j=1}^{k} \|x_{n_j}\|^2 + 2\,\mathrm{Re} \sum_{1 \le i \le j \le k} (x_{n_i}, x_{n_j}) \right)$$
$$\le \frac{1}{k^2} \left(kM^2 + 2\left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{k-1}{k} \right) \right)$$
$$\le \frac{M^2 + 2}{k},$$

i.e., the running average strongly converges to zero. ■

Corollary 1.61 — Strongly closed + convex implies weakly closed. Let $(\mathscr{H}, (\cdot,\cdot))$ be a Hilbert space. Every closed and convex set $\mathscr{C} \subset \mathscr{H}$ is also closed with respect to sequential weak convergence. That is, if (x_n) is a subsequence in $\mathscr{C}$ with weak limit $x \in \mathscr{H}$, then $x \in \mathscr{C}$.

Comment 1.17 “Closed" means closed with respect to strong convergence. A closed set is not necessarily closed with respect to weak convergence, unless it is convex.

Proof. Suppose that x_n is a sequence in $\mathscr{C}$ with weak limit $x \in \mathscr{H}$. By the Banach-Sack theorem there exists a subsequence whose running averages (strongly) converge to x. Because $\mathscr{C}$ is convex, the running averages are also in $\mathscr{C}$, and because $\mathscr{C}$ is closed, $x \in \mathscr{C}$. ■

Definition 1.23 A real-valued functional $f : \mathscr{H} \to \mathbb{R}$ is called **lower-semicontinuous** (רציף למחצה מלרע) at a point x if for every sequence $x_n \to x$,

$$f(x) \le \liminf_{n\to\infty} f(x_n).$$

Definition 1.24 A real-valued functional defined on a convex set $\mathscr{C}$ is called **convex** if

$$\forall 0 \le t \le 1 \qquad f(tx + (1-t)y) \le t\, f(x) + (1-t)\, f(y).$$

■ **Example 1.7** For every $p \ge 1$ the real-valued functional $f(x) = \|x\|^p$ is convex. ■

Theorem 1.62 Let $(\mathscr{H}, (\cdot,\cdot))$ be a Hilbert space. Let $\mathscr{C} \subset \mathscr{H}$ be a closed, bounded, convex set. Let $f : \mathscr{C} \to \mathbb{R}$ be bounded from below, convex and lower-semicontinuous. Then f has a minimum in $\mathscr{C}$.

Proof. Let

$$m = \inf\{f(x) \mid x \in \mathscr{C}\},$$

which is finite because f is bounded from below. Let (x_n) be a sequence in $\mathscr{C}$ such that

$$\lim_{n\to\infty} f(x_n) = m.$$

Since $\mathscr{C}$ is bounded there exists a subsequence x_{n_k} that weakly converges to x. By the previous corollary, since $\mathscr{C}$ is closed and convex, $x \in \mathscr{C}$.

From the Banach-Saks theorem follows that there exists a sub-subsequence (heck! no relabeling) whose running average,

$$s_k = \frac{1}{k}\sum_{j=1}^{k} x_{n_j}$$

strongly converges to x. Since f is convex, it follows inductively that

$$f(s_k) \le \frac{1}{k}\sum_{j=1}^{k} f(x_{n_j}).$$

Since f is moreover lower-semicontinuous,

$$m \le f(x) \le \liminf_{k\to\infty} f(s_k) \le \liminf_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} f(x_{n_j}) = m,$$

i.e., x is a minimizer of f. ■

1.5.1 The weak topology

We have seen that bounded linear functional are continuous with respect to the strong topology. Weak convergence is defined such that bounded linear functionals are continuous with respect to weak convergence, namely, if $x_n \rightharpoonup x$ then

$$\forall T \in \mathscr{H}^* \qquad T x_n \to T x$$

(this is the definition of weak convergence).

What is the topology underlying the concept of weak convergence? It is the coarsest topology for which bounded linear functionals are continuous. Consider all the topologies on $\mathscr{H}$ with respect to which all bounded linear functionals are continuous; this set is not empty as it includes the norm topology. The intersection of all these topologies is a topology—it is the weak topology.

The open sets in the weak topology are generated by the **subbase**:

$$\{T^{-1}(U) \mid T \in \mathscr{H}^*,\ U \text{ open in } \mathbb{C}\},$$

or equivalently,

$$\{x \mid (x,y) \in U,\ y \in \mathscr{H},\ U \text{ open in } \mathbb{C}\},$$

That is, a set is open with respect to the weak topology if and only if it can be written as a union of sets that are finite intersections of sets of the form $T^{-1}(U)$.

1.6 Approximation by polynomials

This section deals with the approximation of functions by polynomials. There are main two reasons to consider this: (i) a tool for proving theorems, (ii) a computational tool for approximating solutions to equations that cannot be solved by analytical means.

In the following we denote by $\Pi_n[a,b]$ the set of polynomials over $[a,b]$ of degree n or less, and

$$\Pi[a,b] = \bigcup_{n=1}^{\infty} \Pi_n[a,b].$$

The following well-known theorem is usually taught in the first year of undergraduate studies:

Theorem 1.63 — Weierstraß approximation theorem. Every continuous function on an interval $[a,b]$ can be uniformly approximated by polynomials. That is, for every $f \in C[a,b]$ and every $\varepsilon > 0$ there exists a polynomial $p \in \Pi[a,b]$ for which

$$\sup_{a \le x \le b} |f(x) - p(x)| < \varepsilon.$$

Proof. Not in this course. ∎

Let K be a compact hausdorff space (i.e., every two distinct points have disjoint neighborhoods). We denote by $C(K)$ the space of continuous scalar-valued functions on K, and define the norm,

$$\|f\|_\infty = \max_{x \in K} |f(x)|.$$

The maximum exists because K is compact.

Definition 1.25 A linear subspace of $C(K)$ that is closed also under multiplication is called an **algebra** (of continuous functions).

■ **Example 1.8** $\Pi(K) \subset C(K)$ is an algebra of continuous functions. ■

Definition 1.26 A collection A of functions on K is called **point-separating** (מפרידה בין נקודות) if for every $x, y \in K$ there exists a function $f \in A$, such that

$$f(x) \neq f(y).$$

Theorem 1.64 — Stone-Weierstraß. Let A be a point-separating algebra of continuous real-valued functions on a compact Hausdorff space K. Suppose furthermore that includes the function 1. Then A is dense in $C(K)$ with respect to the maximum norm.

Comment 1.18 The Weierstraß approximation theorem is a particular case.

Comment 1.19 This theorem has nothing to do with Hilbert spaces but we will use it in the context of Hilbert spaces.

Proof. The uniform limit of continuous functions is continuous, hence the closure of A (with respect to the infinity norm) is also an algebra of continuous functions[9]. Thus, we can assume that A is closed and prove that $A = C(K)$.

We first show that if $f \in A$ then $|f| \in A$. Let $f \in A$ be given. By the Weierstraß approximation theorem,

$$(\forall \varepsilon > 0)(\exists p \in \Pi[-\|f\|_\infty, \|f\|_\infty]) : \max_{-\|f\|_\infty \leq t \leq \|f\|_\infty} ||t| - p(t)| < \varepsilon.$$

Every polynomial of f is also in A (because A is an algebra), and

$$\max_{x \in K} ||f(x)| - p(f(x))| < \varepsilon,$$

i.e., $|f|$ is in the closure of A, and since A is closed, then $|f| \in A$.

Next, we show that if $f, g \in A$ then so are $f \wedge g$ and $f \vee g$, as

$$f \wedge g = \frac{1}{2}(f+g) - \frac{1}{2}|f-g| \qquad \text{and} \qquad f \vee g = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|.$$

Next, we show that

$$(\forall F \in C(K),\ x \in K,\ \varepsilon > 0)(\exists f \in A) : (f(x) = F(x)) \wedge (f \leq F + \varepsilon).$$

[9] If $f, g \in \bar{A}$ then there exist sequences $f_n, g_n \in A$ such that $f_n \to f$ and $g_n \to g$. Then $f_n g_n \to fg$, hence $fg \in \bar{A}$.

Because A separates between points and $1 \in A$, there exists for every $x, y \in A$ and every $\alpha, \beta \in \mathbb{R}$ an element $g \in A$ such that $g(x) = \alpha$ and $g(y) = \beta$ (very easy to show). Thus, given F, x, and ε, then

$$(\forall y \in K)(\exists g_y \in A) : (g_y(x) = F(x)) \wedge (g_y(y) = F(y)).$$

Since both F and g_y are continuous, there exists an open neighborhood U_y of y such that

$$g_y|_{U_y} \le F|_{U_y} + \varepsilon.$$

The collection $\{U_y\}$ is an open covering of K, and since K is compact, there exists a finite open sub-covering $\{U_{y_i}\}_{i=1}^n$. The function

$$f = g_{y_1} \wedge g_{y_2} \wedge \dots g_{y_n}$$

satisfies the required properties.

Let $F \in C(K)$ and $\varepsilon > 0$ be given. We have seen that

$$(\forall x \in K)(\exists f_x \in A) : (f_x(x) = F(x)) \wedge (f_x \le F + \varepsilon).$$

Relying again on continuity, there exists an open neighborhood V_x of x, such that

$$f_x|_{V_x} \ge F|_{V_x} - \varepsilon.$$

By the compactness of K, K can be covered by a finite number of $\{V_{x_i}\}_{i=1}^n$. The function

$$f = f_{x_1} \vee f_{x_2} \vee \dots f_{x_n}$$

satisfies,

$$\|f - F\|_\infty \le \varepsilon,$$

which completes the proof. ■

This theorem, as is, does not apply to complex-valued functions. For this, we need the following modification:

Theorem 1.65 — Stone-Weierstraß for complex functions. Let A be a point-separating algebra of continuous complex-valued functions on a compact Hausdorff space K, that includes the function 1 and is self-conjugate (**צמודה לעצמה**) in the sense that $f \in A$ implies $\overline{f} \in A$. Then A is dense in $C(K)$ with respect to the maximum norm.

Proof. Let $A_R \subset A$ be the subset of real-valued functions. Noting that

$$\operatorname{Re} f = \frac{f + \overline{f}}{2} \qquad \text{and} \qquad \operatorname{Im} f = \frac{f - \overline{f}}{2\imath}$$

are both in A_R, it follows that A_R is point-separating. Thus, we can uniformly approximate both $\operatorname{Re} f$ and $\operatorname{Im} f$. ■

Corollary 1.66 Let $K \subset \mathbb{R}^n$ be compact, then every function $f \in C(K)$ can be uniformly approximated by a polynomial in n variables.

We return now to Hilbert spaces. Consider the space $L^2[a,b]$. Since the polynomials $\Pi[a,b]$ are dense in $L^2[a,b]$ (any uniform ε-approximation is an $(b-a)\varepsilon$-L^2 approximation), it follows that a Gram-Schmidt orthonormalization of the sequence of vectors

$$1, x, x^2, \dots$$

yields a complete orthonormal system.

■ **Example 1.9** Consider the following sequence of polynomials in $[-1,1]$:

$$p_0(x) = 1 \qquad p_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n,$$

which are called the **Legendre polynomials**. For example,

$$p_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x,$$

and

$$p_2(x) = \frac{1}{8}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}2x(x^2-1) = \frac{1}{4}(3x^2-1),$$

and it is clear that $p_n \in \Pi_n$.

We claim that the p_n are orthogonal. For $m < n$,

$$\int_{-1}^{1} x^m p_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1} x^m \left(\frac{d^n}{dx^n}(x^2-1)^n\right) dx.$$

Integrating by parts n times we get zero (do you see why the boundary terms vanish?). Thus,

$$\int_{-1}^{1} p_m(x) p_n(x)\,dx = 0.$$

What about their norms?

$$\int_{-1}^{1} p_n^2(x)\,dx = \left(\frac{1}{2^n n!}\right)^2 \int_{-1}^{1} \left(\frac{d^n}{dx^n}(x^2-1)^n\right)\left(\frac{d^n}{dx^n}(x^2-1)^n\right) dx.$$

We integrate n times by parts (again the boundary term vanishes) and get

$$\begin{aligned}
\int_{-1}^{1} p_n^2(x)\,dx &= \left(\frac{1}{2^n n!}\right)^2 (-1)^n \int_{-1}^{1} \left(\frac{d^{2n}}{dx^{2n}}(x^2-1)^n\right)(x^2-1)^n\,dx \\
&= \left(\frac{1}{2^n n!}\right)^2 (-1)^n (2n!) \int_{-1}^{1} (x^2-1)^n\,dx = \frac{2}{n+1}
\end{aligned}$$

(the last identity can be proved inductively). ■

1.7 Fourier series and transform

1.7.1 Definitions

Consider the Hilbert space $\mathscr{H} = L^2[0,2\pi]$ and the (complex) trigonometric functions

$$\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{\imath nx} \qquad n \in \mathbb{Z}.$$

Proposition 1.67 The trigonometric functions $\{\phi_n\}$ form a complete orthonormal system.

Proof. It is easy to see that the $\{\phi_n\}$ are orthonormal as for $m \neq n$,

$$(\phi_m, \phi_n) = \frac{1}{2\pi} \int_0^{2\pi} e^{\imath mx} e^{-\imath nx}\, dx = \frac{1}{2\pi} \left. \frac{e^{\imath(m-n)x}}{\imath(m-n)} \right|_0^{2\pi} = 0.$$

Why is this system complete? Consider the linear space of **trigonometric polynomials**:

$$A = \operatorname{Span}\{\phi_n \mid n \in \mathbb{N}\}.$$

It is a self-adjoint algebra of continuous functions that includes the function 1. There is only one problem: it does not separate the points 0 and 2π. Thus, we view A as an algebra of continuous functions on a circle, S (i.e., we identify the points 0 and 2π). A is point separating for functions on S, hence it is dense in $C(S)$. On the other hand, $C(S)$ is dense in $L^2[0,2\pi]$, which proves the completeness of the system of the trigonometric functions. ■

Corollary 1.68 For every $f \in L^2[0,2\pi]$,

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{f}(k) e^{\imath nx},$$

where the series converges in $L^2[0,2\pi]$, and

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-\imath nx}\, dx.$$

Moreover,

$$\int_0^{2\pi} |f(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |\hat{f}(k)|^2.$$

Proof. This follows immediately from the properties of a complete orthonormal system. ■

Comment 1.20 It is customary to define the Fourier components $\hat{f}(k)$ divided by an additional factor of $1/\sqrt{2\pi}$, in which case we get

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(k) e^{\imath n x},$$

where

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-\imath n x} \, dx,$$

and,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |\hat{f}(k)|^2.$$

The trigonometric series equal to f is called its **Fourier series** (**טור פורייה**).

Real-valued Fourier series

A related complete orthonormal system is the system

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \sin 2x, \dots, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \cos 2x, \dots \right\}.$$

(The proof of its completeness is again based on the Stone-Weierstraß theorem.) Then, every $L^2[0,2\pi]$ function has the following expansion:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx \qquad a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx,$$

and

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx.$$

The Parseval identity is

$$\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx = |a_0|^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2 \right).$$

Clearly, if f is real-valued then so are the Fourier coefficients.

1.7.2 Two applications

The first application of the trigonometric functions uses only their completeness, but not the Fourier series.

Definition 1.27 A sequence of points (x_n) in $[0,2\pi]$ is said to be **equi-distributed** (מפולגת במידה שווה) if for every a,b:

$$\lim_{n\to\infty} \frac{|\{k \le n \mid a \le x_k \le b\}|}{n} = \frac{b-a}{2\pi}.$$

Note that

$$\frac{|\{k \le n \mid a \le x_k \le b\}|}{n} = \frac{1}{n}\sum_{k=1}^{n} 1_{[a,b]}(x_k),$$

hence a sequence is equi-distributed if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} 1_{[a,b]}(x_k) = \frac{1}{2\pi}\int_0^{2\pi} 1_{[a,b]}(x)\,dx.$$

(Empirical average converges to “ensemble average".) If follows that the sequence is equi-distributed if and only if for every step function f,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(x_k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx.$$

Proposition 1.69 — Hermann Weyl's criterion for equi-distribution. A sequence (x_n) in $[0,2\pi]$ is equi-distributed if and only if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} e^{\imath m x_k} = \frac{1}{2\pi}\int_0^{2\pi} e^{\imath m x}\,dx = \delta_{m,0}.$$

Proof. Suppose that the points are equi-distributed. For every m and $\varepsilon > 0$ there exist step functions f_1, f_2, such that

$$\forall x \in [0,2\pi] \qquad f_1(x) \le \cos(mx) \le f_2(x),$$

and $\|f_1 - f_2\|_\infty < \varepsilon$. Now,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \cos(mx_k) \le \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f_2(x_k) = \frac{1}{2\pi}\int_0^{2\pi} f_2(x)\,dx,$$

whereas

$$\frac{1}{2\pi}\int_0^{2\pi} \cos(mx)\,dx \le \frac{1}{2\pi}\int_0^{2\pi} f_2(x)\,dx.$$

Similarly,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \cos(mx_k) \ge \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f_1(x_k) = \frac{1}{2\pi}\int_0^{2\pi} f_1(x)\,dx,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(mx)\,dx \geq \frac{1}{2\pi}\int_0^{2\pi} f_1(x)\,dx.$$

It follows that

$$\left|\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\cos(mx_k) - \frac{1}{2\pi}\int_0^{2\pi}\cos(mx)\,dx\right| \leq \frac{1}{2\pi}\int_0^{2\pi}(f_2(x)-f_1(x))\,dx \leq \varepsilon.$$

We proceed similarly for the imaginary part of $e^{\imath mx}$.

Conversely, suppose that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} e^{\imath mx_k} = \frac{1}{2\pi}\int_0^{2\pi} e^{\imath mx}\,dx.$$

Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(x_k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

for every trigonometric polynomial f. Let F be a continuous function. Then for every $\varepsilon > 0$ there exists a trigonometric polynomial f such that $\|F - f\|_\infty < \varepsilon$ (relying again on the Stone-Weierstraß theorem). Then,

$$\left|\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} F(x_k) - \underbrace{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(x_k)}_{\frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx}\right| \leq \varepsilon,$$

and

$$\left|\frac{1}{2\pi}\int_0^{2\pi} F(x)\,dx - \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx\right| \leq \varepsilon,$$

which proves that $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} F(x_k) = \frac{1}{2\pi}\int_0^{2\pi} F(x)\,dx$ for every continuous F. It follows that this is true also for step functions since we can uniformly approximate step functions by continuous functions.

■

Corollary 1.70 — Piers Bohl-Wacław Sierpinsky theorem, 1910. If $\alpha/2\pi$ is irrational then the sequence

$$(n\alpha \mod 2\pi)$$

is equi-distributed in $[0, 2\pi]$.

Proof. This proof is due to Weyl (the standard theorem is for the interval $[0,1]$). For every $m \neq 0$:

$$\frac{1}{n}\sum_{k=1}^{n} e^{\imath m x_k} = \frac{1}{n}\sum_{k=1}^{n} e^{\imath m k\alpha} = \frac{1}{n} e^{\imath m\alpha}\frac{e^{\imath n m\alpha}-1}{e^{\imath m\alpha}-1}$$

Because $\alpha/2\pi$ is irrational the denominator is never zero, and it follows as once that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} e^{\imath m x_k} = 0.$$

∎

Comment 1.21 We showed in fact that if a sequence (x_n) is equi-distributed in $[0,2\pi]$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(x_n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

for every continuous function f. Does it apply also for all $f \in L^1[0,2\pi]$? No. Take for example an equi-distributed sequence whose elements are rational and take f to be the Dirichlet function.

Comment 1.22 In 1931 George Birkhoff proved the celebrated **ergodic theorem**: Let $(\Omega,\mathscr{B},\mu)$ be a measure space, and let $T:\Omega\to\Omega$ be a **measure preserving transformation**, namely,

$$(\forall A\in\mathscr{B}) \qquad \mu(A) = \mu(T^{-1}(A)),$$

where T^{-1} is to be understood as a set function. For $f\in L^1(\Omega)$ and $x\in\Omega$, consider the **long-time average**,

$$F(x) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{\infty} f(T^k x).$$

The (strong) ergodic theorem states that $F(x)$ exists for almost every x, and it T invariant, namely

$$F\circ T = F.$$

T is said to be **ergodic** if for every $E\in\mathscr{B}$ for which $T^{-1}(E) = E$ (i.e., for every T-invariant set), either $\mu(E) = 0$ or $\mu(E) = 1$ (only trivial sets are T-invariant). If T is ergodic (and the measure is finite) then F is constant and equal to

$$F(x) = \frac{\int_\Omega f\,d\mu}{\mu(\Omega)}.$$

Back to our example: for $\Omega = [0,2\pi]$ the transformation

$$Tx = x+\alpha \mod 2\pi$$

with $\alpha/2\pi$ irrational can be shown to be ergodic, hence

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(x_n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

for every $L^1[0,2\pi]$ function.

The second application uses Fourier series:

Theorem 1.71 — Isoperimetric inequality. Let γ be a closed simple planar curve of length L and enclosed area S. Then

$$S \le \frac{L^2}{4\pi}.$$

(For those with an inclination to geometry: we get an equality for the circle.)

Proof. The proof here will be for the special case where the curve is piecewise continuously differentiable. WLOG we may assume that the curve is parametrized on $[0,2\pi]$, and that the parametrization is proportional to arclength. That is, if $\gamma : [0,2\pi]$ is of the form

$$\gamma(t) = (x(t), y(t)),$$

then γ differentiable everywhere except at most at a finite number of points, and

$$[x'(t)]^2 + [y'(t)]^2 = \left(\frac{L}{2\pi}\right)^2.$$

We expand the curve in Fourier series:

$$x(t) = \sum_{n=0}^{\infty}(a_n \cos nt + b_n \sin nt) \qquad \text{and} \qquad y(t) = \sum_{n=0}^{\infty}(c_n \cos nt + d_n \sin nt),$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos nt\,dt \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin nt\,dt$$
$$c_n = \frac{1}{\pi}\int_0^{2\pi} y(t)\cos nt\,dt \qquad d_n = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin nt\,dt.$$

Since we assume that $x(t)$ and $y(t)$ are continuously differentiable, their derivative (which is in particular in $L^2[0,2\pi]$) can also be expanded as a Fourier series. Setting:

$$x'(t) = \sum_{n=1}^{\infty}(\alpha_n \cos nt + \beta_n \sin nt) \qquad \text{and} \qquad y'(t) = \sum_{n=1}^{\infty}(\gamma_n \cos nt + \delta_n \sin nt),$$

we have

$$\alpha_n = \frac{1}{\pi}\int_0^{2\pi} x'(t)\cos nt\,dt \qquad \beta_n = \frac{1}{\pi}\int_0^{2\pi} x'(t)\sin nt\,dt$$
$$\gamma_n = \frac{1}{\pi}\int_0^{2\pi} y'(t)\cos nt\,dt \qquad \delta_n = \frac{1}{\pi}\int_0^{2\pi} y'(t)\sin nt\,dt.$$

Integrating by parts and using the periodicity of $x(t)$, $y(t)$, we find right away that

$$\alpha_n = nb_n \quad \beta_n = -na_n \quad \gamma_n = nd_n \quad \text{and} \quad \delta_n = -nc_n,$$

namely

$$x'(t) = \sum_{n=0}^{\infty} n(-a_n \sin nt + b_n \cos nt) \qquad \text{and} \qquad y'(t) = \sum_{n=0}^{\infty} n(-c_n \sin nt + d_n \cos nt).$$

By the Parseval identity:

$$\frac{1}{2\pi}\int_0^{2\pi} |x'(t)|^2\,dt = \frac{1}{2}\sum_{n=1}^{\infty} n^2\left(a_n^2 + b_n^2\right)$$
$$\frac{1}{2\pi}\int_0^{2\pi} |y'(t)|^2\,dt = \frac{1}{2}\sum_{n=1}^{\infty} n^2\left(c_n^2 + d_n^2\right),$$

and therefore

$$\frac{L^2}{4\pi^2} = \frac{1}{2}\sum_{n=1}^{\infty} n^2\left(a_n^2 + b_n^2 + c_n^2 + d_n^2\right).$$

The enclosed area, on the other hand, is given by

$$S = \int_0^{2\pi} x(t)y'(t)\,dt = 2\pi\,(x, y'),$$

and by the generalized Parseval identity:

$$\frac{S}{2\pi} = \frac{1}{2}\sum_{n=1}^{\infty} n\,(a_n d_n - b_n c_n).$$

Using the inequalities

$$na_n d_n \le \frac{n}{2}\left(a_n^2 + d_n^2\right) \le \frac{n^2}{2}\left(a_n^2 + d_n^2\right) \quad \text{and} \quad -nb_n c_n \le \frac{n}{2}\left(b_n^2 + c_n^2\right) \le \frac{n^2}{2}\left(b_n^2 + c_n^2\right),$$

we get

$$\frac{S}{2\pi} \le \frac{1}{4}\sum_{n=1}^{\infty} n^2\left(a_n^2 + b_n^2 + c_n^2 + d_n^2\right) = \frac{L^2}{8\pi^2},$$

which completes the proof.

Under what conditions do we get an equality? The inequality $n \le n^2$ is an equality only for $n = 1$, which means that $n = 1$ is the only non-zero Fourier component. The inequalities $2a_n d_n \le a_n^2 + d_n^2$ and $2b_n c_n \le b_n^2 + c_n^2$ are equalities only for $a_n = d_n$ and $b_n = c_n$, which is a circle. ∎

1.7.3 Pointwise convergence of Fourier series

Set $\mathscr{H} = L^2[0,2\pi]$. We define a sequence of operators $S_n : \mathscr{H} \to \mathscr{H}$ by

$$(S_n f)(x) = \sum_{k=-n}^{n} \hat{f}(k) e^{\imath kx}.$$

The functions $S_n f$ are trigonometric functions. We know that $S_n f \to f$ in $\mathscr{H}$. The question in under what conditions they is converge pointwise.

Recall the Weierstraß M-test. Since $|\hat{f}(k)e^{\imath kx}| = |\hat{f}(k)|$ is follows that if

$$\sum_{k=-\infty}^{\infty} |\hat{f}(k)| < \infty$$

then the sequence $S_n f$ converges uniformly, and in particular converges pointwise to f (since uniform convergence implies convergence in L^2 and the limit is unique). In this case we say that f has an **absolutely convergent** Fourier series.

Recall the following:

Definition 1.28 A function f is said to be **absolutely continuous** (רציפה בהחלט) if for every $\varepsilon > 0$ there exists a $\delta > 0$, such that whenever a finite sequence of disjoint intervals (x_k, y_k) satisfies

$$\sum_{k=1}^{s} |x_k - y_k| < \delta,$$

then

$$\sum_{k=1}^{s} |f(x_k) - f(y_k)| < \varepsilon.$$

Moreover,

Theorem 1.72 For a function f defined on an interval $[a,b]$, absolute continuity is equivalent to being differentiable a.e., with a Lebesgue integrable derivative f' satisfying

$$f(x) = f(a) + \int_a^x f'(t)\, dt.$$

This leads us to the following:

Theorem 1.73 Let f be absolutely continuous on $[0,2\pi]$ such that $f(0)=f(2\pi)$ and $f'\in L^2[0,2\pi]$, then the Fourier series of f converges absolutely, and moreover,

$$\lim_{n\to\infty}\|S_n f - f\|_\infty = 0.$$

Proof. Since $f'\in L^2[0,2\pi]$, then using the properties of f:

$$\widehat{f'}(n) = \frac{1}{2\pi}\int_0^{2\pi} f'(x)e^{-\imath nx}\,dx = \frac{-\imath n}{2\pi}\int_0^{2\pi} f(x)e^{-\imath nx}\,dx = -\imath n\hat{f}(n),$$

hence using the Parseval identity,

$$\begin{aligned}\sum_{n=-\infty}^{\infty}|\hat{f}(n)| &= |\hat{f}(0)| + \sum_{n\neq 0}\frac{1}{n}|\widehat{f'}(n)| \\ &\le |\hat{f}(0)| + \left(\sum_{n\neq 0}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n\neq 0}|\widehat{f'}(n)|^2\right)^{1/2} \\ &= |\hat{f}(0)| + C\|f'\|,\end{aligned}$$

which proves that the series converges uniformly. ■

Thus, we found a class of functions (which in particular includes C^1 functions) for which the convergence of the Fourier series is pointwise (and even uniform).

Fourier series of $L^1[0,2\pi]$ functions

Note that in the definition of $\hat{f}(n)$ we do not even need f to be square integrable; it only needs to be in $L^1[0,2\pi]$. Thus, we consider as of now Fourier series of L^1 functions (which however do not necessarily exist, since the series of $|\hat{f}(n)|^2$ is not necessarily summable).

Note that

$$\begin{aligned}S_n f(x) &= \sum_{k=-n}^{n}\left(\frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-\imath kt}\,dt\right)e^{\imath kx} \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\left(\sum_{k=-n}^{n} e^{-\imath k(t-x)}\right)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t)D_n(x-t)\,dt,\end{aligned}$$

where

$$D_n(x) \equiv \sum_{k=-n}^{n} e^{-\imath kx} = e^{\imath nx}\frac{e^{-\imath(2n+1)x}-1}{e^{-\imath x}-1}$$
$$= \frac{e^{-\imath(n+1)x}-e^{\imath nx}}{e^{-\imath x}-1}$$
$$= \frac{e^{-\frac{1}{2}\imath x}}{e^{-\frac{1}{2}\imath x}}\frac{e^{-\imath(n+\frac{1}{2})x}-e^{\imath(n+\frac{1}{2})x}}{e^{-\frac{1}{2}\imath x}-e^{\frac{1}{2}\imath x}}$$
$$= \frac{\sin(n+\frac{1}{2})x}{\sin\frac{1}{2}x}.$$

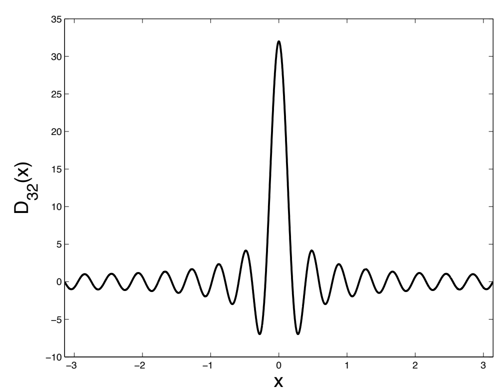
The functions $D_n(x)$ are called the **Dirichlet kernels** (**גרעיני דיריכלה**); D_n is symmetric in x, is defined on the entire line, and is 2π periodic. Also, setting $f \equiv 1$ we get

$$\frac{1}{2\pi}\int_0^{2\pi} D_n(x)\,dx = 1.$$

Changing variables, $x-t=y$, and extending f on the entire like periodically, we can get as well:

$$S_nf(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-y)D_n(y)\,dy.$$

Consider the graph of $D_n(x)$ for $n=32$:



This graph is "centered" at $x=0$, and oscillates farther away, insinuating that for large n, $S_nf(x)$ is some "local average" of f in the vicinity of the point x.

Proposition 1.74 — Localization principle. Let f,g be $L^1[0,2\pi]$ functions, such that $f=g$ in some neighborhood of a point x. Then,

$$\lim_{n\to\infty}\left(S_nf(x)-S_ng(x)\right)=0.$$

Comment 1.23 It is not implied that the limits of $S_n f(x)$ and $S_n g(x)$ exist.

Proof. We have

$$S_n f(x) - S_n g(x) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(x-y)-g(x-y)}{\sin\frac{1}{2}y}\sin\left[\left(n+\tfrac{1}{2}\right)y\right] dy.$$

It is given that $f(x-y)-g(x-y)$ vanishes in a neighborhood of $y=0$. Hence, the function

$$\frac{f(x-y)-g(x-y)}{\sin\frac{1}{2}y}$$

is integrable. The $n \to 0$ limit follows from the Riemann-Lebesgue lemma.[10] ∎

Theorem 1.75 — Dini criterion. Let $f \in L^1[0,2\pi]$ and $x \in [0,2\pi]$. Suppose there exists a scalar ℓ such that

$$\int_0^{\pi} \frac{1}{t}\left|\frac{f(x+t)+f(x-t)}{2} - \ell\right| dt < \infty,$$

then $\lim_{n\to\infty} S_n f(x) = \ell$.

Proof. We know that

$$S_n f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t) D_n(t)\, dt.$$

Since D_n is symmetric, we also have

$$S_n f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+t) D_n(t)\, dt,$$

and since D_n is normalized,

$$S_n f(x) - \ell = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{f(x-t)+f(x-t)}{2} - \ell\right) D_n(t)\, dt.$$

[10] The Riemann-Lebesgue lemma states that for every $f \in L^1[0,2\pi]$:

$$\lim_{n\to\infty}\int_0^{2\pi} f(x) e^{inx}\, dx = 0.$$

The proof is simple. A direct calculation shows that for every interval I,

$$\lim_{n\to\infty}\int_I e^{inx}\, dx = 0,$$

hence this is true for every step function, and by dominated convergence for every positive integrable functions, and finally for every integrable function.

Now,

$$S_n f(x) - \ell = \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\frac{1}{t}\left(\frac{f(x-t)+f(x-t)}{2} - \ell\right)}_{\text{in } L^1} \underbrace{\frac{t}{\sin\frac{1}{2}t}}_{\text{bounded}} \sin\left[\left(n+\frac{1}{2}\right)t\right] dt,$$

and this expression tends to zero by the Riemann-Lebesgue lemma. ■

Comment 1.24 This criterion holds, for example, if

$$\frac{f(x+t)+f(x-t)}{2} - \ell = O(t^\alpha)$$

for some $\alpha > 1$, i.e., the functions needs only to be "a little more than differentiable" at x.

The general question

The basic question remains: under what conditions does the Fourier series of a function (if it exists) converge pointwise to the function? The question was asked already by Fourier himself in the beginning of the 19th century. Dirichlet proved that the Fourier series of continuously differentiable functions converges everywhere. It turns out, however, that continuity is not enough for pointwise convergence everywhere (Paul du Bois-Reymond, showed in 1876 that there is a continuous function whose Fourier series diverges at one point). In 1966, Lennart Carleson proved that for every $f \in L^2[0, 2\pi]$,

$$\lim_{n\to\infty} S_n f(x) = f(x) \qquad \text{almost everywhere.}$$

(This was conjectured in 1915 by Nikolai Lusin; the proof is by no means easy.) Two years later Richard Hunt extended the proof for every $L^p[0, 2\pi]$ function for $p > 1$. On the other hand, Andrey Kolmogorov showed back in the 1920s that there exist $L^1[0, 2\pi]$ functions whose Fourier series nowhere converges.

Fejér sums

Rather than looking at the partial sums $S_n f$, we can consider their running average:

$$\sigma_n f = \frac{1}{n+1} \sum_{k=0}^{n} S_n f.$$

Such sums are called after Lipót Fejér. It turns out that Fejér sums are much better behaved than the Fourier partial sums.

Note that it is trivial that if $S_n f(x)$ converges as $n \to \infty$, then so does $\sigma_n f(x)$, and to the same limit.

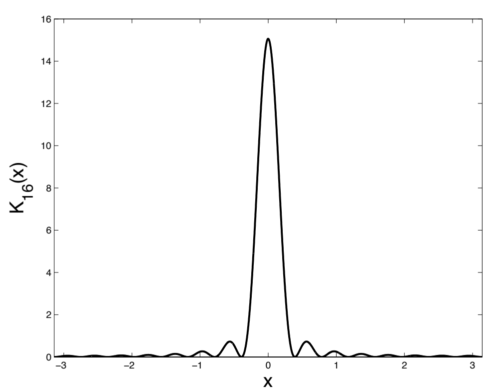
Consider the Fejér sums in explicit form:

$$\begin{aligned}\sigma_n f(x) &= \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{2\pi}\int_0^{2\pi} f(x-y)D_k(y)\,dy\\ &= \frac{1}{2\pi}\int_0^{2\pi} f(x-y)\left(\frac{1}{n+1}\sum_{k=0}^{n} D_k(y)\right)dy\\ &\equiv \frac{1}{2\pi}\int_0^{2\pi} f(x-y)K_n(y)\,dy,\end{aligned}$$

where,

$$\begin{aligned}\frac{1}{n+1}\sum_{k=0}^{n} D_k(y) &= \frac{1}{n+1}\sum_{k=0}^{n}\frac{\sin\left(k+\frac{1}{2}\right)y}{\sin\frac{1}{2}y}\\ &= \frac{1}{n+1}\frac{1}{\sin\frac{1}{2}y}\operatorname{Im}\sum_{k=0}^{n} e^{\imath(k+\frac{1}{2})y}\\ &= \frac{1}{n+1}\frac{1}{\sin\frac{1}{2}y}\operatorname{Im} e^{\frac{1}{2}\imath y}\frac{e^{\imath ny}-1}{e^{\imath y}-1}\\ &= \frac{1}{n+1}\frac{1}{\sin\frac{1}{2}y}\operatorname{Im} e^{\frac{1}{2}\imath y}\frac{e^{\frac{1}{2}\imath ny}}{e^{\frac{1}{2}\imath y}}\frac{\sin\frac{1}{2}ny}{\sin\frac{1}{2}y}\\ &= \frac{1}{n+1}\frac{\sin^2\frac{1}{2}ny}{\sin^2\frac{1}{2}y}.\end{aligned}$$

The function $K_n(x)$ is known as the **Fejér kernel**.



The Fejér kernel is also normalized, as found by setting $f \equiv 1$:

$$1 = \sigma_n f(x) = \frac{1}{2\pi}\int_0^{2\pi} K_n(y)\,dy.$$

It differs a lot from the Dirichlet kernel in that it is non-negative.

Like the Dirichlet kernel it is "centered" at zero. That is, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \int_{\varepsilon}^{2\pi-\varepsilon} K_n(x)\,dx = \lim_{n\to\infty} \frac{1}{n+1} \int_{\varepsilon}^{2\pi-\varepsilon} \frac{\sin^2 \frac{1}{2}nx}{\sin^2 \frac{1}{2}x}\,dx$$
$$\le \lim_{n\to\infty} \frac{1}{n+1} \int_0^{2\pi} \frac{dx}{\sin^2 \frac{1}{2}\delta} = 0.$$

Theorem 1.76 — Fejér. For every $f \in C[0,2\pi]$ such that $f(0) = f(2\pi)$:

$$\lim_{n\to\infty} \|\sigma_n f - f\|_\infty = 0.$$

Moreover, if $f \in L^1[0,2\pi]$ is continuous at x then $\lim_{n\to\infty} \sigma_n f(x) = f(x)$.

Proof. Let $f \in C[0,2\pi]$. Since continuous functions on compact intervals are uniformly continuous,

$$(\forall \varepsilon > 0)(\exists \delta > 0) : (\forall x, y : |x-y| < \delta)(|f(x) - f(y)| < \varepsilon).$$

Using the normalization of the Fejér kernel:

$$|\sigma_n f(x) - f(x)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-y) - f(x)) K_n(y)\,dy \right|$$
$$= \left| \frac{1}{2\pi} \int_{|x|\ge\delta} (f(x-y) - f(x)) K_n(y)\,dy + \frac{1}{2\pi} \int_{|x|<\delta} (f(x-y) - f(x)) K_n(y)\,dy \right|$$
$$\le 2\|f\|_\infty \frac{1}{2\pi} \int_{|x|\ge\delta} K_n(y)\,dy + \frac{\varepsilon}{2\pi} \int_0^{2\pi} K_n(y)\,dy,$$

and we used the fact that the Fejér kernel is non-negative. For sufficiently large n the right hand side can be made smaller than a constant (independent of x) times ε, which proves the first part.

The proof of the second part is similar. Continuity at x means local boundedness. By the localization principle we may therefore replace f by a bounded function that coincides with f in some neighborhood of x. ■

1.7.4 The Fourier transform

The Fourier series represents a function on $[0,2\pi]$ (or equivalently a 2π-periodic function on $\mathbb{R}$) as a sum of trigonometric functions $e^{\imath nx}$. We now wish to extend the treatment to the Hilbert space $L^2(\mathbb{R})$. We still want to expand the functions as a sum of trigonometric functions $e^{\imath\xi x}$, but this time we need to consider $\xi \in \mathbb{R}$ (otherwise we'll only represent 2π-periodic functions).

Can this be done? Note that these "basis" functions are not in $L^2(\mathbb{R})$. Moreover, they form an uncountable set of functions, whereas we know that $L^2(\mathbb{R})$ is separable, and hence every orthonormal basis is countable.

Definition 1.29 Let $f \in L^1(\mathbb{R})$. The function $\hat{f}:\mathbb{R}\to\mathbb{C}$ defined by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-\imath\xi x}\, dx$$

is called the **Fourier transform** (טרנספורם פורייה) of f.

Proposition 1.77 For $f \in L^1(\mathbb{R})$, $\hat{f}$ is continuous.

Proof. This is an immediate consequence of dominated convergence. Let $\xi_n \to \xi$. Then,

$$\hat{f}(\xi_n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \underbrace{f(x) e^{-\imath\xi_n x}}_{g_n(x)}\, dx.$$

The sequence $g_n(x)$ converges pointwise to $f(x)e^{-\imath\xi x}$, and

$$|g_n(x)| = |f(x)|,$$

and $f \in L^1(\mathbb{R})$. It follows that $\hat{f}(\xi_n) \to \hat{f}(\xi)$. ■

Proposition 1.78 The mapping $f \mapsto \hat{f}$ is a bounded linear operator $L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$, and

$$\|\hat{f}\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}} \|f\|_{L^1(\mathbb{R})}.$$

Proof. Obvious. ■

Lemma 1.79 — Riemann-Lebesgue. For every $f \in L^1(\mathbb{R})$:

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0.$$

Proof. Let $f = 1_{[a,b]}$, then

$$|\hat{f}(\xi)| = \frac{1}{\sqrt{2\pi}} \left| \int_a^b e^{-\imath\xi x}\, dx \right| = \frac{1}{\sqrt{2\pi}} \left| \frac{1}{-\imath\xi} \left(e^{-\imath\xi b} - e^{-\imath\xi a} \right) \right| \le \frac{2}{\sqrt{2\pi}|\xi|},$$

which satisfies the desired property. Thus, the lemma holds for every step function. Let $f \in L^1(\mathbb{R})$ and let $\varepsilon > 0$. There exists a step function g such that

$$\|f(x) - g(x)\|_{L^1(\mathbb{R})} < \varepsilon.$$

Then, by the previous proposition

$$\|\hat{f} - \hat{g}\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2\pi}} \|f - g\|_{L^1(\mathbb{R})} \le \frac{\varepsilon}{\sqrt{2\pi}}.$$

Since g is a step function, there exists an L such that

$$(\forall |\xi| > L) \qquad |\hat{g}(\xi)| \le \varepsilon.$$

Then for every $|\xi| > L$,

$$|\hat{f}(\xi)| \le |\hat{g}(\xi)| + \frac{\varepsilon}{\sqrt{2\pi}} \le \left(1 + \frac{1}{\sqrt{2\pi}}\right)\varepsilon.$$

■

Proposition 1.80 Let $f \in L^1(\mathbb{R})$.

① For $\mathbb{R} \ni \lambda \ne 0$ and $t \in \mathbb{R}$ set $g(x) = f(\lambda x + t)$. Then

$$\hat{g}(\xi) = \frac{1}{\lambda} e^{\imath \xi t/\lambda} \hat{f}\left(\frac{\xi}{\lambda}\right).$$

② If $\mathrm{Id} \cdot f \in L^1(\mathbb{R})$ then $\hat{f}$ is differentiable and

$$\hat{f}'(\xi) = -\imath \widehat{\mathrm{Id}\, f}(\xi).$$

Proof. For the first part, just follow the definition. g is clearly in $L^1(\mathbb{R})$, and

$$\begin{aligned}
\hat{g}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x) e^{-\imath \xi x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\lambda x + t) e^{-\imath \xi x}\, dx \\
&= \frac{1}{\lambda} e^{\imath \xi t/\lambda} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(\lambda x + t) e^{-\imath \xi (\lambda x + t)/\lambda}\, \lambda dx \\
&= \frac{1}{\lambda} e^{\imath \xi t/\lambda} \hat{f}(\xi/\lambda).
\end{aligned}$$

Next,

$$\frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{h} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \underbrace{f(x)e^{-\imath\xi x}\left(\frac{e^{-\imath hx}-1}{h}\right)}_{\ell^h(x)} dx$$

Observing that

$$\left|\frac{e^{-\imath hx}-1}{h}\right| \le |x| \qquad \text{and} \qquad \lim_{h\to 0}\ell^h(x) = f(x)e^{-\imath\xi x}(-\imath x),$$

we apply Lebesgue's dominated convergence theorem to conclude that

$$\lim_{h\to 0}\frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{h} = -\imath\,\widehat{\mathrm{Id}\,f}.$$

■

Comment 1.25 If the opposite true? Does $f\in L^1(\mathbb{R})$ such that $\hat{f}\in L^1(\mathbb{R})$ is differentiable almost everywhere imply that $\imath x f \in L^1(\mathbb{R})$?

The Fourier transform in $L^2(\mathbb{R})$

So far, we defined the Fourier transform for functions in $L^1(\mathbb{R})$. Recall that unlike for bounded intervals, $L^2(\mathbb{R}) \not\subset L^1(\mathbb{R})$, hence the Fourier transform is as of now not defined for functions in $L^2(\mathbb{R})$.

Lemma 1.81 Let $f\in L^2(\mathbb{R})$ have **compact support** (i.e., it vanishes outside a segment $[a,b]$), then $\hat{f}(\xi)$ exists and

$$\int_{\mathbb{R}} |f(x)|^2\,dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2\,d\xi.$$

Proof. Since $f|_{[a,b]}\in L^2[a,b]$ then it is integrable, and

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_a^b f(x)e^{-\imath\xi x}\,dx.$$

Thus,

$$|\hat{f}(\xi)|^2 = \frac{1}{2\pi}\int_a^b\int_a^b f(x)\overline{f}(y)e^{-\imath\xi(x-y)}\,dxdy.$$

Integrating from $-L$ to L:

$$\begin{aligned}\int_{-L}^{L} |\hat{f}(\xi)|^2\,d\xi &= \frac{1}{2\pi}\int_a^b\int_a^b f(x)\overline{f}(y)\left(\int_{-L}^{L} e^{-\imath\xi(x-y)}d\xi\right)dxdy \\ &= \frac{1}{2\pi}\int_a^b\int_a^b f(x)\overline{f}(y)\frac{2\sin L(x-y)}{x-y}\,dxdy.\end{aligned}$$

We are definitely stuck... although physicists would say that the ratio tends as $L \to \infty$ to the "delta-function" $\delta(x-y)$.

Let's be serious... first we'll show that we may well assume that the support of f is $[0,2\pi]$. Define

$$g(x) = f\left(\frac{2\pi}{b-a}x - \frac{2\pi a}{b-a}\right).$$

The function g has support in $[0,2\pi]$. By the previous proposition:

$$\hat{g}(\xi) = \frac{1}{\lambda} e^{\imath \text{stuff}} \hat{f}(\xi/\lambda),$$

where $\lambda = 2\pi/(b-a)$, hence,

$$|\hat{g}(\xi)|^2 = \frac{1}{\lambda^2}|\hat{f}(\xi/\lambda)|^2.$$

Now,

$$\int_{\mathbb{R}} |\hat{g}(\xi)|^2 \, d\xi = \frac{1}{\lambda^2} \int_{\mathbb{R}} |\hat{f}(\xi/\lambda)|^2 \, d\xi = \frac{1}{\lambda} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \, d\xi,$$

and

$$\int_{\mathbb{R}} |g(x)|^2 \, dx = \int_{\mathbb{R}} |f(\lambda x + \text{stuff})|^2 \, dx = \frac{1}{\lambda} \int_{\mathbb{R}} |f(x)|^2 \, dx,$$

which means that we can assume WLOG that the support of the function is in $[0,2\pi]$.

We are in the realm of functions in $L^2[0,2\pi]$. Note first that for every $t \in \mathbb{R}$:

$$\widehat{e^{-\imath tx} g}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-\imath tx} g(x) e^{-\imath nx} \, dx = \frac{1}{\sqrt{2\pi}} \hat{g}(n+t).$$

Applying the Parseval identity we have for every $t \in \mathbb{R}$:

$$\frac{1}{2\pi} \int_0^{2\pi} |g(x)|^2 \, dx = \frac{1}{2\pi} \int_0^{2\pi} |e^{-\imath xt} g(x)|^2 \, dx = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} |\hat{g}(n+t)|^2.$$

Integrating over $t \in [0,1]$ we get

$$\int_{\mathbb{R}} |g(x)|^2 \, dx = \int_{\mathbb{R}} |\hat{g}(\xi)|^2 \, d\xi.$$

■

Theorem 1.82 — Plancherel. There exists a unique bounded linear operator $\mathcal{F}$ in $L^2(\mathbb{R})$, such that

$$(\forall f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})) \qquad \mathcal{F}(f) = \hat{f}.$$

The operator $\mathcal{F}$ is an isometry, namely,

$$\|\mathcal{F}(f)\| = \|f\|.$$

Proof. Let $f \in L^2(\mathbb{R})$. By the previous lemma:

$$\widehat{f_n}(\xi) \equiv \widehat{f \cdot 1_{[-n,n]}}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} f(x) e^{-\imath \xi x}\, dx$$

exists, and

$$\int_{\mathbb{R}} |\widehat{f_n}(\xi)|^2 \, d\xi = \int_{-n}^{n} |f(x)|^2 \, dx.$$

In fact, the same Lemma implies that for $n > m$:

$$\int_{\mathbb{R}} |\widehat{f_n}(\xi) - \widehat{f_m}(\xi)|^2 \, d\xi = \int_{\mathbb{R}} |f_n(x) - f_m(x)|^2 \, dx = \int_{-n}^{n} |f(x)|^2 \, dx - \int_{-m}^{m} |f(x)|^2 \, dx,$$

which means that $\widehat{f_n}$ is a Cauchy sequence in $L^2(\mathbb{R})$, and hence converges.

Define:

$$\mathcal{F}(f) = \lim_{n\to\infty} \widehat{f_n},$$

where the limit is in $L^2(\mathbb{R})$. By the continuity of the norm $\|\mathcal{F}(f)\| = \|f\|$. Also, for $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, $\mathcal{F}(f) = \hat{f}$. Uniqueness is immediate because $\mathcal{F}(f)$ is specified on a dense subset and the operator $\mathcal{F}$ is bounded. ■

Comment 1.26 We have thus shown that the Fourier transform can be "naturally" extended to an operator on $L^2(\mathbb{R})$. From now on $\hat{f}$ will denote the extension $\mathcal{F}(f)$.

Hermite functions

Consider the sequence of elements in $L^2(\mathbb{R})$:

$$\{x^n e^{-x^2/2} \mid n \in \mathbb{N}\}.$$

If we perform a Gram-Schmidt orthonormalization on this sequence, we get the following orthonormal system:

$$h_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-x^2/2} H_n(x),$$

where $h_n(x)$ are known as the **Hermite functions** and

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

are known as the **Hermite polynomials** (H_n is of degree n). Also, the H_n are alternately odd/even.

Comment 1.27 The details of the Hermite functions are not important for our purposes. We only care that they are obtained from the functions $\{x^n e^{-x^2/2}$ by Gram-Schmidt orthonormalization.

Proposition 1.83 The Hermite functions h_n form a basis in $L^2(\mathbb{R})$.

Proof. The orthonormality can be checked by a direct calculation. Suppose that $f \in L^2(\mathbb{R})$ was orthogonal to all Hermite functions, namely

$$(\forall n \in \mathbb{N}) \qquad \int_{\mathbb{R}} f(x) h_n(x)\, dx = 0.$$

This amounts to having

$$(\forall n \in \mathbb{N}) \qquad \int_{\mathbb{R}} f(x) x^n e^{-x^2/2}\, dx = 0.$$

Consider the function

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\imath t z} e^{-t^2/2} f(t)\, dt.$$

This function is analytic on the whole complex plane. Note that

$$\varphi^{(n)}(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (\imath t)^n e^{\imath t z} e^{-t^2/2} f(t)\, dt = 0,$$

from which we conclude that $\varphi \equiv 0$. In particular, for $z = \xi$,

$$\widehat{e^{-t^2/2} f}(\xi) = 0.$$

Because the Fourier transform is an isometry, $e^{-t^2/2} f \equiv 0$, which completes the proof. ∎

Consider next the Fourier transform of the function $e^{-x^2/2}$:

$$\mathcal{F}[e^{-x^2/2}](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x^2/2} e^{-\imath \xi x}\, dx = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} \int_{\mathbb{R}} e^{-(x+\imath\xi)^2/2}\, dx.$$

Using Cauchy's theorem for the function $e^{-z^2/2}$, for rectangles $z = [-R, R]$ and $z = [-R + \imath\xi, R + \imath\xi]$ and letting $R \to \infty$,

$$\mathcal{F}[e^{-x^2/2}](\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} \int_{\mathbb{R}} e^{-x^2/2}\, dx = e^{-\xi^2/2}.$$

That is, the function $e^{-x^2/2}$ is invariant under the Fourier transform.

Next, recalling that $\mathcal{F}[f]' = \mathcal{F}[-\imath x f]$(if the latter exists), it follows that

$$\frac{d^n}{d\xi^n} e^{-\xi^2/2} = \mathcal{F}\left[(-\imath x)^n e^{-x^2/2}\right](\xi),$$

and hence

$$\begin{aligned}\mathcal{F}[h_n](\xi) &= \mathcal{F}\left[\sum_{k=0}^{n} a_{n,k} x^k e^{-x^2/2}\right](\xi)\\ &= \sum_{k=0}^{n} a_{n,k} \imath^k \mathcal{F}\left[(-\imath x)^n e^{-x^2/2}\right](\xi)\\ &= \sum_{k=0}^{n} a_{n,k} \imath^k \frac{d^n}{d\xi^n} e^{-\xi^2/2}.\end{aligned}$$

The right hand side is a polynomial times $e^{-\xi^2/2}$, hence it can be expanded in the form:

$$\mathcal{F}[h_n](\xi) = \sum_{k=0}^{n} b_{n,k} h_k(\xi).$$

Moreover, since $\mathcal{F}$ is an isometry, the functions $\mathcal{F}[h_n]$ form an orthonormal system (an isometry is also an isomorphism). It follows that both the h_n and the $\mathcal{F}[h_n]$ are orthonormalizations of the same set, however, the Gram-Schmidt orthonormalization is unique up to prefactors of modulus 1, namely,

$$\mathcal{F}[h_n] = \lambda_n h_n,$$

where $|\lambda_n| = 1$. To find λ_n it is only needed to look at the prefactor of x^n, and it is easy to see that[11]

$$\mathcal{F}[h_n] = (-\imath)^n h^n.$$

For any $f \in L^2(\mathbb{R})$,

$$f = \sum_{n=0}^{\infty} (f, h_n) h_n,$$

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$$\mathcal{F}[x^n e^{-x^2/2}] = \imath^n \frac{d^n}{d\xi^n} e^{-\xi^2/2} = \imath^n (-\xi)^n e^{-\xi^2/2} + \dots.$$

hence

$$\mathcal{F}[f] = \sum_{n=0}^{\infty}(f,h_n)\mathcal{F}[h_n] = \sum_{n=0}^{\infty}(-\imath)^n(f,h_n)h_n,$$

and

$$\mathcal{F}[\mathcal{F}[f]] = \sum_{n=0}^{\infty}(-1)^n(f,h_n)h_n.$$

Since $h_n(-x) = (-1)^n h_n(x)$, it follows that

$$\mathcal{F}[\mathcal{F}[f]](x) = \sum_{n=0}^{\infty}(f,h_n)h_n(-x) = f(-x).$$

That is, the Fourier transform is almost its own inverse. In particular, if $\hat{f} \in L^1(\mathbb{R})$, then

$$f(-x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\hat{f}(\xi)e^{-\imath\xi x}\,d\xi.$$

To summarize:

Theorem 1.84 — Inverse Fourier transform. Let $f \in L^2(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$, then

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\hat{f}(\xi)e^{\imath\xi x}\,d\xi.$$