Lecture Notes in Asymptotic Methods

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Contents

1 Ordinary differential equations **3**

1.1 Introduction 3
1.2 Homogeneous linear equations 6
1.3 Inhomogeneous linear equations 15
1.4 Various nonlinear equations 18
1.5 Eigenvalue problems 20
1.6 Differential equations in the complex plane 21

2 Local analysis I: Linear differential equations **25**

2.1 Classification of singular points 26
2.2 Local solution near an ordinary point 29
2.3 Local solution near a regular singular point 34
2.4 Local solution near irregular singular points 43
2.5 Asymptotic series 53
2.6 Irregular singular points at infinity 61

3 Local analysis II: Nonlinear differential equations **69**

3.1 Spontaneous singularities 69

4 Evaluation of integrals **73**

4.1 Motivation 73
4.2 Some examples 75

4.3 Integration by parts 78
4.4 Laplace's method 81

- Perturbation and asymptotics: find an approximate solution to problems which cannot be solved analytically, by identifying them as "close" to some solvable problems.
- Emphasis of course: how to.
- Textbook: Bender and Orszag.

Chapter 1

Ordinary differential equations

1.1 Introduction

A differential equation is a functional relation between a function and its derivatives. An n-th order differential equation for a function $y(x)$ is a relation of the form

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x)). \tag{1.1}$$

A solution to (1.1) is a function $y(x)$ for which this equation is satisfied.

Linear equations An n-th order differential equation is called ***linear*** if the function f is linear in y and its derivatives (it needs not be linear in x). A linear equation can be brought to the form

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = g(x).$$

It is called ***homogeneous*** if $g(x) \equiv 0$ and ***inhomogeneous*** otherwise (to be precise, it is only the homogeneous equation that is linear; the inhomogeneous equation should be called "affine").

n-th order equations as first order systems Any n-th order equation can be rewritten as a first-order equation for a vector-valued function,

$$\boldsymbol{y}(x) = (y_1(x), \dots, y_n(x)).$$

To do this, we define

$$y_1(x) = y(x) \quad y_2(x) = y'(x) \quad \cdots \quad y_n(x) = y^{(n-1)}(x),$$

and note that

$$\begin{aligned} y_1'(x) &= y_2(x) \\ y_2'(x) &= y_3(x) \\ \vdots &= \vdots \\ y_{n-1}'(x) &= y_n(x) \\ y_n'(x) &= f(x, y_1(x), \ldots, y_{n-1}(x)). \end{aligned}$$

This relation can be written in vector form,

$$\boldsymbol{y}'(x) = \boldsymbol{f}(x, \boldsymbol{y}(x)).$$

Initial-value problems versus boundary value problems In general, a differential equation does not have a unique solution. The set of solutions can usually be parametrized by a set of "integration constants". To obtain a problem that has a unique solution it is necessary to prescribe additional constraints, such as the value of the function, and perhaps some of its derivatives at certain points.

Example: The solutions to the first-order differential equation

$$y'(x) = 6\, y(x)$$

form the set

$$\left\{ y(x) = c\, e^{6x} : \; c \in \mathbb{R} \right\}.$$

It we require, in addition, that $y(1) = 2$, then the unique solution is

$$y(x) = \frac{2}{e^6} e^{6x}.$$

▲▲▲

Normally, the number of pointwise constraints is equal to the order n of the equation (and in the case of a first-order system to the size of the vector $\boldsymbol{y}$). If all the pointwise constraints are prescribed at the same point we call the resulting problem an ***initial-value problem*** (IVP). Otherwise, if the pointwise constraints are prescribed at two or more points, we call the resulting problem a ***boundary-value problem*** (BVP).

The term initial-value problem comes from applications where the independent variable x is identified as ***time***. In such cases it is more common to denote it by t, i.e., $y = y(t)$. In such applications, y (or the vector-valued $\boldsymbol{y}$), is prescribed at an initial time t_0, and the goal is to find the "trajectory" of the system $y(t)$ at later times, $t > t_0$ (of course, we can also look for the trajectory in the "past").

Example: Newton's mechanics can be viewed as an IVP. The vector $\boldsymbol{y}(t)$ describes the time evolution of the three components of the position of a point particle. Newton's equations are

$$m\boldsymbol{y}''(t) = \boldsymbol{f}(t, \boldsymbol{y}(t), \boldsymbol{y}'(t)).$$

The right-hand side is the force vector acting on the particle. It may depend on both the position and velocity of the particle (e.g., in electro-magnetism). If one prescribes initial data,

$$\boldsymbol{y}(t_0) = \boldsymbol{a} \qquad \text{and} \qquad \boldsymbol{y}'(t_0) = \boldsymbol{b},$$

then, assuming that $\boldsymbol{f}$ is nice enough, there exists a unique "future" (cf. the Laplace paradox). ▲▲▲

The term boundary-value problem comes from situations where the independent parameter x represents position along a line. The equation is defined in a domain, and the pointwise conditions are prescribed at the boundary of this domain.

Initial-value problems are in many respects simpler than boundary-value problems. Notably, there exists a general theorem of existence and uniqueness due to Cauchy:

Theorem 1.1 Consider the vector-valued IVP

$$\boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y}(x)) \qquad \boldsymbol{y}(t_0) = \boldsymbol{y}_0.$$

If there exists an open set that contains $(t_0, \boldsymbol{y}_0)$ *in which function* $\boldsymbol{f}(t, \boldsymbol{y})$ *is continuous in t and Lipschitz-continuous in* $\boldsymbol{y}$*, then there exists a time interval* $(t-\tau_1, t+\tau_2)$ *in which the initial-value problem has a unique solution.*

Comment: The fact that we have a general existence and uniqueness theorem does not imply that we can solve the equation (i.e., express its solution in terms of known functions). Most equations cannot be solved, which is precisely why we need approximation methods.

Example: Cauchy's theorem only guarantees ***short-time existence***. Consider the IVP

$$y'(t) = y^2(t) \qquad y(0) = 1.$$

The unique solution in the vicinity of the point $(t = 0, y = 1)$ is

$$y(t) = \frac{1}{1-t}.$$

This solution diverges at time $t = 1$ (***finite-time blowup***). ▲▲▲

In fact, as long as $f(t, y)$ satisfies the continuity requirements, finite-time blowup is the only way existence and uniqueness may cease to hold.

Example: To see how Lipschitz-continuity is required for uniqueness consider the IVP

$$y'(t) = y^{1/3}(t) \qquad y(0) = 0.$$

Clearly, $y(t) \equiv 0$ is a solution. On the other hand,

$$y(t) = \left(\frac{2t}{3}\right)^{3/2}$$

is also a solution. This does not contradict Cauchy's theorem since $f(t, y) = y^{1/3}$ is not Lipschitz-continuous at $y = 0$, i.e., there exists no constant L and $\delta > 0$ such that

$$|y^{1/3}| \leq L|y|$$

for all $|y| < \delta$. ▲▲▲

In the rest of this chapter we study particular types of equations which can either be solved analytically, or for which we can at least make general statements. The reason why this is important for this course is that we will always need to identify solvable problems that are close enough to the (non-solvable) problems of interest. We therefore need to develop skills for solving a variety of solvable problems.

1.2 Homogeneous linear equations

Recall that a homogeneous n-th order linear equation is of the form

$$\mathcal{L}[y](x) = y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0.$$

At this point, we do not distinguish between IVPs and BVPs. Note that the trivial solution $y(x) \equiv 0$ is always a solution to a homogeneous linear equation.

Definition 1.1 *A collection of functions $y_1(x), \dots, y_k(x)$ is said to be* ***linearly dependent*** *if there exists constant $c_1, \dots, c_k$ such that*

$$\sum_{i=1}^{k} c_i\, y_i(x) \equiv 0.$$

Otherwise, this collection is ***linearly independent****. Note that this sum has to be zero for all x (perhaps restricted to some interval). There is no meaning to dependence/independence at a point.*

Homogeneous linear equations have the property that if $y(x)$ and $z(x)$ are solutions, so is every linear combination,

$$a\, y(x) + b\, z(x).$$

In other words, the set of solutions forms a vector space in the space of functions, with respect to pointwise addition and pointwise scalar multiplication. Like any vector space, its dimension is the maximum number of independent elements.

The remarkable fact is that a linear homogeneous n-order equation has exactly n independent solutions (the set of solutions is an n-dimensional vector space). Obviously, this set is not unique (as the basis of a vector space is never unique). What we do know is that if

$$y_1(x), \dots, y_n(x)$$

forms such a basis, then any solution is necessarily of the form

$$y(x) = \sum_{i=1}^{n} c_i\, y_i(x).$$

Theorem 1.2 *Consider an n-th order linear homogeneous equation such that the existence and uniqueness theorem holds on some interval I (for that it is sufficient that the coefficients $a_i(t)$ be continuous). Then the set of solutions on I forms an n-dimensional vector space.*

Proof: Let x_0 be a point in I, and consider the sequence of IVPS,

$$\mathcal{L}[y_k] = 0, \qquad y_k(x_0) = 0,\ y_k'(x_0) = 0, \dots, y_k^{(k)}(x_0) = 1, \dots, y_k^{(n-1)}(x_0) = 0,$$

with $k = 0, 1, 2, \ldots, n-1$. Each of these IVPs has a unique solution. These solutions are independent since suppose they were dependent, i.e., there exist constant c_i, not all zero, such that

$$\sum_{k=0}^{n-1} c_k y_k(x) \equiv 0.$$

Let $c_k \neq 0$, then differentiating k times and setting $x = x_0$ we get a contradiction. To show that these functions span the space of solutions, let $y(x)$ be an arbitrary solution. Then

$$z(x) = \sum_{k=0}^{n-1} y^{(k)}(x_0) y_k(x)$$

is a linear combination of our basis satisfying $z^{(k)}(x_0) = y^{(k)}(x_0)$ for $k = 0, \ldots, n-1$ and by uniqueness $y(x) = z(x)$. ■

The Wronskian Suppose that $y_1(x), \ldots, y_k(x)$ are functions defined on some interval. At this stage we do not care whether they are solutions of some differential equations, or just a set of God-given functions. How can we determine whether they are independent? This is important because we often can find n solutions to an n-th order equation, but we must be sure that they are independent in order to claim that the most general solution is a linear combination of this collection.

There is a simple test for the linear independence of functions. We define the ***Wronskian***, which is a real-valued function,

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_k(x) \\ y_1'(x) & y_2'(x) & \cdots & y_k'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(k-1)}(x) & y_2^{(k-1)}(x) & \cdots & y_k^{(k-1)}(x) \end{vmatrix}$$

Proposition 1.1 The collection of functions is linearly independent on an interval if $W(x) \equiv 0$ *on that interval.*

Proof: Obvious ■

Note that the opposite is not true. A vanishing Wronskian does not imply dependence.

Example: The set of functions e^x, e^{-x} and $\cosh x$ is linearly dependent. ▲▲▲

Example: The set of functions x, x^2 is linearly independent. ▲▲▲

Abel's formula Let now $y_1(x), \dots, y_n(x)$ be solutions of a homogeneous linear n-th order equation. Differentiating the Wronskian, we obtain that

$$W'(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)}(x) & y_2^{(n)}(x) & \cdots & y_n^{(n)}(x) \end{vmatrix}.$$

(Why did we differentiate only the last row?). Substituting the equations that each $y_i(x)$ satisfies we get that (why?)

$$W'(x) = -a_{n-1}(x)\, W(x),$$

which is a first-order linear homogeneous equation. Its solution is of the form

$$W(x) = c\, \exp\left(- \int^x a_{n-1}(u)\, du \right).$$

Remarkably, we have an explicit expression for the Wronskian even without knowing the solutions. In particular, if the Wronskian is non-zero at a point x_0 and a_{n-1} is integrable in the interval $[x_0, x]$, then the Wronskian remains non-zero throughout the interval.

Initial-value problems Suppose that we are given the value of y and its $n-1$ first derivatives at a point t_0,

$$y^{(k)}(t_0) = b_k, \qquad i = 0, 1, \dots, n-1.$$

If we have n independent solutions $y_i(t)$, then the general solution is of the form

$$y(t) = \sum_{i=1}^n c_i\, y_i(t),$$

and we need to find n constants c_i such that

$$\sum_{i=1}^n c_i\, y_i^{(k)}(t_0) = b_k$$

This is an n-dimensional linear system for the c_i and the matrix of coefficients is precisely $W(t_0)$. This system has a unique solution if and only if $W(t_0) \neq 0$. Thus the Wronskian has a double role. Its non-vanishing *in an interval* guarantees the independence of solutions, whereas its non-vanishing *at a point* guarantees that initial data prescribed at that point yield a unique solution.

Example: Consider the IVP

$$y'' - \frac{1+t}{t}y' + \frac{1}{t}y = 0, \qquad y(1) = 1,\ y'(1) = 2.$$

First, we can easily verify that

$$y_1(t) = e^t \qquad \text{and} \qquad y_2(t) = 1 + t$$

are solutions. Moreover,

$$W(t) = \begin{vmatrix} e^t & 1+t \\ e^t & 1 \end{vmatrix} = -t\,e^t$$

vanishes only at the point $t = 0$, hence the solutions are independent in any interval on the line. Since $W(1) \neq 0$ this IVP has a unique solution. If, however, we prescribe initial data at $t = 0$, for example,

$$y(0) = 1,\ y'(0) = 2,$$

then if we express the general solution as

$$y(t) = c_1 e^t + c_2(1 + t),$$

it follows that

$$c_1 + c_2 = 1 \qquad \text{and} \qquad c_1 + c_2 = 2,$$

which has no solution. If instead,

$$y(0) = 1,\ y'(0) = 1,$$

then there are infinitely many solutions. ▲▲▲

Boundary-value problems There is no general existence and uniqueness theorem for BVPS.

Example: Consider the second-order linear homogeneous BVP

$$y''(x) + y(x) = 0, \qquad y(0) = 0,\ y'(\pi/2) = 1.$$

The general solution to this differential equation is

$$y(x) = c_1 \sin x + c_2 \cos x.$$

Since $y(0) = 0$ it follows that $c_2 = 0$. The second condition

$$y'(\pi/2) = c_1 \cos\frac{\pi}{2} = 1$$

has no solution. ▲▲▲

Constant coefficient equations The simplest (and fully solvable) case of homogeneous linear equations is when the coefficients $a_i(x)$ are constant, i.e., the differential equation is

$$\mathcal{L}[y](x) = y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = 0.$$

Recall that in order to find the general solution we need to find n independent solutions. We then look for solutions of the form

$$y(x) = e^{px}.$$

Substituting into the equation we get a polynomial equation for p,

$$p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0 = 0.$$

If this polynomial has n distinct roots, then we have n independent solutions and the general solution is found. Note that some of the roots may be complex-valued, however if p is a complex root, so is $\bar{p}$, and from the independent solutions e^{px} and $e^{\bar{p}x}$ we may form a new pair of independent real-valued solutions,

$$\frac{e^{px} + e^{\bar{p}x}}{2} = e^{(\Re e p)x} \cos[(\Im m) p\, x] \quad \text{and} \quad \frac{e^{px} - e^{\bar{p}x}}{2\imath} = e^{(\Re e p)x} \sin[(\Im m) p\, x].$$

It may also be that p is an m-fold degenerate root, in which case we lack $m-1$ solutions. The remaining solutions can be constructed as follows. For every q,

$$\mathcal{L}[e^{qx}] = e^{qx} P_n(q) = e^{qx} (q-p)^m Q_{n-m}(q),$$

where the Q_n are polynomials of degree n. Differentiating both sides with respect to q and setting $q = p$ we get

$$\mathcal{L}[x\, e^{px}] = 0,$$

i.e., $x\, e^{px}$ is also a solution. We may differentiate again up to the $(m-1)$st time and find that

$$e^{px},\ x\, e^{px},\ x^2\, e^{px}\ \ldots, x^{m-1}\, e^{px}$$

are all independent solutions. Thus, we end up with a set of n independent solutions.

Comment: Recall that an n-th order equation can be rewritten as a first-order system for a vector-valued function $\boldsymbol{y}(t)$. In the case of a linear constant-coefficient equation the linear system is

$$\frac{d\boldsymbol{y}}{dx} = A\boldsymbol{y},$$

where A is an n-by-n constant matrix. The general solution of this vector equation is

$$\boldsymbol{y}(x) = e^{Ax}\boldsymbol{c},$$

where $\boldsymbol{c}$ is a constant vector. While this looks much simpler, the evaluation of a matrix exponential, which is defined by

$$e^{Ax} = \sum_{k=0}^{\infty} \frac{(Ax)^k}{k!}$$

requires the diagonalization (more precisely, transformation to Jordan canonical form) of A, which amounts to an eigenvalue analysis.

Example: A harmonic oscillator in Newtonian mechanics satisfies the second-order equations

$$m\, y''(t) + k\, y(t) = 0.$$

Looking for a solution of the form $y(t) = e^{pt}$ we get a quadratic equation for p,

$$mp^2 + k = 0,$$

whose solution is $p = \pm\imath\sqrt{k/m}$. From these two independent complex-valued solutions we may construct the real-valued solutions,

$$y_1(t) = \cos(\sqrt{k/m}\, t) \qquad \text{and} \qquad y_2(t) = \sin(\sqrt{k/m}\, t).$$

▲▲▲

Example: Consider now the third-order equation

$$y'''(t) - 3y''(t) + 3y'(t) - 1 = 0.$$

Substituting $y(t) = e^{pt}$ gives the polynomial equation $(p-1)^3 = 0$, i.e., $p = 1$ is a triple root. In this case, the general solution is

$$y(t) = c_1\, e^t + c_2\, te^t + c_3\, t^2 e^t.$$

▲▲▲

Euler equations Another case where the equation is solvable is when

$$a_i(x) = \frac{a_i}{x^{n-i}},$$

i.e.,

$$y^{(n)}(x) + \frac{a_{n-1}}{x} y^{(n-1)}(x) + \cdots + \frac{a_1}{x^{n-1}} y'(x) + \frac{a_0}{x^n} y(x) = 0.$$

This equation can be solved by a change of variable, $x = e^t$. Then,

$$x \frac{d}{dx} = \frac{d}{dt}$$
$$x^2 \frac{d^2}{dx^2} = x \frac{d}{dx} x \frac{d}{dx} - x \frac{d}{dx} = \frac{d^2}{dt^2} - \frac{d}{dt},$$

and so on, and we end up with an equation with constant coefficients for $y(t) = y(\log x)$. Alternatively, we may look for solutions of the form $y(x) = x^p$ and obtain that p satisfies the polynomial equation,

$$[p(p-1)\dots 1] + a_{n-1}[p(p-1)\dots 2] + a_{n-2}[p(p-1)\dots 3] + \cdots + a_1 p + a_0 = 0.$$

Reduction of order Suppose we have a homogeneous linear equation

$$\mathcal{L}[y] = 0$$

and we happen to know one solution $y_1(x)$. We then look for more solutions of the form

$$y(x) = y_1(x) u(x).$$

It turns out that $u'(x)$ satisfies a homogeneous linear equation of order $n-1$, which may make the remaining problem solvable.

Example: Consider once again the equation

$$y''(x) - \frac{1+x}{x} y'(x) + \frac{1}{x} y(x) = 0.$$

Suppose we (by chance) observe that $y_1(x) = e^x$ is a solution. We then set $y(x) = e^x u(x)$, which gives

$$y'(x) = e^x \left[u(x) + u'(x)\right] \qquad \text{and} \qquad y''(x) = e^x \left[u(x) + 2u'(x) + u''(x)\right].$$

Substituting into the equation we get

$$u(x) + 2u'(x) + u''(x) - \frac{1+x}{x}\left[u(x) + u'(x)\right] + \frac{1}{x}u(x) = u''(x) - \frac{1-x}{x}u'(x) = 0.$$

Thus, $v(x) = u'(x)$ satisfies the equation

$$\frac{d}{dx}\log v(x) = \frac{v'(x)}{v(x)} = \frac{1-x}{x}.$$

Integrating both sides we get

$$\log v(x) = \log x - x,$$

or,

$$v(x) = xe^{-x}.$$

Integrating we get

$$u(x) = e^{-x}(x+1),$$

hence,

$$y(x) = x + 1$$

is a second solution independent of e^x. We didn't care about constants of integration because we are content with finding any additional (independent) solution. ▲▲▲

"Known" equations In a general, the solution to a homogeneous linear equation cannot be expressed in terms of elementary functions. First-order linear equations,

$$y'(x) + a_0(x)y(x) = 0$$

can always be solved (***reduced to "quadratures"***) by separation of variables,

$$\frac{d}{dx}\log y(x) = -a_0(x),$$

hence

$$y(x) = c\,\exp\left[-\int^x a_0(u)\,du\right].$$

Second-order equations,

$$y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0$$

do not have closed form solutions.

Yet, there are many such equations which occur repeatedly in the various sciences, and as a result their solutions have been studied extensively. They have been tabulated, and their asymptotic properties have been studied. Some of them can also be computed by certain recurrence relations. Thus, to some extent we may say that we know the solution to those equations (to the same extent as we "know" what a sine function or a logarithmic function is!).

Example: The solutions to the second-order equation

$$y''(x) = x\, y(x)$$

is called the ***Airy function***. Similarly, the family of equations,

$$y''(x) + \frac{1}{x} y'(x) + \left(1 - \frac{\nu^2}{x^2}\right) y(x) = 0,$$

with $\nu \geq 0$ is called the ***Bessel equations***, and their solutions are known as the ***Bessel functions***. There are two sets of Airy and Bessel functions since there are two independent solutions. ▲▲▲

1.3 Inhomogeneous linear equations

Consider now inhomogeneous linear equations of the form

$$\mathcal{L}[y](x) = g(x).$$

Suppose that $u(x)$ is a (particular) solution of this equation. Setting

$$y(x) = u(x) + z(x),$$

and substituting into the equation we get

$$g(x) = \mathcal{L}[y](x) = \mathcal{L}[u + z](x) = \mathcal{L}[u](x) + \mathcal{L}[z](x) = g(x) + \mathcal{L}[z](x).$$

That is, $z(x)$ solves the corresponding homogeneous equation. Thus, *the general solution to an inhomogeneous linear equation can be expressed as the sum of any particular solution and the general solution to the homogeneous equation*. This means that all we need is to find one particular solution.

Example: Consider the first order equation,

$$y'(x) + y(x) = 6.$$

$y(x) = 6$ is a particular solution, hence the general solution is

$$y(x) = 6 + c\, e^{-x}.$$

▲▲▲

An elegant solution can be found when the equation is written as a first-order system. Then, the inhomogeneous system takes the form

$$\frac{d}{dx}\boldsymbol{y}(x) = A\boldsymbol{y}(x) + \boldsymbol{g}(x).$$

Recall that the columns matrix-valued function e^{Ax} are independent solutions to the homogeneous system. Consider now the particular function

$$\boldsymbol{y}(x) = \int_0^x e^{A(x-s)}\, \boldsymbol{g}(s)\, ds.$$

We claim that this function solves the inhomogeneous system. Indeed,

$$\frac{d}{dx}\boldsymbol{y}(x) = \boldsymbol{g}(s) + A \int_0^x e^{A(x-s)}\, \boldsymbol{g}(s)\, ds = A\boldsymbol{y}(x) + \boldsymbol{g}(x).$$

The general method for solving the nth-order inhomogeneous linear equation is the so-called ***variation of constants***. If $y_1(x), \dots, y_n(x)$ are independent solutions to homogeneous equation, then so is any linear combination,

$$c_1 y_1(x) + \cdots + c_n y_n(x).$$

We then look for a solution to the inhomogeneous equation of the form

$$y(x) = c_1(x) y_1(x) + \cdots + c_n(x) y_n(x),$$

and try to find coefficient functions for which the equation is satisfied. This procedure turns out to yield a substantial reduction of order.

Let's examine this procedure for a general second-order equation,

$$y''(x) + a_1 y'(x) + a_0 y(x) = g(x).$$

Suppose that $y_1(x)$ and $y_2(x)$ are independent solutions to the homogeneous equation, and look for a solution of the form

$$y(x) = c_1(x)y_1(x) + c_2(x)y_2(x).$$

Substituting into the equation we get

$$\begin{aligned} g(x) &= (c_1''y_1 + 2c_1'y_1' + c_1y_1'') + (c_2''y_2 + 2c_2'y_2' + c_2y_2'') \\ &+ a_1\,(c_1'y_1 + c_1y_1') + a_1\,(c_2'y_2 + c_2y_2') \\ &+ a_2c_1y_1 + a_2c_2y_2. \end{aligned}$$

Using the fact that y_1, y_2 solve the homogeneous equation, we remains with

$$g(x) = (c_1''y_1 + 2c_1'y_1') + (c_2''y_2 + 2c_2'y_2') + a_1\,(c_1'y_1) + a_1\,(c_2'y_2)\,.$$

Now let's try to impose an additional constraint on the coefficient functions, namely, that

$$c_1'(x)y_1(x) + c_2'(x)y_2(x) \equiv 0.$$

This also implies upon differentiation that

$$c_1''(x)y_1(x) + c_2''(x)y_2(x) + c_1'(x)y_1'(x) + c_2'(x)y_2'(x) \equiv 0.$$

Substituting, we remain with

$$c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = g(x).$$

This last equation an the constraint yield a system

$$\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix} \begin{pmatrix} c_1'(x) \\ c_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix},$$

which is solvable if the Wronskian is non-zero (i.e., if $y_1(x), y_2(x)$ are indeed independent). Thus we solve for $c_1'(x)$ and $c_2'(x)$ and this means that the problem has been solved up to quadratures.

✎ *Exercise 1.1* Solve the second-order inhomogeneous linear equation

$$y''(x) - 3y'(x) + 2y(x) = e^{4x},$$

using variation of constants (find the general solution).

✎ *Exercise 1.2* Use reduction of order to find the general solution of

$$x^2y''(x) - 4xy'(x) + 6y(x) = x^4\ \sin x,$$

after observing that $y(x) = x^2$ is a solution of the homogeneous equation.

1.4 Various nonlinear equations

There are no general methods to handle nonlinear equations, yet there are certain equations for which the solution is known. In this section we cover a few such examples.

Separable first-order equations A first-order equation is called separable if it is of the form

$$y'(x) = f(x)g(y(x)).$$

In such case we have

$$\int_{x_0}^{x} \frac{y'(u)}{g(y(u))}\, du = \int_{x_0}^{x} f(u)\, du.$$

If F denotes the primitive function of f and G denotes the primitive function of $1/g$, then changing variables $y(u) = z$ on the left-hand side

$$\int_{y(x_0)}^{y(x)} \frac{dz}{g(z)} = G(y(x)) - G(y(x_0)) = F(x) - F(x_0),$$

or in explicit form

$$y(x) = G^{-1}\left(F(x) - F(x_0) + G(y(x_0))\right).$$

Example: Take the IVP

$$y'(t) = t\sqrt{y(t)}, \qquad y(1) = 1.$$

By the above procedure

$$\int_{1}^{t} \frac{y'(s)}{\sqrt{y(s)}}\, ds = \int_{1}^{t} s\, ds = \frac{1}{2}(t^2 - 1).$$

Changing variables $y(s) = z$ on the left-hand side

$$\int_{y(1)}^{y(t)} \frac{dz}{\sqrt{z}} = 2\sqrt{z}\Big|_{1}^{y(t)} = \frac{1}{2}(t^2 - 1),$$

and

$$y(t) = \left[1 + \frac{1}{4}(t^2 - 1)\right]^2.$$

▲▲▲

✎ *Exercise 1.3* Solve the IVP

$$y'(t) = y^{2/3}(t)\,\sin t, \qquad y(1) = 1.$$

✎ *Exercise 1.4* Find the general solution of the differential equation

$$y'(x) = e^{x+y(x)}.$$

Bernoulli equations are of the form

$$y'(x) = a(x)y(x) + b(x)y^p(x),$$

where p is any number. The cases $p = 0$ is a linear equation, whereas $p = 1$ is separable. For arbitrary p we first divide by y^p,

$$(-p+1)^{-1}\frac{d}{dx}y^{-p+1}(x) = y^{-p}y'(x) = a(x)y^{-p+1}(x) + b(x),$$

and thus, $u(x) = y^{-p+1}(x)$ satisfies the linear equation

$$u'(x) = (-p+1)a(x)u(x) + (-p+1)b(x).$$

Riccati equations are of the form

$$y'(x) = a(x)y^2(x) + b(x)y(x) + c(x).$$

There is no general solution to a Riccati equation, but if one solution $y_1(x)$ is known, then we set

$$y(x) = y_1(x) + u(x).$$

Substituting we get

$$y_1'(x) + u'(x) = a(x)\left[y_1^2(x) + 2y_1(x)u(x) + u^2(x)\right] + b(x)\left[y_1(x) + u(x)\right] + c(x),$$

and

$$u'(x) = a(x)u^2(x) + \left[b(x) + 2a(x)y_1(x)\right]u(x).$$

This equation is solvable by the following substitution,

$$u(x) = -\frac{w'(x)}{a(x)w(x)},$$

i.e.,

$$u'(x) = -\frac{w''(x)}{a(x)w(x)} + \frac{w'(x)}{a^2(x)w^2(x)}\left[a'(x)w(x) + a(x)w'(x)\right],$$

hence

$$-\frac{w''(x)}{a(x)w(x)}+\frac{w'(x)}{a^2(x)w^2(x)}\left[a'(x)w(x) + a(x)w'(x)\right] = a(x)\frac{[w'(x)]^2}{a^2(x)w^2(x)}-\left[b(x) + 2a(x)y_1(x)\right]\frac{w'(x)}{a(x)w(x)},$$

from which remains

$$w''(x) + \frac{w'(x)}{a(x)}a'(x) = -\left[b(x) + 2a(x)y_1(x)\right]w'(x),$$

which is a solvable equation.

Example: Solve the Riccati equation

$$y'(x) = y^2(x) - xy(x) + 1$$

noting that $y(x) = x$ is a particular solution. ▲▲▲

✎ *Exercise 1.5* Find a closed-form solution to the Riccati equation

$$xy'(x) + xy^2(x) + \tfrac{1}{2}x^2 = \tfrac{1}{4}.$$

1.5 Eigenvalue problems

An eigenvalue problem in the context of differential equations is a boundary-value problem that has non-trivial solutions when a parameter λ in the equation takes special values, which we call the eigenvalues. Note the analogy to eigenvalue problems in linear algebra,

$$A\boldsymbol{x} = \lambda\boldsymbol{x}.$$

For a linear algebraic system we know that the solution (the values of λ) is generally non-unique. This is also the case when the BVP is linear and homogeneous.

Example: Consider the BVP

$$y''(x) + \lambda y(x) = 0 \qquad y(0) = y(1) = 0.$$

This BVP has a trivial solution for all value of λ. Note that the general solution of this differential equation is

$$y(x) = a \cos(\sqrt{\lambda} x) + b \sin(\sqrt{\lambda} x).$$

Since it vanishes at zero $a = 0$. The boundary condition at $x = 1$ implies that

$$\sin(\sqrt{\lambda} x) = 0,$$

i.e., that

$$\lambda = k^2 \pi^2, \qquad k = 1, 2, \dots.$$

▲▲▲

✎ *Exercise 1.6* Solve the eigenvalue problem

$$y''(x) + \lambda y(x) = 0 \qquad y(0) = y'(1) = 0.$$

Eigenvalue problems can also be formulated on infinite domains:

Example: Consider the eigenvalue problem

$$y''(x) + (\lambda - x^2/4) y(x) = 0, \qquad \lim_{x \to \pm\infty} y(x) = 0.$$

This eigenvalue problem arises in quantum mechanics as the solution to the wave-functions of the harmonic oscillator. The eigenvalues turn out to be

$$\lambda = k + \frac{1}{2}, \qquad k = 0, 1, 2, \dots,$$

and the corresponding eigenfunctions are

$$y(x) = H_n(x) e^{-x^2/4},$$

where the $H_n(x)$ are Hermite polynomials. ▲▲▲

1.6 Differential equations in the complex plane

Although our interest is in real-valued differential equations, their analysis relies heavily on the properties of their complex-valued counterparts. We first remain the definitions of analyticity:

Definition 1.2 A function $f(x)$ is called ***(real)-analytic*** *in an open set $U \subset \mathbb{R}$ if it is infinitely-differentiable in U and for every $x_0 \in U$ its Taylor series about x_0,*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

converges and equals to $f(x)$ in some neighborhood of x_0. A complex valued function $f(z)$ is called ***analytic*** *in an open set $U \subset \mathbb{C}$ if it satisfies the corresponding conditions within the complex context.*

Comments:

① Both real- and complex analytic functions have the properties that sums and products of analytic functions are analytic.

② The reciprocal of a non-vanishing analytic function is analytic.

③ The inverse of an analytic function that has a non-vanishing derivative is analytic. Finally, uniform limits of analytic functions are analytic.

There are also differences between real- and complex-analytic functions: differentiability in the complex plane is much more restrictive than differentiability in a real vector space. Recall that a complex-valued function $f(z)$ is ***differentiable*** at a point z_0 if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \equiv f'(z_0)$$

exists (independently on how z approaches z_0). A function is said to be ***homomorphic*** in a domain if its derivative exists in the domain.

Comments:

① If a function is holomorphic in a domain then it is infinitely-differentiable and it is analytic (complex-differentiability in an open set implies complex-analyticity in that set).

② The radius of convergence of its Taylor series about a point z_0 equals to the distance of its nearest singularity to z_0 (in particular, $f(z)$ is analytic in an open ball B around z_0 then its Taylor series about z_0 converges in B).

③ The analogous statement does not hold for real-analytic functions.

④ Real-valued functions that are analytic in an open segment can be extended to a complex-valued function that is analytic on some open set in the complex plane (the extension is simply through the Taylor series). It is not necessarily true, however, that a function that is (real-)analytic on the whole line has an extension that is analytic in the entire complex plane.

With this in mind, we may consider systems of complex-valued differential equations of the form

$$y'(z) = f(z, y(z)),$$

where $y \in \mathbb{C}^n$ and $f : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$.

Theorem 1.3 Suppose that f is analytic in an open set $D \subset \mathbb{C} \times \mathbb{C}^n$ that contains the point (z_0, y_0) (this means that every component of f is analytic in each of its arguments). Then there exists an open ball $|z - z_0| < a$ in which the IVP

$$y'(z) = f(z, y(z)) \qquad y(z_0) = y_0$$

has a unique solution, which is analytic.

One can prove this theorem by Picard's iterations defining

$$\phi_0(z) = y_0$$
$$\phi_{k+1}(z) = y_0 + \int_{z_0}^{z} f(\zeta, \phi_k(\zeta))\, d\zeta,$$

where the integration path can, for example, be taken to be the line segment between z_0 and z. One can show that all the ϕ_k are analytic, and that by choosing a sufficiently small a, this sequence converges uniformly in the open ball. Finally, one has to show that this limit satisfies the differential equation.

✎ *Exercise 1.7* Complete the existence proof for ODEs in the complex plane. Show that if f is analytic in an open set $D \subset \mathbb{C} \times \mathbb{C}^n$ that contains the point (z_0, y_0) (this means that every component of f is analytic in each of its arguments). Then there exists an open ball $|z - z_0| < a$ in which the IVP

$$y'(z) = f(z, y(z)) \qquad y(z_0) = y_0$$

has a unique solution, which is analytic. Follow Picard's method of successive approximations.

Chapter 2

Local analysis I: Linear differential equations

Perturbation and asymptotic methods can be divided into two main categories: ***local*** and ***global*** analysis. In local analysis one approximates a function in a neighborhood of some point, whereas in global analysis one approximates a function throughout the domain. IVPS can be treated by either approach whereas BVPS require inherently a global approach. Local analysis is easier and is therefore the first approach we learn. This chapter is devoted to the local analysis of solutions of linear differential equations. In cases where the equation is solvable we can explicitly assess the accuracy of the approximation by comparing the exact and approximate solutions.

Example: The fourth-order differential equation

$$\frac{d^4y}{dx^4}(x) = (x^4 + \sin x)\, y(x),$$

cannot be solved in terms of elementary functions. Yet, we will be able to determine very easily (Bender and Orszag claim that on the back of a stamp) that as $x \to \infty$ the solution is well-approximated by a linear combination of the functions

$$x^{-3/2} e^{\pm x^2/2}, \quad x^{-3/2} \sin(x^2/2) \quad \text{and} \quad x^{-3/2} \cos(x^2/2).$$

▲▲▲

2.1 Classification of singular points

We are concerned with homogeneous linear equations of the form

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0. \tag{2.1}$$

In local analysis we approximate its solution near a point x_0.

Definition 2.1 A point x_0 is called an ***ordinary point*** *of* (2.1) *if the coefficient functions $a_i(x)$ are (real) analytic in a neighborhood of x_0. (Recall that real-analyticity implies the complex analyticity of its analytic continuation.)*

Example: Consider the equation

$$y''(x) = e^x y(x).$$

Every point $x_0 \in \mathbb{R}$ is an ordinary point because the function e^z is entire. ▲▲▲

It was proved in 1866 (Fuchs) that all n independent solutions of (2.1) are analytic in the neighborhood of an ordinary point. Moreover, if these solutions are Taylor expanded about x_0 then the radius of convergence is at least as the distance of the nearest singularity of the coefficient functions to x_0. This is not surprising. We know that an analytic equation has analytic solutions. In the case of linear equations, this solution can be continued indefinitely as long as no singularity has been encountered.

Example: Consider the equation

$$y'(x) + \frac{2x}{1+x^2} y(x) = 0.$$

The point $x = 0$ is an ordinary point. The complexified coefficient function

$$\frac{2z}{1+z^2}$$

has singularities at $z = \pm\imath$, i.e., at a distance of 1 from the origin. The general solution is

$$y(x) = \frac{c}{1+x^2} = c \sum_{n=0}^{\infty} (\imath x)^{2n}$$

and the Taylor series has a radius of convergence of 1. Note that the solution $y(x) = 1/(1+x^2)$ is analytic on the whole of $\mathbb{R}$, yet the radius of convergence of the Taylor series is bounded. ▲▲▲

*Definition 2.2 A point x_0 is called a **regular singular point** of (2.1) if it is not an ordinary point, but the functions*

$$(x - x_0)^n a_0(x),\ (x - x_0)^{n-1} a_1(x), \dots, (x - x_0) a_{n-1}(x)$$

are analytic in a (complex) neighborhood of x_0. Alternatively, x_0 is a regular singular point if the equation is of the form

$$y^{(n)}(x) + \frac{b_{n-1}(x)}{x - x_0} y^{(n-1)}(x) + \cdots + \frac{b_1(x)}{(x - x_0)^{n-1}} y'(x) + \frac{b_0(x)}{(x - x_0)^n} y(x) = 0,$$

and the coefficient functions $b(x)$ are analytic at x_0.

Example: Consider the equation

$$y'(x) = \frac{y(x)}{x - 1}.$$

The point $x = 1$ is a regular singular point. However, the point $x = 0$ is *not* a regular singular point of the equation

$$y'(x) = \frac{x + 1}{x^3} y(x).$$

▲▲▲

It was proved (still Fuchs) that a solution *may be* analytic at a regular singular point. If it is not, then its singularity can only be either a pole, or a branch point (algebraic or logarithmic). Moreover, there always exists at least one solution of the form

$$(x - x_0)^\alpha g(x),$$

where α is called the ***indical exponent*** and $g(x)$ is analytic at x_0. For equations of order two and higher, there exists another independent solution either of the form

$$(x - x_0)^\beta h(x),$$

or of the form

$$(x - x_0)^\alpha g(x) \log(x - x_0) + (x - x_0)^\beta h(x),$$

where $h(x)$ is analytic at x_0. This process can be continued.

Example: For the case where the coefficients $b_i(x)$ are constant the equation is of Euler type, and we know that the solutions are indeed of this type, with $g(x)$ and $h(x)$ constant. ▲▲▲

Example: Consider the equation

$$y'(x) = \frac{y(x)}{\sinh x}.$$

It has a regular singular point at $x = 0$. The general solution is

$$y(x) = c \tanh \frac{x}{2},$$

which is analytic at $x = 0$. It's Taylor series at zero, which involves the ***Bernoulli numbers***, has a radius of π, which is the distance of the nearest singularity of the coefficient function (at $z = \pm\imath\pi$). ▲▲▲

*Definition 2.3 A point x_0 is called an **irregular singular point** of* (2.1) *if it is neither an ordinary point nor a regular singular point.*

There are no general properties of solutions near such point.

Finally, we consider also the point $x = \infty$ by changing the dependent variable into $t = 1/x$ and looking at $t = 0$. The point $x = \infty$ inherits the classification of the point $t = 0$.

Examples:

1. Consider the three equations

$$y'(x) - \frac{1}{2}y(x) = 0$$
$$y'(x) - \frac{1}{2x}y(x) = 0$$
$$y'(x) - \frac{1}{2x^2}y(x) = 0.$$

   Changing variables $t = 1/x$ we have

$$-y'(t) - \frac{1}{2t^2}y(t) = 0$$
$$-y'(t) - \frac{1}{2t}y(t) = 0$$
$$-y'(t) - \frac{1}{2}y(t) = 0.$$

Thus, every point $x \neq 0$ is an ordinary point of the first equation, the point $x = 0$ is a regular singular point of the second equation and an irregular singular point of the third equation. At $x = \infty$ it is the exact opposite. The solutions are

$$y(x) = c\, e^{x/2}, \qquad y(x) = c\sqrt{x}, \quad \text{and} \quad y(x) = c\, e^{-1/2x},$$

respectively. Note that the second solution has branch cuts at $x = 0$ and $x = \infty$, whereas the third solution has an essential singularity at $x = 0$.

✎ *Exercise 2.1* Classify all the singular points (finite and infinite) of the following equations

1. $y'' = xy$ (Airy equation).
2. $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$ (Bessel equation).
3. $y'' + (h - 2\theta \cos 2x)y = 0$ (Mathieu equation).

2.2 Local solution near an ordinary point

In the vicinity of an ordinary point, the solution to a linear differential equation can be sought by explicitly constructing its Taylor series. The latter is guaranteed to converge in a ball whose radius is a property of the coefficient functions; that is, it can be determined directly from the problem, without need to solve it. While this method can (in principle) provide the full solution, we are interested in it as a ***perturbation series***, i.e., as an ***approximation*** to the exact solution as a power series of the "small" parameter $x - x_0$. We will discuss perturbation series in more generality later on.

Example: Consider the IVP

$$y'(x) = 2x\, y(x) \qquad y(0) = 1.$$

The exact solution is $y(x) = ce^{x^2}$, but we will ignore it, and obtain it via a power series expansion.

Note first that the coefficient function $2z$ is entire, hence the Taylor series of the solution has an infinite radius of convergence. We now seek a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Substituting into the equation we get

$$\sum_{n=0}^{\infty} n a_n x^{n-1} = 2 \sum_{n=0}^{\infty} a_n x^{n+1}.$$

Since the Taylor series is unique the two sides must have equal coefficients, hence we resort to a term by term identification. Equating the x^0 terms we get that $a_1 = 0$. Then,

$$\sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+2} = 2 \sum_{n=0}^{\infty} a_n x^{n+1},$$

and

$$a_{n+2} = \frac{2a_n}{n+2}, \qquad n = 0, 1, \ldots.$$

The first coefficients a_0 is determined by the initial data $a_0 = 1$. All the odd coefficients vanish. For the even coefficients

$$a_2 = \frac{2}{2}, \quad a_4 = \frac{2^2}{2 \cdot 4}, \qquad a_6 = \frac{2^3}{2 \cdot 4 \cdot 6},$$

and thus,

$$y(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!},$$

which is the right solution.

Suppose that we interested in $y(1)$ within an error of 10^{-8}. If we retain n terms, then the error is

$$\sum_{k=n+1}^{\infty} \frac{1}{k!} = \frac{1}{(n+1)!} \left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots \right)$$
$$\leq \frac{1}{(n+1)!} \left(1 + \frac{1}{n} + \frac{1}{n^2} + \cdots \right) = \frac{1}{(n+1)!} \frac{n}{n-1} \leq \frac{2}{(n+1)!}.$$

Taking $n = 11$ will do. ▲▲▲

The Gamma function Before going to the next example, we remind ourselves the properties of the ***Gamma function***. It is a complex-valued function defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt.$$

This function is analytic in the upper half-plane. If has the property that

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^{z}\, dt = z\,\Gamma(z).$$

In particular,

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1,$$

which which follows that $\Gamma(2) = 1$, $\Gamma(3) = 2$, $\Gamma(4) = 6$, and more generally, for n integer,

$$\Gamma(n) = (n-1)!.$$

In fact, the Gamma-function can be viewed as the analytic extension of the factorial. The Gamma function can be used to shorten notations as

$$x(x+1)(x+2)\dots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}.$$

Example: Consider the ***Airy equation***[1]

$$y''(x) = x\,y(x).$$

Our goal is to analyze the solutions near the ordinary point $x = 0$. Here again, the solutions are guaranteed to be entire.

Again, we seek for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

[1]The Airy function is a special function named after the British astronomer George Biddell Airy (1838). The Airy equation is the simplest second-order linear differential equation with a turning point (a point where the character of the solutions changes from oscillatory to exponential). The Airy function describes the appearance of a star–a point source of light–as it appears in a telescope. The ideal point image becomes a series of concentric ripples because of the limited aperture and the wave nature of light.

Substituting we get

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n = \sum_{n=0}^{\infty} a_n x^{n+1}.$$

The coefficient a_0, a_1 remain undetermined, and are in fact the two integration constants. Equating the x^0 terms we get $a_2 = 0$. Then

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)}, \qquad n = 0, 1, 2, \dots.$$

This recursion relation can be solved: first the multiple of three,

$$a_3 = \frac{a_0}{2\cdot 3}, \quad a_6 = \frac{a_0}{2\cdot 3\cdot 5\cdot 6},$$

and more generally,

$$a_{3n} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\dots(3n-1)3n} = \frac{a_0}{3^{2n}\frac{2}{3}\cdot 1\cdot(1+\frac{2}{3})\cdot 2\dots(n-1+\frac{2}{3})n} = \frac{a_0\,\Gamma(\frac{2}{3})}{3^{2n}n!\Gamma(n+\frac{2}{3})}.$$

Similarly,

$$a_4 = \frac{a_1}{3\cdot 4}, \quad a_7 = \frac{a_1}{3\cdot 4\cdot 6\cdot 7},$$

hence

$$a_{3n+1} = \frac{a_0}{3\cdot 4\cdot 6\cdot 7\dots 3n(3n+1)} = \frac{a_1}{3^{2n}1\cdot(1+\frac{1}{3})\cdot 2\cdot(2+\frac{1}{3})\dots n(n+\frac{1}{3})} = \frac{a_0\,\Gamma(\frac{4}{3})}{3^{2n}n!\Gamma(n+\frac{4}{3})}.$$

Thus, the general solution is

$$y(x) = a_0\sum_{n=0}^{\infty}\frac{x^{3n}}{9^n n!\Gamma(n+\frac{2}{3})} + a_1\sum_{n=0}^{\infty}\frac{x^{3n+1}}{9^n n!\Gamma(n+\frac{4}{3})},$$

where we absorbed the constants into the coefficients.

These are very rapidly converging series, and their radius of convergence is infinite. An approximate solution is obtained by truncating this series. To solve an initial value problem one has to determine the coefficients a_0, a_1 first.

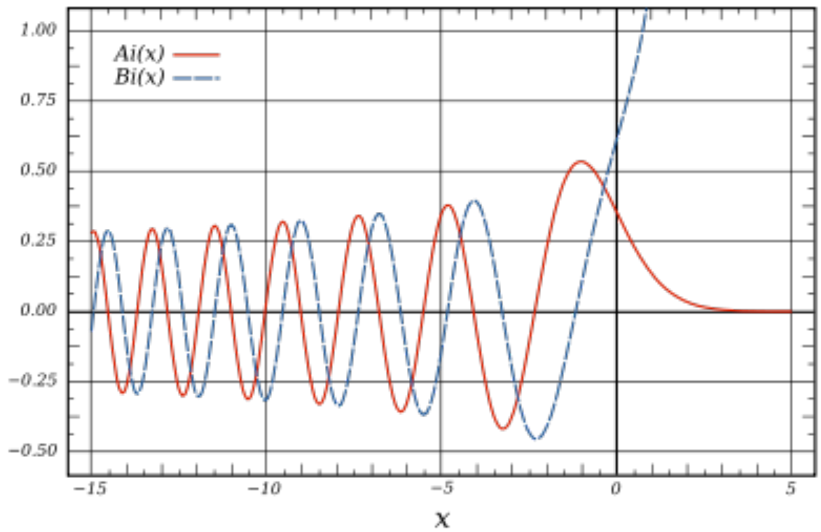

Figure 2.1: The Airy functions

The two terms we obtained are independent solutions. There is arbitrariness in the choice of independent solutions. It is customary to the refer to ***Airy functions*** as the two special (independent) choices of

$$\mathrm{Ai}(x) = 3^{-2/3}\sum_{n=0}^{\infty}\frac{x^{3n}}{9^n n!\Gamma(n+\frac{2}{3})} - 3^{-4/3}\sum_{n=0}^{\infty}\frac{x^{3n+1}}{9^n n!\Gamma(n+\frac{4}{3})}$$

$$\mathrm{Bi}(x) = 3^{-1/6}\sum_{n=0}^{\infty}\frac{x^{3n}}{9^n n!\Gamma(n+\frac{2}{3})} + 3^{-5/6}\sum_{n=0}^{\infty}\frac{x^{3n+1}}{9^n n!\Gamma(n+\frac{4}{3})}.$$

▲▲▲

✎ *Exercise 2.2* Find the Taylor expansion about $x = 0$ of the solution to the initial value problem

$$(x-1)(x-2)y''(x) + (4x-6)y'(x) + 2y(x) = 0, \qquad y(0) = 2,\ y'(0) = 1.$$

For which values of x we should expect the series to converge? What is its actual radius of convergence?

✎ *Exercise 2.3* Estimate the number of terms in the Taylor series need to estimate the Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ to three decimal digits at $x = \pm 1$, $x = \pm 100$ and $x = \pm 10,000$.

2.3 Local solution near a regular singular point

Let us first see what may happen if we Taylor expand the solution about a regular singular point:

Example: Consider the Euler equation,

$$y''(x) + \frac{y(x)}{4x^2} = 0, \qquad x_0 = 0.$$

Substituting a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

we get

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \frac{1}{4} \sum_{n=0}^{\infty} a_n x^{n-2} = 0,$$

i.e.,

$$(n - \tfrac{1}{2})^2 a_n = 0.$$

This gives $a_n = 0$ for all n, i.e., we only find the trivial solution. The general solution, however, is of the form $y(x) = c_1 \sqrt{x} + c_2 \sqrt{x} \log x$. ▲▲▲

The problem is that Taylor series are not general enough for this kind of problems. Yet, we know from Fuchs' theory that there exists at least one solution of the form

$$y(x) = (x - x_0)^\alpha g(x),$$

where $g(x)$ is analytic at x_0. This suggest to expand the solution in a series known as a ***Frobenius series***,

$$y(x) = (x - x_0)^\alpha \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

To remove indeterminacy we require $a_0 \neq 0$.

Example: Going back to the previous example, we search a solution of the form

$$y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n.$$

Substituting we get

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n x^{n+\alpha-2} + \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{n+\alpha-2} = 0,$$

i.e.,

$$\left[(n+\alpha)(n+\alpha-1)+\tfrac{1}{4}\right]a_n = 0.$$

Since we require $a_0 \neq 0$, the indical exponent α satisfies the quadratic equation

$$P(\alpha) = (\alpha - \tfrac{1}{2})^2 = 0.$$

This equation has a double root at $\alpha = 1/2$. For $n = 1, 2, \dots$ we have $a_n = 0$, hence we found an exact solution, $y(x) = \sqrt{x}$. On the other hand, this method does not allow us, for the moment, to find a second independent solution. ▲▲▲

We will discuss now, in generality, local expansions about regular singular points of second-order equations,

$$y''(x) + \frac{p(x)}{x-x_0} + \frac{q(x)}{(x-x_0)^2} = 0.$$

We assume that the functions $p(x), q(x)$ are analytic at x_0, i.e., they can be locally expanded as

$$p(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^n$$

$$q(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^n.$$

We then substitute into the equation a Frobenius series,

$$y(x) = (x-x_0)^{\alpha} \sum_{n=0}^{\infty} a_n (x-x_0)^n.$$

This gives

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n(x-x_0)^{n+\alpha-2}$$

$$+ \left(\sum_{k=0}^{\infty} p_n (x-x_0)^k\right)\sum_{n=0}^{\infty}(n+\alpha)a_n(x-x_0)^{n+\alpha-2}$$

$$+ \left(\sum_{k=0}^{\infty} q_n (x-x_0)^k\right)\sum_{n=0}^{\infty} a_n(x-x_0)^{n+\alpha-2} = 0.$$

Equating same powers of $(x - x_0)$ we get

$$(n+\alpha)(n+\alpha-1)a_n + \sum_{k=0}^{n} \left[p_k(n-k+\alpha) + q_k\right] a_{n-k} = 0.$$

Separating the $k = 0$ term we get

$$\left[(n+\alpha)^2 + (p_0 - 1)(n+\alpha) + q_0\right] a_n = -\sum_{k=1}^{n} \left[p_k(n-k+\alpha) + q_k\right] a_{n-k}.$$

We write the left-hand side as $P(n+\alpha)\, a_n$. The requirement that $a_0 \neq 0$ implies that $P(\alpha) = 0$, i.e., α is a solution of a quadratic equation. a_0 is indeterminate (integration constant), whereas the other a_n are then given by a recursion relation,

$$a_n = -\frac{1}{P(\alpha+n)} \sum_{k=1}^{n} \left[p_k(n-k+\alpha) + q_k\right] a_{n-k}, \qquad n = 1, 2, \ldots.$$

A number of problems arise right away: (1) α may be a double root in which case we're lacking a solution. (2) The recursion relation may break down if for some $n \in \mathbb{N}$, $P(\alpha + n) = 0$. Yet, if α_1, α_2 are the two roots of the indical equation, and $\Re\alpha_1 \geq \Re\alpha_2$, then it is guaranteed that $P(\alpha_1 + n) \neq 0$, and the recursion relation can be continued indefinitely. This is why there is always at least one solution in the form of a Frobenius series. More generally, we have the following possible scenarios:

1. $\alpha_1 \neq \alpha_2$ and $\alpha_1 - \alpha_2 \notin \mathbb{Z}$. In this case there are two solutions in the form of Frobenius series.
2. (a) $\alpha_1 = \alpha_2$. There is one solution in the form of a Frobenius series and we will see how to construct a second independent solution.

   (b) $\alpha_1 - \alpha_2 = N$, $N \in \mathbb{N}$:

      i. If $\sum_{k=1}^{N} \left[p_k(\alpha_1 - k) + q_k\right] a_{N-k} = 0$ then $a_N = 0$ and the series can be continued past the "bad" index.

      ii. Otherwise, there is only one solution in the form of a Frobenius series. We will see how to construct another independent solution.

✎ *Exercise 2.4* Find series expansions about $x = 0$ for the following differential equations. Try to sum (if possible) the infinite series.

① $2xy''(x) - y'(x) + x^2 y(x) = 0.$

② $x(x+2)y''(x) + (x+1)y'(x) - 4y(x) = 0.$

③ $x(1-x)y''(x) - 3xy'(x) - y(x) = 0.$

④ $\sin x\, y''(x) - 2\cos x\, y'(x) - \sin x\, y(x) = 0.$

Example: We start with an example of type 1. Consider the modified Bessel equation

$$y''(x) + \frac{1}{x}y(x) - \left(1 + \frac{\nu^2}{x^2}\right)y(x) = 0.$$

The point $x = 0$ is a regular singular point, hence we substitute the Frobenius series

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}.$$

This gives

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n x^{n+\alpha-2} + \sum_{n=0}^{\infty}(n+\alpha)a_n x^{n+\alpha-2} - \sum_{n=0}^{\infty} a_n x^{n+\alpha} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+\alpha-2} = 0.$$

Equating powers of x we get

$$\left[(n+\alpha)^2 - \nu^2\right] a_n = a_{n-2}.$$

For $n = 0$ we get the indical equation

$$P(\alpha) = \alpha^2 - \nu^2 = 0,$$

i.e., $\alpha = \pm\nu$. Take first $\alpha = \nu > 0$, in which case $P(\alpha + n) > 0$.

For $n = 1$, since $P(\nu + 1) \neq 0$ we have $a_1 = 0$. For $n \geq 0$,

$$a_n = \frac{a_{n-2}}{(n+\nu)^2 - \nu^2} = \frac{a_{n-2}}{n(n+2\nu)}.$$

That is,

$$a_{2n} = \frac{a_0}{2n(2n-2)\dots 2 \cdot (2n+2\nu)(2n+2\nu-2)\dots(4+2\nu)}.$$

We can express this using the Γ function,

$$a_{2n} = \frac{\Gamma(\nu+1)}{4^n n!\, \Gamma(n+\nu+1)} a_0.$$

Thus, a first solution is

$$y(x) = \Gamma(\nu + 1) \sum_{n=0}^{\infty} \frac{x^{2n+\nu}}{4^n n!\, \Gamma(n + \nu + 1)}.$$

It is conventional to define the ***modified Bessel function***

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}x)^{2n+\nu}}{n!\, \Gamma(n + \nu + 1)}.$$

This series has an infinite radius of convergence, as expected from the analyticity of the coefficients.

A second solution can be found by setting $\alpha = -\nu$. In order for $P(-\nu + n) \neq 0$ we need 2ν not to be an integer. Note however that $I_{-\nu}(x)$ given by the above power series is well defined as long as 2ν is not an even integer, i.e., $I_{-1/2}(x)$, $I_{-3/2}(x)$ and so on are well-defined and form a second independent solution.

▲▲▲

✎ *Exercise 2.5* Show that all the solutions of the modified Bessel equation

$$y''(x) + \frac{y'(x)}{x} - \left(1 + \frac{\nu^2}{x^2}\right) y(x) = 0.$$

with $\nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, can be expanded in Frobenius series.

Case 2b(i) This is the simplest "bad" case, where nevertheless as second solution in the form of a Frobenius series can be constructed.

Example: You will be asked as homework to examine the half-integer modified Bessel equation. ▲▲▲

Case 2a This is that case where α is a double root, $\alpha_1 = \alpha_2 = \alpha$. Recall that when we substitute in the equation a Frobenius series, we get

$$P(n + \alpha) a_n = -\sum_{k=1}^{n} [p_k(n - k + \alpha) + q_k] a_{n-k},$$

where

$$P(s) = s^2 + (p_0 - 1)s + p_0.$$

In the present case we have

$$P(s) = (s - \alpha)^2.$$

One solution can be obtained by this procedure. We solve iteratively for the a_n and have

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\alpha}.$$

We may generalize this type of solutions by replacing α by an arbitrary β, i.e., form a function

$$y(x;\beta) = \sum_{n=0}^{\infty} a_n(\beta)(x - x_0)^{n+\beta},$$

where the coefficients $a_n(\beta)$ satisfy the recursion relations,

$$P(n + \beta)a_n(\beta) = -\sum_{k=1}^{n} [p_k(n - k + \beta) + q_k] a_{n-k}(\beta).$$

Of course, this is a solution only for $\beta = \alpha$.

Let's see now what happens if we substitute $y(x;\beta)$ into the differential equation,

$$\mathcal{L}[y](x) = y''(x) + \frac{p(x)}{x - x_0} y'(x) + \frac{q(x)}{(x - x_0)^2} y(x) = 0.$$

We get

$$\begin{aligned}\mathcal{L}[y(\cdot;\beta)] = &\sum_{n=0}^{\infty} (n + \beta)(n + \beta - 1) a_n(\beta)(x - x_0)^{n+\beta-2} \\ &+ p(x) \sum_{n=0}^{\infty} (n + \beta) a_n(\beta)(x - x_0)^{n+\beta-2} \\ &+ q(x) \sum_{n=0}^{\infty} a_n(\beta)(x - x_0)^{n+\beta-2}.\end{aligned}$$

If we substitute the series expansions for $p(x)$, $q(x)$, we find that almost all the terms vanish, because the $a_n(\beta)$ satisfy the correct recursion relations. The only terms that do not vanish are those proportional to $x^{\beta-2}$,

$$\mathcal{L}[y(\cdot;\beta)](x) = a_0 \left[\beta^2 + (p_0 - 1)\beta + q_0\right] x^{\beta-2} = a_0 P(\beta) x^{\beta-2}.$$

Indeed, this vanishes if and only if $\beta = \alpha$. If we now differentiate both sides with respect to β and set $\beta = \alpha$ the right-hand side vanishes because α is a double root of P. Thus,

$$\mathcal{L}\left[\frac{\partial}{\partial\beta} y(\cdot;\beta)\bigg|_{\beta=\alpha}\right] = 0,$$

that is we found another independent solution,

$$\frac{\partial}{\partial\beta} y(\cdot;\beta)\bigg|_{\beta=\alpha} = \sum_{n=0}^{\infty} \frac{da_n}{d\beta}\bigg|_{\beta=\alpha} (x-x_0)^{n+\alpha} + \log(x-x_0) \sum_{n=0}^{\infty} a_n(\alpha)(x-x_0)^{n+\alpha},$$

where we have used the fact that

$$\frac{\partial}{\partial\beta} x^\beta = x^\beta \log x.$$

We write it in the more compact form,

$$y_2(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+\alpha} + \log(x-x_0)\, y_1(x), \qquad b_n = \frac{da_n}{d\beta}\bigg|_{\beta=\alpha}.$$

Example: Consider the modified Bessel for $\nu = 0$. Recall that we get the recursion relation

$$(n+\alpha)^2 a_n = a_{n-2},$$

and therefore conclude that $\alpha = 0$ and that

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n (n!)^2}$$

is a first solution. We then define the coefficients $a_n(\beta)$ by the recursion relation

$$(n+\beta)^2 a_n(\beta) = a_{n-2}(\beta),$$

i.e.,

$$a_{2n}(\beta) = \frac{a_{2n-2}(\beta)}{(2n+\beta)^2} = \frac{a_0}{(2n+\beta)^2(2n-2+\beta)^2 \ldots (2+\beta)^2}.$$

Differentiating with respect to β and setting $\beta = \alpha = 0$ we get

$$b_{2n} = -a_{2n}(0)\left(\frac{1}{n} + \frac{1}{n-1} + \ldots 1\right) = -\frac{a_0}{4^n (n!)^2}\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + 1\right).$$

Thus we have found another (independent solution)

$$y_2(x) = a_0 \log x \sum_{n=0}^{\infty} \frac{(\frac{1}{2}x)^{2n}}{(n!)^2} - a_0 \sum_{n=0}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \frac{(\frac{1}{2}x)^{2n}}{(n!)^2}.$$

It is conventional to choose for other independent function a linear combination of $y_2(x)$ and $I_0(x)$ (it is called $K_0(x)$). ▲▲▲

Case 2b(ii) We are left with the case where

$$P(s) = (s - \alpha_1)(s - \alpha_2),$$

with $\alpha_1 - \alpha_2 = N \in \mathbb{N}$, and no "miracle" occurs. As before, using $y(x; \beta)$ we have

$$\mathcal{L}[y(\cdot;\beta) = a_0 P(\beta) x^{\beta-2}.$$

If we try to do again as before, differentiating both sides with respect to β and setting $\beta = \alpha_1$ we find

$$\mathcal{L}\left[\frac{\partial}{\partial\beta} y(\cdot,\beta)\bigg|_{\beta=\alpha_1}\right] = a_0 N x^{\alpha_1 - 2} = a_0 N x^{\alpha_2 + N - 2}.$$

In other words,

$$\frac{\partial}{\partial\beta} y(\cdot,\beta)\bigg|_{\beta=\alpha_1}$$

satisfies the inhomogeneous equation

$$\mathcal{L}[y](x) = a_0 N x^{\alpha_2 + N - 2}.$$

A way to obtain a solution to the homogeneous equation is to subtract any particular solution to this inhomogeneous equation. Can we find such? It turns out that we can find a solution in the form of a Frobenius series.

Setting

$$z(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+\alpha_2},$$

and substituting into the inhomogeneous equation, we get

$$\sum_{n=0}^{\infty}(n+\alpha_2)(n+\alpha_2-1)c_n(x-x_0)^{n+\alpha_2-2}$$
$$+\left(\sum_{k=0}^{\infty} p_n(x-x_0)^k\right)\sum_{n=0}^{\infty}(n+\alpha_2)c_n(x-x_0)^{n+\alpha_2-2}$$
$$+\left(\sum_{k=0}^{\infty} q_n(x-x_0)^k\right)\sum_{n=0}^{\infty} c_n(x-x_0)^{n+\alpha_2-2} = a_0 N x^{\alpha_2+N-2}.$$

Equating the coefficients of $(x-x_0)^{\alpha_2-2}$ we get

$$\left[\alpha_2^2+(p_0-1)\alpha_2+q_0\right]c_0 = 0,$$

which is indeed satisfies since the pre-factor is $P(\alpha_2)$. In particular, c_0 is not (yet) determined. For all powers of $(x-x_0)$ other than α_2+N-2 we have the usual recursion relation,

$$P(n+\alpha_2)c_n = -\sum_{k=1}^{n}[p_k(n-k+\alpha_2)+q_k]c_{n-k}.$$

Since $n \neq N$ there is no problem. Remain the terms proportional to $(x-x_0)$ to the power α_2+N-2, which give

$$P(N+\alpha_2)c_N = -\sum_{k=1}^{N}[p_k(N-k+\alpha_2)+q_k]c_{N-k} + a_0 N.$$

While the left hand side is zero, we can view this equation as determining c_0 (i.e., relating it to a_0). Then, c_N is left arbitrary, but that is not a problem. Particular solutions are not unique. Thus, we have constructed a second (independent) solution which is

$$y_2(x) = \left.\frac{\partial}{\partial\beta}y(\cdot,\beta)\right|_{\beta=\alpha_1} - z(x),$$

or,

$$y_2(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^{n+\alpha_1} + \log(x-x_0)y_1(x) - \sum_{n=0}^{\infty} c_n(x-x_0)^{n+\alpha_2}.$$

✎ *Exercise 2.6* Find a second solution for the modified Bessel equation with $\nu = 1$.

2.4 Local solution near irregular singular points

No far everything was very straightforward (though sometimes tedious). Rigorous methods to find local solutions, always guaranteed to work, reflecting the fact that the theory of local solutions near ordinary and regular singular points is complete. In the presence of irregular singular points, no such theory exists, and one has to build up approximation methods that are often based on heuristics and intuition.

In the same way as we examined the breakdown of Taylor series near regular singular points, let's examine the breakdown of Frobenius series near irregular singular points.

Example: Let's start with a non-dramatic example,

$$y'(x) = x^{1/2} y(x).$$

The point $x = 0$ is am irregular singular points (note that nothing diverges). The general solution is obtained by separation of variables,

$$\frac{d}{dx} \log y(x) = \frac{2}{3} \frac{d}{dx} x^{3/2},$$

i.e.,

$$y(x) = c\, e^{\frac{2}{3} x^{3/2}}.$$

This can be written as a series,

$$y(x) = c \sum_{n=0}^{\infty} \frac{(\frac{2}{3} x^{3/2})^n}{n!},$$

that has nothing of a Frobenius series, where all powers must be of the form $n + \alpha$.
▲▲▲

Example: We next consider the equation

$$x^3 y''(x) = y(x),$$

where $x = 0$ is clearly an irregular singular point. If we attempt a Frobenius series,

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha},$$

we get

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n x^{n+\alpha+1} = \sum_{n=0}^{\infty} a_n x^{n+\alpha}.$$

The first equation is $a_0 = 0$, which is a contradiction. ▲▲▲

Example: This third example is much more interesting,

$$x^2 y''(x) + (1+3x)y'(x) + y(x) = 0.$$

Substituting a Frobenius series gives

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n x^{n+\alpha} + \sum_{n=0}^{\infty}(n+\alpha)a_n x^{n+\alpha-1} + 3\sum_{n=0}^{\infty}(n+\alpha)a_n x^{n+\alpha} + \sum_{n=0}^{\infty} a_n x^{n+\alpha} = 0.$$

Equating the coefficients of $\alpha - 1$ we get $\alpha a_n = 0$, i.e., $\alpha = 0$, which means that the Frobenius series is in fact a power series. Then,

$$n a_n = -\left[(n-1)(n-2) + 3(n-1) + 1\right] a_{n-1} = -n^2 a_{n-1},$$

from which we get that $a_n = (-1)^n n!\, a_0$, *i.e.*, the solution is

$$y(x) = a_0 \sum_{n=0}^{\infty}(-1)^n n! x^n.$$

This is a series whose radius of convergence is zero! Thus, it does not look as a solution at all. This is indeed a divergent series, on the other hand, it is a perfectly good ***asymptotic series***, something we are going to explore in depth. If we truncate this series at some n, it provides a very good approximation for small x. ▲▲▲

Some definitions We will address the notion of asymptotic series later on, and at this stage work "mechanically" in a way we will learn.

*Definition 2.4 We say that $f(x)$ is **much smaller** than $g(x)$ as $x \to x_0$,*

$$f(x) \ll g(x) \qquad \text{as } x \to x_0,$$

if

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

*We say that $f(x)$ is **asymptotic** to $g(x)$ as $x \to x_0$,*

$$f(x) \sim g(x) \qquad \text{as } x \to x_0,$$

if

$$f(x) - g(x) \ll g(x).$$

Note that asymptoticity is symmetric as $f(x) \sim g(x)$ implies that

$$\lim_{x\to x_0} \frac{f(x) - g(x)}{g(x)} = \lim_{x\to x_0} \frac{f(x)}{g(x)} - 1 = 0,$$

i.e., that

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1.$$

Examples:

1. $x \ll 1/x$ as $x \to 0$.
2. $x^{1/2} \ll x^{1/3}$ as $x \to 0^+$.
3. $(log x)^5 \ll x^{1/4}$ as $x \to \infty$.
4. $e^x + x \sim e^x$ as $x \to \infty$.
5. $x^2 \nsim x$ as $x \to 0$.
6. A function can never be asymptotic to zero!
7. $x \ll -1$ as $x \to 0^+$ even though $x > -1$ for all $x > 0$.

In the following, until we do it systematically, we will assume that asymptotic relations can be added, multiplied, integrated and differentiated. Don't worry about justifications at this point.

Example: Let's return to the example

$$x^3 y''(x) = y(x),$$

for which we were unable to construct a Frobenius series. It turns out that as $x \to 0$, the two independent solutions have the asymptotic behavior,

$$y_1(x) \sim c_1 x^{3/4} e^{2x^{-1/2}}$$
$$y_2(x) \sim c_2 x^{3/4} e^{-2x^{-1/2}}.$$

▲▲▲

Example: Recall the other example,

$$x^2 y''(x) + (1 + 3x)y'(x) + y(x) = 0,$$

for which we were able to construct only one series, which was divergent everywhere. It turns out that the second solution has the asymptotic behavior,

$$y_2(x) \sim c^2 \frac{1}{x} e^{1/x} \qquad \text{as } x \to 0^+.$$

▲▲▲

All these solutions exhibit an exponential of a function that diverges at the singular point. This turns out to be typical. These asymptotic expressions turn out to be the most significant terms in an infinite series expansion of the solution. We call these terms the ***leading terms*** (it is not clear how well-defined this concept is). Each of these leading terms is itself a product of functions among which we can identify one which is the "most significant"—the ***controlling factor***. Identifying the controlling factor is the first step in finding the leading term of the solution.

Comment: Note that if $z(x)$ is a leading term for $y(x)$ it does not mean that their difference is small; only that their ratio tends to one.

Consider now a linear second-order equation,

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0,$$

where x_0 is an irregular singular point. The first step in approximating the solution near x_0 is to substitute,

$$y(x) = \exp S(x),$$

which gives,

$$S''(x) + [S'(x)]^2 + p(x)S'(x) + q(x) = 0.$$

This substitution goes back to Carlini (1817), Liouville (1837) and Green (1837).

The resulting equation is of course as complicated as the original one. It turns out, however, that it is typical for

$$S''(x) \ll [S'(x)]^2 \qquad \text{as } x \to x_0.$$

(Check for all above examples.) Then, it implies that

$$[S'(x)]^2 \sim -p(x)S'(x) + q(x) \qquad \text{as } x \to x_0.$$

Note that we moved two terms to the right-hand side since no function can be asymptotic to zero. We then proceed to integrate this relation (ignoring again any justification) to find $S(x)$.

Example: Let us explore the example

$$x^3 y''(x) = y(x)$$

in depth. The substitution $y(x) = \exp S(x)$ yields,

$$x^3 S''(x) + x^3 [S'(x)]^2 = 1,$$

which is no more solvable than the original equation (even less as it is nonlinear). Assuming that $S''(x) \ll [S'(x)]^2$ we get

$$[S'(x)]^2 \sim x^{-3},$$

hence

$$S'(x) \sim \pm x^{-3/2}.$$

If this were an equation we would have $S(x) = \mp 2x^{-1/2} + c$. Here this constant could depend on x, i.e.,

$$S(x) = \mp 2x^{-1/2} + c_\pm(x),$$

as long as

$$S'(x) = \pm x^{-3/2} + c'_\pm(x) \sim \pm x^{-3/2},$$

i.e., $c'_\pm(x) \ll x^{-3/2}$ as $x \to 0^+$. Note that this is consistent with the assumption that

$$S''(x) \sim - \mp \frac{3}{2} x^{-5/2} \ll [S'(x)] \sim x^{-3}.$$

Let us focus on the positive solution and see if this result can be refined. We have the ansatz

$$S(x) = 2x^{-1/2} + c(x), \qquad c'(x) \ll x^{-3/2},$$

which substituted into the (full) equation for $S(x)$ gives

$$\frac{3}{2}x^3x^{-5/2} + x^3c''(x) + x^3(-x^{3/2} + c'(x))^2 = 1,$$

or,

$$\frac{3}{2}x^{1/2} + x^3c''(x) - 2x^{3/2}c'(x) + x^3[c'(x)]^2 = 0.$$

Since $c'(x) \ll x^{-3/2}$ the last term is much smaller than the third. Moreover, assuming that we can differentiate the asymptotic relation,

$$c''(x) \ll -\frac{3}{2}x^{-5/2},$$

we remain with $\frac{3}{2}x^{1/2} \sim 2x^{3/2}c'(x)$, or,

$$c'(x) \sim \frac{3}{4x}.$$

Again, if this were an equality we would get $c(x) = \frac{3}{4}\log x$. Here we have

$$c(x) = \frac{3}{4}\log x + d(x),$$

where $d'(x) \ll \frac{3}{4x}$.

We proceed, this time with

$$S(x) = 2x^{-1/2} + \frac{3}{4}\log x + d(x).$$

This time,

$$\frac{3}{2}x^{-5/2} - \frac{3}{4x^2} + d''(x) + \left(-x^{-3/2} + \frac{3}{4x} + d'(x)\right)^2 = \frac{1}{x^3},$$

which simplifies into

$$-\frac{3}{16x^2} + d''(x) + [d'(x)]^2 - 2x^{-3/2}d'(x) + \frac{3}{2x}d'(x) = 0.$$

Since $x^{-1} \ll x^{-3/2}$ and $d'(x) \ll 3/4x$ (from which we conclude that $d''(x) \ll x^{-2}$), we remain with the asymptotic relation,

$$-\frac{3}{16x^2} \sim 2x^{-3/2}d'(x),$$

i.e.,

$$d'(x) \sim -\frac{3}{32}x^{-1/2}.$$

From this we deduce that

$$d(x) \sim \frac{3}{16}x^{1/2} + \delta(x),$$

where $\delta'(x) \ll x^{-1/2}$. This time, the leading term vanishes as $x \to 0$. Thus, we conclude that

$$S(x) \sim 2x^{-1/2} + \frac{3}{4}\log x + c,$$

which gives that

$$y(x) \sim c\, x^{3/4} e^{2x^{-1/2}}.$$

This was long, exhausting, and relying on shaky grounds!

Numerical validation Since we do not have a theoretical justification to the above procedure, let us try to evaluate the quality of the approximation numerically. On the one hand, let us solve the equation $x^3 y''(x) = y(x)$ numerically, with the initial date

$$y(1) = 1 \qquad \text{and} \qquad y'(1) = 0.$$

The solution is shown in Figure 2.2*a* on a log-log scale.

We know, on the other hand that the solution is *asymptotically* a linear combination of the form

$$y(x) = c_1 x^{3/4} e^{2x^{-1/2}}(1 + \epsilon_1(x)) + c_2 x^{3/4} e^{-2x^{-1/2}}(1 + \epsilon_2(x)),$$

where $\epsilon_{1,2}(x)$ tend to zero as $x \to 0$. Since one of the two solutions terns to zero as $x \to 0$, we expect that

$$\frac{y(x)}{x^{3/4}e^{2x^{-1/2}}} \sim c_1(1 + \epsilon_2(x)).$$

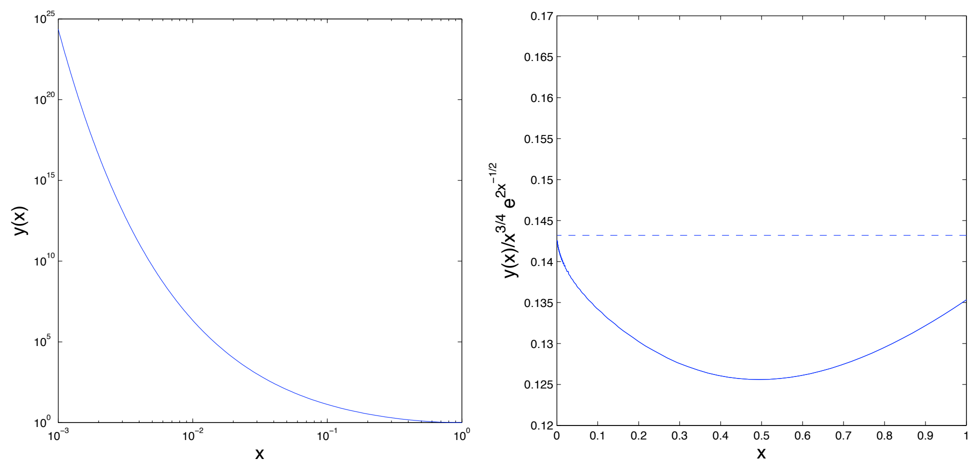

Figure 2.2:

In Figure 2.2*a* we show $y(x)/x^{3/4}e^{2x^{-1/2}}$ versus x. The deviation from the constant $c_1 \approx 0.1432$ is $c_1\epsilon_1(x)$. ▲▲▲

The technique which we have used above is called the ***method of dominant balance***. It based on (i) identifying the terms that appear to be small, dropping them, thus replacing the equation by an asymptotic relation. (ii) We then replace the asymptotic sign by an equality and solve the differential equation. (iii) We check that the result is consistent and allow for additional weaker variations. (iv) We iterate this procedure.

Example: We go back once more to our running example and try to improve the approximation. At this stage we have

$$y(x) = x^{3/4}e^{2x^{-1/2}}[1 + \epsilon(x)].$$

We will substitute into the equation and try to solve for $\epsilon(x)$ as a power series of x. Setting $w(x) = 1 + \epsilon(x)$, we have

$$y'(x) = x^{3/4}e^{2x^{-1/2}}\left[\frac{3}{4x}w(x) - x^{-3/2}w(x) + w'(x)\right],$$

and

$$y''(x) = x^{3/4}e^{2x^{-1/2}}\left[\frac{3}{4x} - x^{-3/2}\right]\left[\frac{3}{4x}w(x) - x^{-3/2}w(x) + w'(x)\right]$$
$$+ x^{3/4}e^{2x^{-1/2}}\left[\frac{3}{4x}w'(x) - x^{-3/2}w'(x) + w''(x) - \frac{3}{4x^2}w(x) + \frac{3}{2}x^{-5/2}w(x)\right].$$

This equals $x^{3/4}e^{2x^{-1/2}}w(x)x^{-3}$, which leaves us with the equation,

$$x^{-3}w(x) = \left[\frac{3}{4x} - x^{-3/2}\right]\left[\frac{3}{4x}w(x) - x^{-3/2}w(x) + w'(x)\right]$$
$$+ \left[\frac{3}{4x}w'(x) - x^{-3/2}w'(x) + w''(x) - \frac{3}{4x^2}w(x) + \frac{3}{2}x^{-5/2}w(x)\right].$$

This further simplifies into

$$w''(x) + \left(\frac{3}{2x} - 2x^{-3/2}\right)w'(x) - \frac{3}{16x^2}w(x) = 0.$$

This is a linear equation. Since we have extracted the singular parts of the solution, there is hope that this remaining equation can be dealt with by simpler means. This does not mean that the resulting equation no longer has an irregular singularity at zero. The only gain is that $w(x)$ does not diverge at the origin.

We proceed to solve this equation by the method of dominant balance. The equation for $\epsilon(x)$ is

$$\epsilon''(x) + \left(\frac{3}{2x} - 2x^{-3/2}\right)\epsilon'(x) - \frac{3}{16x^2}(1 + \epsilon(x)) = 0.$$

Since $\epsilon(x) \ll 1$ as $x \to 0$ we remain with

$$\epsilon''(x) - 2x^{-3/2}\epsilon'(x) \sim \frac{3}{16x^2},$$

subject to the constraint that $\epsilon(x) \to 0$. We do not know whether among the three remaining terms there are some greater than the others. We therefore need to investigate four possible scenarios:

① $\epsilon'' \sim 3/16x^2$ and $x^{-3/2}\epsilon' \ll \epsilon''$. In this case we get that

$$\epsilon(x) \sim -\frac{3}{16}\log x + ax + b,$$

which can't possible vanish at the origin.

② $\epsilon'' \sim 2x^{-3/2}\epsilon'$ and $3/16x^2 \ll \epsilon''$. In this case,

$$(\log \epsilon(x))' \sim 2x^{-3/2},$$

i.e.,

$$\log \epsilon(x) \sim -4x^{-1/2} + c,$$

which again violates the condition at the origin.

③ All three terms are of equal importance. This means that

$$(e^{-4x^{-1/2}}\epsilon'(x))' \sim \frac{3}{16x^2}e^{-4x^{-1/2}}.$$

Integrating we get again divergence at the origin.

④ This leaves for unique possibility that $-2x^{-3/2}\epsilon' \sim \frac{3}{16x^2}$ and $\epsilon'' \ll -2x^{-3/2}\epsilon'$. Then

$$\epsilon'(x) \sim -\frac{3}{32}x^{-1/2},$$

i.e.,

$$\epsilon(x) \sim -\frac{3}{16}x^{1/2} + \epsilon_1(x),$$

where $\epsilon_1'(x) \ll x^{-1/2}$.

Thus, we already have for approximation

$$y(x) = x^{3/4}e^{2x^{-1/2}}\left[1 - \frac{3}{16}x^{1/2} + \epsilon_1(x)\right].$$

See Figure 2.3*a*.

We can proceed further, discovering doing so that each correction is of power of $x^{1/2}$ greater than its predecessor. At this stage we may well substitute the asymptotic series,

$$w(x) = \sum_{n=0}^{\infty} a_n x^{n/2},$$

and $a_0 = 1$. This yields after some manipulations,

$$y(x) \sim x^{3/4}e^{2x^{-1/2}}\sum_{n=0}^{\infty}\frac{\Gamma(n-\frac{1}{2})\,\Gamma(n+\frac{3}{2})}{\pi 4^n\, n!}x^{n/2}.$$

In Figure 2.3*b* we show the ratio between the exact solution and the approximate solution truncated at $n = 10$. Note how the solution becomes more accurate near the origin, although it becomes less accurate further away, reflecting the fact that the series has a vanishing radius of convergence. ▲▲▲

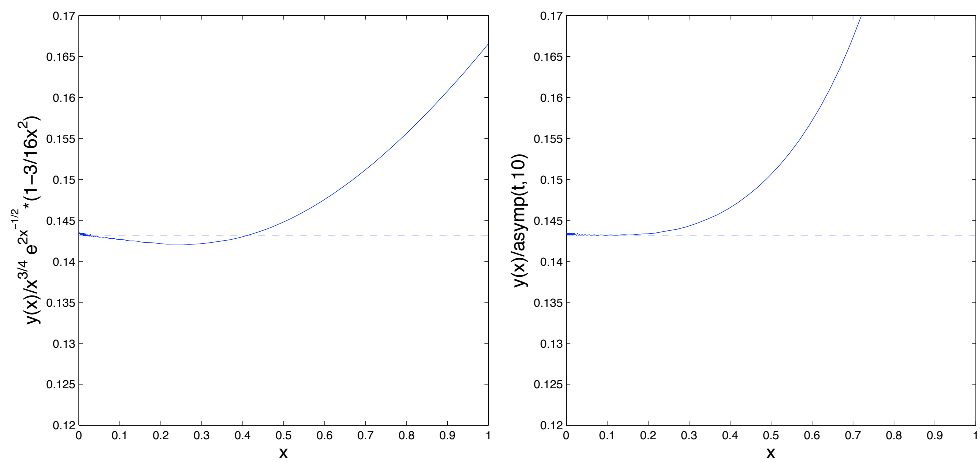

Figure 2.3:

✎ *Exercise 2.7* Using the method of dominant balance, investigate the second solution to the equation

$$x^2 y''(x) + (1 + 3x)y'(x) + y(x) = 0.$$

Try to imitate all the steps followed in class. You should actually end up with an exact solution!

✎ *Exercise 2.8* Find the leading behavior, as $x \to 0^+$, of the following equations:

① $y''(x) = \sqrt{x}\, y(x)$.
② $y''(x) = e^{-3/x} y(x)$.

2.5 Asymptotic series

Definition 2.5 The power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is said to be ***asymptotic to the function*** *$y(x)$ as $x \to x_0$, denoted*

$$y(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad x \to x_0,$$

if for every N

$$y(x) - \sum_{n=0}^{N} a_n(x-x_0)^n \ll (x-x_0)^N, \qquad x \to x_0.$$

This does not require the series to be convergent.

Comment: The asymptotic series does not need to be in integer powers of $x - x_0$. For example,

$$y(x) \sim \sum_{n=0}^{\infty} a_n(x-x_0)^{\alpha n}, \qquad x \to x_0$$

$\alpha > 0$, if for every N

$$y(x) - \sum_{n=0}^{N} a_n(x-x_0)^{\alpha n} \ll (x-x_0)^{\alpha N}, \qquad x \to x_0.$$

For $x_0 = \infty$ the definition is that

$$y(x) \sim \sum_{n=0}^{\infty} a_n x^{-\alpha n}, \qquad x \to \infty,$$

if for every N

$$y(x) - \sum_{n=0}^{N} a_n x^{-\alpha n} \ll x^{-\alpha N}, \qquad x \to \infty.$$

Not all functions can be expanded in asymptotic series:

Example: Not all functions have asymptotic series expansions. The function $1/x$ does not have a asymptotic series expansion at $x_0 = 0$ because it diverges. Similarly, the function e^x does not have an asymptotic series expansion at $x_0 = \infty$. ▲▲▲

The difference between a convergent series and an asymptotic series is worth stressing. Recall that a series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is convergent if

$$\lim_{N\to\infty} \sum_{n=N+1}^{\infty} a_n(x-x_0)^n = 0, \qquad \text{for } x \text{ fixed.}$$

Convergence is an absolute property. A series is either convergent or not, and convergence can be determined regardless of whether we know what the limit is. In contrast, a series is asymptotic to a function $f(x)$ if

$$f(x) - \sum_{n=0}^{N} a_n (x - x_0)^n \ll (x - x_0)^N, \qquad \text{for } N \text{ fixed.}$$

Asymptoticity is relative to a function. It makes no sense to ask whether a series is asymptotic. In fact, every power series is asymptotic to some function at x_0.

Proposition 2.1 Let (a_n) be a sequence of numbers. Then there exists a function $y(x)$ such that

$$y(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad x \to x_0.$$

Proof: Without loss of generality, let us take $x_0 = 0$. We define the following continuous function,

$$\phi(x; \alpha) = \begin{cases} 1 & |x| \leq \frac{1}{2}\alpha \\ 2\left(1 - \frac{|x|}{\alpha}\right) & \frac{1}{2}\alpha < |x| < \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Then we set

$$\alpha_n = \min(1/|a_n|^2, 2^{-n}),$$

and

$$y(x) = \sum_{n=0}^{\infty} a_n \phi(x; \alpha_n) x^n.$$

For every x this series is finite and continuous because it truncates after a finite number of terms. On the other hand we show that

$$y(x) \sim \sum_{n=0}^{\infty} a_n x^n.$$

Indeed, fixing N we can find a $\delta > 0$ such that

$$\phi(x; \alpha_n) = 1, \qquad \text{for all } n = 0, 1, \dots, N \text{ for } |x| < \delta.$$

Thus,

$$\frac{y(x) - \sum_{n=0}^{N} a_n x^n}{x^N} = \sum_{n=N+1}^{\infty} a_n \phi(x; \alpha_n) x^{n-N}.$$

It remains to show that the right hand side tends to zero as $x \to 0$. For $|x| \leq \delta$ we only get contributions from n's such that

$$n < \frac{\log x}{\log 2} \qquad \text{and} \qquad |a_n| < \frac{1}{\sqrt{x}}.$$

Hence,

$$\left| \sum_{n=N+1}^{\infty} a_n \phi(x; \alpha_n) x^{n-N} \right| \leq \frac{\log x}{\log 2} \sqrt{x} \to 0.$$

■

Before demonstrating the properties of asymptotic series, let us show that solutions to differential equations can indeed be represented by asymptotic series:

Example: Recall that we "found" that a solution to the differential equation

$$x^2 y''(x) + (1 + 3x) y'(x) + y(x) = 0.$$

is a (diverging) power series

$$y(x) = \sum_{n=0}^{\infty} (-x)^n n!.$$

This is of course meaningless. We will now *prove* that this series is indeed asymptotic to the solution.

We start by noting that

$$n! = \int_0^\infty e^{-t} t^n \, dt$$

(recall the definition of the Γ function). We then do formal manipulations, which we do not justify,

$$y(x) = \sum_{n=0}^{\infty} (-x)^n \int_0^\infty e^{-t} t^n \, dt = \int_0^\infty e^{-t} \sum_{n=0}^{\infty} (-x)^n t^n \, dt = \int_0^\infty \frac{e^{-t}}{1 + xt} \, dt.$$

This integral exists, and in fact defines an analytic function (it is called a ***Stieltjes integral***). Moreover, we can check directly that this integral solves the differential equation.

We will now show that this solution has the above asymptotic expansion. Integrating by parts we have

$$
\begin{aligned}
y(x) &= \int_0^\infty \frac{e^{-t}}{1+xt}\,dt = -\left.(1+xt)^{-1}\right|_0^\infty - x\int_0^\infty \frac{e^{-t}}{(1+xt)^2}\,dt \\
&= 1 - x\int_0^\infty \frac{e^{-t}}{(1+xt)^2}\,dt.
\end{aligned}
$$

We may proceed integrating by parts to get

$$
y(x) = 1 - x - 2x^2\int_0^\infty \frac{e^{-t}}{(1+xt)^3}\,dt,
$$

and after N steps,

$$
y(x) = \sum_{n=1}^{N} n!(-x)^n + (N+1)!\,(-x)^{N+1}\int_0^\infty \frac{e^{-t}}{(1+xt)^{N+1}}\,dt.
$$

Since the integral is bounded by 1, we get that

$$
y(x) - \sum_{n=1}^{N} n!(-x)^n \le (N+1)!\,(-x)^{N+1} \ll x^N, \qquad x \to 0.
$$

A more interesting question is how many terms to need to take for the approximation to be optimal. *It is not true that the more the better!* We may rewrite the error as follows

$$
\begin{aligned}
\epsilon_n = y(x) - \sum_{n=0}^{N} n!(-x)^n &= \int_0^\infty \frac{e^{-t}}{1+xt}\,dt - \sum_{n=0}^{N}\int_0^\infty e^{-t}(-xt)^n\,dt \\
&= \int_0^\infty e^{-t}\left(\frac{1}{1+xt} - \sum_{n=0}^{N}(-xt)^n\right) \\
&= \int_0^\infty e^{-t}\frac{(-xt)^N}{1+xt}\,dt.
\end{aligned}
$$

What is the optimal N? Note that the coefficients of the series are alternating in sign and their ratio is $(-nx)$. As long as this ratio is less that 1 in absolute value, the error decreases, otherwise it increases. The optimal N is therefore the largest integer less than $1/x$. An evaluation of the error at the optimal N gives

$$\epsilon_N \sim \frac{\pi}{2x} e^{-1/x}.$$

▲▲▲

We are now in measure to prove the properties of asymptotic series:

Proposition 2.2 (Non-uniqueness) Let

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad x \to x_0.$$

Then there exists a function $g(x) \neq f(x)$ asymptotic to the same series.

Proof: Take

$$g(x) = f(x) + e^{-1/(x-x_0)^2}.$$

This follows from the fact that

$$e^{-1/x^2} \ll x^n, \qquad x \to 0$$

for every n. The function e^{-1/x^2} is said to be ***subdominant***. ■

Proposition 2.3 (Uniqueness) If a function $y(x)$ has an asymptotic series expansion at x_0 then the series is unique.

Proof: By definition,

$$y(x) - \sum_{n=0}^{N-1} a_n (x - x_0)^{\alpha n} - a_N (x - x_0)^{\alpha N} \ll (x - x_0)^{\alpha N},$$

hence,

$$a_N = \lim_{x \to x_0} \frac{y(x) - \sum_{n=0}^{N-1} a_n (x - x_0)^{\alpha n}}{(x - x_0)^{\alpha N}},$$

which is a constructive definition of the coefficients. ■

Comment: It follows that if two sides of an equation are have asymptotic series expansions we can equate the coefficients term by term.

Proposition 2.4 (Arithmetic operations) Let

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad \text{and} \qquad g(x) \sim \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

then

$$\alpha f(x) + \beta g(x) \sim \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(x - x_0)^n,$$

and

$$f(x) g(x) \sim \sum_{n=0}^{\infty} c_n (x - x_0)^n,$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Proof: This follows directly from the definitions. ■

✎ *Exercise 2.9* Show that if

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad \text{and} \qquad g(x) \sim \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

then

$$f(x) g(x) \sim \sum_{n=0}^{\infty} c_n (x - x_0)^n,$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Proposition 2.5 (Integration) Let

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

If f is integrable near x_0 then

$$\int_{x_0}^{x} f(t)\, dt \sim \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1}.$$

Proof: Set N. By the asymptotic property of f it follows that for every ϵ there exists a δ such that

$$\left| f(x) - \sum_{n=0}^{N} a_n (x-x_0)^n \right| \le \epsilon (x-x_0)^N, \qquad |x| \le \delta.$$

Thus,

$$\left| \int_{x_0}^{x} f(t)\,dt - \sum_{n=0}^{N} \frac{a_n}{n+1}(x-x_0)^{n+1} \right| \le \frac{\epsilon (x-x_0)^{N+1}}{N+1},$$

which proves the claim. ■

Proposition 2.6 (Differentiation 1) Let

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n.$$

Then it is not necessarily true that

$$f'(x) \sim \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1}.$$

Proof: The problem is tightly related to the presence of subdominant functions. Defining

$$g(x) = f(x) + e^{-1/x^2} \sin(e^{1/x^2}),$$

the two functions have the same asymptotic expansion at zero, but not their derivatives. ■

Proposition 2.7 (Differentiation 2) If $f'(x)$ has an asymptotic expansion and is integrable near x_0 then

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x-x_0)^n.$$

implies that

$$f'(x) \sim \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1}.$$

Proof: Set

$$f'(x) \sim \sum_{n=0}^{\infty} b_n (x - x_0)^n.$$

Using the integration Proposition and the uniqueness of the expansion we get the desired result. ■

We come now to the ultimate goal of this section. Suppose we have a differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0.$$

Suppose that $p(x)$ and $q(x)$ have asymptotic expansions at x_0. Does it imply that $y(x)$ has an asymptotic expansion as well, and that its coefficient can be identified by term-by-term formal manipulations? In general this is true.

First we need to assume that $p'(x)$ also has an asymptotic expansion. Then we usually proceed in two steps. First we *assume* that $y(x)$ has an asymptotic expansion. Then, since

$$y''(x) + [p(x)y(x)]' + [q(x) - p'(x)]y(x) = 0,$$

it follows that

$$y'(x) - y'(x_0) + p(x)y(x) - p(x_0)y(x_0) + \int_{x_0}^{x} [q(t) - p'(t)]y(t)\, dt.$$

Hence $y'(x)$ has an asymptotic expansion and so does $y''(x)$ (by the arithmetic properties). We are then allowed to use the arithmetic properties and the uniqueness of the expansion to identify the coefficients.

It remains however to show that $y(x)$ can indeed to expanded in an asymptotic series. In the next section we will demonstrate the standard approach to do so.

2.6 Irregular singular points at infinity

Irregular singular points at infinity are ubiquitous in equations that arise in physical applications (*e.g.*, Bessel, Airy), and the asymptotic behavior at infinity is of major importance in such applications. In principle, the investigation of irregular singular points at infinity can be dealt with by the change of variables $t = 1/x$, yet, we can use the method of dominant balance to study the asymptotic behavior in the original variables.

Example: Consider the function

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2},$$

reminiscent of the Bessel function

$$I_0(x) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}x)^{2n}}{(n!)^2},$$

i.e., $y(x) = I_0(\sqrt{2x})$. This series is convergent everywhere, yet to evaluate it at, say, $x = 10000$ to ten significant digits requires at least,

$$\frac{10000^n}{(n!)^2} < 10^{-10},$$

and using Stirling's formula,

$$n \log 10000 - 2n \log n + 2n < -10 \log 10,$$

this roughly gives $n > 284$. It would be useful to obtain an approximation that does not require the addition of hundreds of numbers.

Consider the following alternative. First, note that

$$y'(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!(n-1)!},$$

hence

$$(xy'(x))' = \sum_{n=1}^{\infty} \frac{x^{n-1}}{[(n-1)!]^2} = y(x),$$

i.e., $y(x)$ is a solution of the differential equation

$$x\, y''(x) + y'(x) = y(x).$$

We are looking for a solution as $x \to \infty$, in the form $y(x) = \exp S(x)$, yielding

$$xS''(x) + x[S'(x)]^2 + S'(x) = 1.$$

As before, we assume that $S''(x) \ll [S'(x)]^2$. We remain with

$$x[S'(x)]^2 + S'(x) \sim 1.$$

This is a quadratic equation, whose solution is

$$S'(x) \sim \frac{-1 \pm \sqrt{1+4x}}{2x} \sim \pm x^{-1/2}, \qquad x \to \infty.$$

Thus,

$$S(x) \sim \pm 2x^{1/2},$$

or

$$S(x) \pm 2x^{1/2} + C(x),$$

where $C'(x) \ll x^{-1/2}$.

Since all the coefficients in the power series are positive, $y(x)$ is an increasing function of x, and the leading behavior must be dominated by the positive sign. We then go to the next equation,

$$x[2x^{1/2} + C(x)]'' + x([2x^{1/2} + C(x)])'^2 + [2x^{1/2} + C(x)]' = 1.$$

Expanding we get

$$x\left[-\frac{1}{2}x^{-3/2} + C''(x)\right] + x\left[x^{-1/2} + C'(x)\right]^2 + \left[x^{-1/2} + C'(x)\right] = 1,$$

and

$$\frac{1}{2}x^{-1/2} + xC''(x) + 2x^{1/2}C'(x) + x[C'(x)]^2 + C'(x) = 0,$$

Recall that $C'(x) \ll x^{-1/2}$ hence $C''(x) \ll x^{-3/2}$, and so we remain with

$$2x^{1/2}C'(x) \sim -\frac{1}{2}x^{-1/2},$$

or

$$C(x) \sim -\frac{1}{4}\log x.$$

The next correction is asymptotic to a constant.

The leading solution is then

$$y(x) \sim c\, x^{-1/4} e^{2x^{1/2}}.$$

We cannot (at this point) evaluate the constant (the equation is homogeneous!), which turns out to be $\frac{1}{2}\pi^{-1/2}$. In Figure 2.4 we show the ratio of this asymptotic

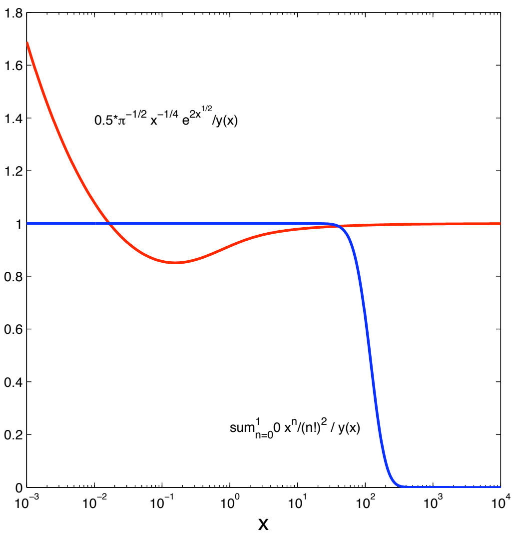

Figure 2.4:

solution and $y(x)$ versus x. Interestingly, the approximation is (relatively) excellent for $x > 100$, whereas the power series truncated at $n = 10$ is very accurate up to that point. Together, the two approximations yield a "uniformly accurate" approximation.

▲▲▲

✎ *Exercise 2.10* Show that the asymptotic behavior at infinity of the solutions to the modified Bessel equation,

$$x^2 y''(x) + xy'(x) - (x^2 + \nu^2)y(x) = 0$$

is

$$y_1(x) \sim c_1 x^{-1/2} e^x$$
$$y_2(x) \sim c_2 x^{-1/2} e^{-x}.$$

Example: The modified Bessel equation

$$x^2 y''(x) + xy'(x) - (x^2 + \nu^2)y(x) = 0$$

has an irregular singular point at $x = \infty$. There are two independent solutions, one which decays at infinity and one which diverges. We will study the behavior of the converging one.

By using the method of dominant balance (see above exercise) we find that

$$y(x) \sim cx^{-1/2}e^{-x}.$$

We "peel off" the leading behavior by setting

$$y(x) = x^{-1/2}e^{-x}w(x).$$

Then,

$$y'(x) = x^{-1/2}e^{-x}\left(-\frac{1}{2}x^{-1}w(x) - w(x) + w'(x)\right)$$

$$y''(x) = x^{-1/2}e^{-x}\left(-\frac{1}{2}x^{-1} - 1\right)\left(-\frac{1}{2}x^{-1}w(x) - w(x) + w'(x)\right) + x^{-1/2}e^{-x}\left(\frac{1}{2}x^{-2}w(x) - \frac{1}{2}x^{-1}w'(x) - w'(x) + w''(x)\right).$$

Substituting we get

$$x^2w''(x) - 2x^2w'(x) + \left(\tfrac{1}{4} - \nu^2\right)w(x) = 0.$$

At this point we construct an asymptotic series for $w(x)$.

$$w(x) \sim \sum_{n=0}^{\infty} a_n x^{-n},$$

and proceed formally. Substituting we get

$$\sum_{n=0}^{\infty} n(n+1)a_n x^{-n} - 2\sum_{n=0}^{\infty} na_n x^{-n+1} + \left(\tfrac{1}{4} - \nu^2\right)a_n x^{-n} = 0.$$

Equating the power of x^{-n} we get

$$n(n+1)a_n - 2(n+1)a_{n+1} + \left(\tfrac{1}{4} - \nu^2\right)a_n = 0,$$

or

$$a_{n+1} = \frac{(n+\frac{1}{2})^2 - \nu^2}{2(n+1)}a_n.$$

Recall that we proved in the previous section that if $w(x)$ assumes a power series expansion, then it is given by the above procedure. We will now prove that this is indeed the case. Setting $\lambda = \frac{1}{4} - \nu^2$ we have

$$w''(x) - 2w'(x) + \frac{\lambda}{x^2}w(x) = 0,$$

and we want to show that there exists a solution that can be expanded about infinity. We first write this equation as an integral equation. First,

$$(e^{-2x}w'(x))' + \frac{\lambda e^{-2x}}{x^2}w(x) = 0,$$

from which we deduce that

$$w'(x) = \lambda \int_x^\infty \frac{e^{-2(s-x)}}{s^2} w(s)\, ds.$$

Note that we chose the integration constant such that $w'(x) \to 0$ at infinity. One more integration yields

$$w(x) = 1 + \lambda \int_x^\infty \int_t^\infty \frac{e^{-2(s-t)}}{s^2} w(s)\, dsdt.$$

Exchanging the order of integration we end up with

$$\begin{aligned} w(x) &= 1 + \lambda \int_x^\infty \int_x^s \frac{e^{-2(s-t)}}{s^2} w(s)\, dtds \\ &= 1 + \frac{\lambda}{2} \int_x^\infty \frac{e^{-2(s-x)} - 1}{s^2} w(s)\, ds \end{aligned}$$

We now claim that the solution to this integral equation is bounded for sufficiently large x. That is, there exist $a, B > 0$ such that $|w(x)| \leq B$ for $x \geq a$. To show that we proceed formally and iterate this integral,

$$w(x) = 1 + \frac{\lambda}{2} \int_x^\infty \frac{K(x,s)}{s^2}\, ds + \left(\frac{\lambda}{2}\right)^2 \int_x^\infty \int_{s_1}^\infty \frac{K(x,s_1)}{s_1^2} \frac{K(s_1,s_s)}{s_s^2}\, ds + \dots,$$

where $K(x,s) = e^{-2(s-x)} - 1$. Since $|K(x,s)| \leq 1$ for $x \geq s$, it follows that the k-th term of this series is bounded by

$$|I_k| \leq \left(\frac{\lambda}{2}\right)^n \int_x^\infty \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty \frac{1}{s_1^2} \cdots \frac{1}{s_n^2}\, ds_n \dots ds_1 \leq \left(\frac{\lambda}{2}\right)^n \frac{x^{-n}}{n!},$$

i.e., the series converges absolutely and is bounded by $e^{-\lambda/2x}$. Since we constructed an absolutely converging series that satisfies an iterative relation satisfied by $w(x)$, it is indeed the solution.

Having proved the boundedness of $w(x)$, it remains to show that it has an asymptotic expansion. We start with $w(x) = 1 + w_1(x)$, and

$$|w_1(x)| = \left|\frac{\lambda}{2}\int_x^\infty \frac{K(x,s)}{s^2} w(s)\,ds\right| \le \frac{\lambda}{2} B \int_s^\infty \frac{ds}{s^2} = \frac{\lambda}{2x} B,$$

i.e., $w(x) \to 1$. Next,

$$\begin{aligned} w_1(x) &= \frac{\lambda}{2}\int_x^\infty \frac{e^{-2(s-x)}-1}{s^2}\,ds + \frac{\lambda}{2}\int_x^\infty \frac{e^{-2(s-x)}-1}{s^2} w_1(s)\,ds \\ &= -\frac{\lambda}{2x} + \frac{\lambda}{2}\int_x^\infty \frac{e^{-2(s-x)}-1}{s^2} w_1(s)\,ds. \end{aligned}$$

Using the bound, $|w_1(x)| \le \frac{\lambda B}{2x}$, we get that

$$\left| w_1(x) + \frac{\lambda}{2x}\right| \le \frac{\lambda^2 B}{4}\int_x^\infty \frac{ds}{s^3} = \frac{\lambda^2 B}{12 x^2},$$

and so on.

▲▲▲

Chapter 3

Local analysis II: Nonlinear differential equations

3.1 Spontaneous singularities

For linear equations the situation is that in regions where the coefficients are regular so is the solution. Singularities of the solution are always associated with a singularity of the equation. In particular, points of singularity can be predicted without solving the equation. This is non the case for nonlinear equations, where singularities can occur "spontaneously" even if the coefficients are regular.

Example: Consider the following linear IVP

$$y'(x) + \frac{y(x)}{x-1} = 0 \qquad y(0) = 1.$$

The point $x = 1$ is a singular regular point and we are not surprised to find that the solution

$$y(x) = \frac{1}{1-x}$$

has a pole at this point. If we replace the initial data into $y(0) = 2$ this does not change.

Consider in contrast the nonlinear IVP

$$y'(x) = y^2(x).$$

The coefficients are analytic everywhere and yet the solution $y(x) = 1/(1-x)$ has a pole at $x = 1$. If the initial data are changed into $y(0) = 2$, the location of the singularity changes,

$$y(x) = \frac{2}{1-2x}.$$

▲▲▲

This example shows that such singularities cannot be predicted by our current tools. Yet, it is easy to show that if the equation is analytic in the vicinity of the initial data, then the solution is analytic in some neighborhood of that point. In principle, one can therefore look for solutions in the form of Taylor series. As long as the series converges there is no singularity.

Example: Consider the following IVP

$$y'(x) = \frac{y^2}{1-xy}, \qquad y(0) = 1.$$

Since the coefficients are analytic at the point $(0, 1)$ we expect the Taylor series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad a_0 = 1,$$

to have a non-zero radius of convergence. On the other hand, we do not expect to be able to find the coefficients (we rarely expect a nonlinear equation to be solvable).

It turns out that this particular problem is solvable. Substituting the Taylor series we get

$$\left(1 - \sum_{n=0}^{\infty} a_n x^{n+1}\right) \sum_{m=0}^{\infty} m a_m x^{m-1} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_n a_m x^{n+m},$$

hence

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (1+m) a_n a_m x^{n+m} = \sum_{m=0}^{\infty} m a_m x^{m-1},$$

which we rewrite as

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} (1+k) a_k a_{n-k} x^n = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

hence

$$a_{n+1} = \frac{\sum_{k=0}^{n}(1+k)a_k a_{n-k}}{n+1}.$$

We claim that the solution is

$$a_n = \frac{(n+1)^{n-1}}{n!}.$$

Indeed, by induction

$$\begin{aligned}
\frac{\sum_{k=0}^{n}(1+k)a_k a_{n-k}}{n+1} &= \frac{\sum_{k=0}^{n}(1+k)\frac{(k+1)^{k-1}}{k!}\frac{(n-k+1)^{n-k-1}}{(n-k)!}}{n+1} \\
&= \frac{1}{n+1}\sum_{k=0}^{n}\frac{(k+1)^k(n-k+1)^{n-k-1}}{k!(n-k)!} \\
&= \frac{1}{(n+1)!}\sum_{k=0}^{n}\binom{n}{k}(k+1)^k(n-k+1)^{n-k-1} \\
&= \frac{(n+2)^n}{(n+1)!},
\end{aligned}$$

where in the last step we used a binomial identity,

The radius of convergence on this Taylor series is obtained by the standard test,

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{(n+1)^{n-1}(n+1)!}{n!(n+2)^n} = \left(\frac{n+1}{n+2}\right)^n = \left(1+\frac{1}{n+1}\right)^{-n} \to \frac{1}{e}.$$

Thus, the solution has a (spontaneous) singularity at a distance of $1/e$ from the origin. It must be on the positive real axis since all the coefficients in the series are positive. This singularity cannot be predicted by an inspection of the equation. ▲▲▲

Chapter 4

Evaluation of integrals

4.1 Motivation

Example: Consider the IVP

$$y''(x) = y(x) \qquad y(0) = 1,\ y'(0) = 0.$$

If we make a local analysis at the irregular singular point $x = \infty$ we find that the solution has two possible behaviors,

$$y(x) \sim a\,e^{x} \qquad \text{and} \qquad y(x) \sim b\,e^{-x}.$$

In the general case, we cannot determine the coefficients when the initial data are given away from the singular point. This case is of course fully solvable so we find $y(x) = \cosh x$.

In contrast, suppose we want the behavior at infinity of the solution to the modified Bessel equation

$$x^2 y'(x) + xy'(x) - (x^2 + 1)y(x) = 0, \qquad y(0) = 1,\ y(\infty) = 0.$$

We know that this equation has solutions

$$y_1(x) \sim c_1 x^{-1/2} e^{x} \qquad \text{and} \qquad y_1(x) \sim c_1 x^{-1/2} e^{-x}.$$

Only the decaying solution remains, but we cannot determine its prefactor by the initial condition at zero. ▲▲▲

In many case, this problem can be remedied by representing the solution to the differential equation as an integral in which x appears as a parameter. The advantage is that the initial data are usually built into the integral, thus it remains to learn how to evaluate such integrals.

Example: Consider the IVP

$$y'(x) = xy(x) + 1, \qquad y(0) = 0.$$

Suppose we are interested in the properties of the solution at the point $x = \infty$. By the method of dominant balance, we expect $1 \ll xy$, hence

$$y'(x) \sim xy(x),$$

which we integrate,

$$y(x) \sim c\, e^{x^2/2}.$$

The problem is that we cannot determine the coefficient c.

Alternatively, we can solve the equation directly,

$$\left(e^{-x^2/2} y(x)\right)' = e^{-x^2/2},$$

hence,

$$y(x) = e^{x^2/2} \int_0^x e^{-t^2/2}\, dt,$$

and we know that the prefactor converges as $x \to \infty$ to $\sqrt{\pi/2}$. ▲▲▲

Example: Consider now a third-order equation

$$x\, y'''(x) + 2y(x) = 0, \qquad y(0) = 1,\ y(\infty) = 0.$$

The point $x = \infty$ is an irregular singular point. Setting $y(x) = e^{S(x)}$ we get

$$x\left([S'(x)]^3 + 3S''(x)S'(x) + S'''(x)\right) + 2 = 0.$$

As usual, we assume that $S'''(x) \ll [S'(x)]^3$ as well as $S''(x) \ll [S'(x)]^2$, then

$$[S'(x)]^3 \sim -\frac{2}{x}.$$

and $S'(x) \sim -\omega_i (x/2)^{-1/3}$, which leads to

$$S(x) \sim -3\omega_i (x/2)^{2/3},$$

where ω_i are the three roots of unity. Since the solution has to decay at infinity we remain with only $\omega = 1$, after which we substitute

$$S(x) = -3(x/2)^{2/3} + C(x),$$

where $C'(x) \ll (x/2)^{-1/3}$. We get

$$\left(-(x/2)^{-1/3} + C'(x)\right)^3 + 3\left(\frac{1}{6}(x/2)^{-4/3} + C''(x)\right)\left(-(x/2)^{-1/3} + C'(x)\right) + \left(-\frac{1}{9}(x/2)^{-7/3} + C'''(x)\right) = -\frac{2}{x}.$$

Retaining only the highest order terms this simplifies into

$$3(-(x/2)^{-1/3})^2 C'(x) \sim \frac{1}{2}(x/2)^{-5/3}, \qquad \text{or} \qquad C'(x) \sim \frac{1}{3x}.$$

Thus,

$$y(x) \sim c\, x^{1/3} e^{-3(x/2)^{2/3}}.$$

The question again is how to find the coefficient.

We note however that the solution has an integral representation,

$$y(x) = \int_0^\infty \exp\left(-t - \frac{x}{\sqrt{t}}\right) dt.$$

Indeed,

$$xy'''(x) = -\int_0^\infty \frac{x}{t^{3/2}} \exp\left(-t - \frac{x}{\sqrt{t}}\right) dt,$$

while upon integration by parts we see that the equation is satisfied. The boundary condition at $x = 0$ is also satisfied, i.e., this is an integral representation of the solution.

▲▲▲

4.2 Some examples

Example: Consider the evaluation of the integral

$$I(x) = \int_0^2 \cos[(xt^2 + x^2 t)^{1/3}]\, dt.$$

It is hard to evaluate, but clearly as $x \to 0$ we have $I(x) \sim 2$. ▲▲▲

More generally, we will be concerned with the evaluation of integrals of the form

$$I(x) = \int_a^b f(x,t)\,dt$$

as $x \to x_0$.

Proposition 4.1 If

$$f(x,t) \sim f_0(t) \qquad x \to x_0,$$

uniformly for all $t \in [a,b]$, which means that

$$\lim_{x\to x_0} \frac{f(x,t)}{f_0(t)} = 1 \qquad \textit{uniformly in } t,$$

then

$$I(x) \sim \int_a^b f_0(t)\,dt,$$

provided that the right hand side is finite and non-zero.

Proof: It is given that for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x,t) - f_0(t)| \le \epsilon |f_0(t)| \qquad |x - x_0| \le \delta.$$

Then,

$$\left| I(x) - \int_a^b f_0(t)\,dt \right| \le \epsilon \int_a^b |f_0(t)|\,dt.$$

If the integral of f_0 is finite and does not vanish, there then exists a constant C such that

$$\left| I(x) - \int_a^b f_0(t)\,dt \right| \le C\epsilon \left| \int_a^b f_0(t)\,dt \right|,$$

where

$$C = \frac{\int_a^b |f_0(t)|\,dt}{\left| \int_a^b f_0(t)\,dt \right|}.$$

(Indeed, we need the numerator to be finite and the denominator to be non-zero.)
■

More generally, if

$$f(t,x) \sim \sum_{n=0}^{\infty} a_n(t)(x-x_0)^{\alpha n},$$

uniformly on $t \in [a,b]$, then

$$\int_a^b f(x,t)\,dt \sim \sum_{n=0}^{\infty}(x-x_0)^{\alpha n}\int_a^b a_n(t)\,dt,$$

provided that all the terms on the right hand side are finite and do not vanish.

✎ *Exercise 4.1* Prove that if

$$f(t,x) \sim \sum_{n=0}^{\infty} a_n(t)(x-x_0)^{\alpha n},$$

uniformly on $t \in [a,b]$, then

$$\int_a^b f(x,t)\,dt \sim \sum_{n=0}^{\infty}(x-x_0)^{\alpha n}\int_a^b a_n(t)\,dt,$$

provided that all the terms on the right hand side are finite and do not vanish.

Example: Consider the integral

$$I(x) = \int_0^1 \frac{\sin xt}{t}\,dt$$

as $x \to 0$. Since

$$\sin xt \sim \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(tx)^{2n+1}}{(2n+1)!}$$

uniformly on $t \in [0,1]$, then

$$I(x) \sim \sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!}\int_0^1 t^{2n}\,dt.$$

▲▲▲

Example: Let

$$I(x) = \int_x^\infty e^{-t^4}\, dt.$$

The expansion

$$e^{-t^4} \sim \sum_{n=0}^\infty \frac{(-1)^n t^{4n}}{n!},$$

yield a divergent series. The solution is to write

$$I(x) = \int_0^\infty e^{-t^4}\, dt - \int_0^x e^{-t^4}\, dt.$$

The first integral is

$$\int_0^\infty e^{-t^4}\, dt = \frac{1}{4}\int_0^\infty e^{-s}\, s^{-3/4}\, ds = \frac{1}{4}\Gamma\left(\frac{1}{4}\right) = \Gamma\left(\frac{5}{4}\right).$$

For the rest we can use term by term integration,

$$I(x) = \Gamma\left(\frac{5}{4}\right) - \sum_{n=0}^\infty \frac{(-1)^n x^{4n+1}}{4\,(n+1)!}$$

▲▲▲

✎ *Exercise 4.2* Find the leading behavior as $x \to 0^+$ of the following integrals:

① $\int_x^1 \cos xt.$
② $\int_0^1 e^{-x/t}\, dt.$

✎ *Exercise 4.3* Find the full asymptotic behavior as $x \to 0^+$ of the integral

$$\int_0^1 \frac{e^{-t}}{1 + x^2 t^3}\, dt.$$

4.3 Integration by parts

Integration by parts is a standard technique for finding asymptotic expansions of integrals.

Example: Suppose that $f(x)$ is infinitely differentiable near $x = 0$, then its asymptotic series at $x = 0$ is Taylor's series. This can be shown as follows:

$$f(x) = f(0) + \int_0^x f'(t)\,dt = f(0) + \int_0^x (x-t)^0 f'(t)\,dt.$$

Integrating by parts,

$$f(x) = f(x) + xf'(x) + \int_0^x (t-x) f''(t)\,dt.$$

Repeating it once more,

$$f(x) = f(x) + xf'(x) + \frac{1}{2}xf''(x) + \frac{1}{2}\int_0^x (t-x)^2 f'''(t)\,dt,$$

and so on. If the remainders exists for sufficiently small x, then we develop an asymptotic series. ▲▲▲

Example: Consider again the integral

$$I(x) = \int_x^\infty e^{-t^4}\,dt$$

as $x \to \infty$. We saw already that

$$I(x) = \Gamma\left(\frac{5}{4}\right) - \sum_{n=0}^\infty \frac{(-1)^n x^{4n+1}}{4\,(n+1)!}.$$

This is an exact solution by tedious to evaluate for large x.

An asymptotic series for large x needs to be in powers of $1/x$. We use for that integration by parts,

$$\begin{aligned}
I(x) &= \int_x^\infty \frac{-4t^3}{-4t^3} e^{-t^4}\,dt = \frac{1}{-4t^3} e^{-t^4}\Big|_x^\infty - \int_x^\infty \frac{3}{4t^4} e^{-t^4}\,dt \\
&= \frac{1}{4x^3} e^{-x^4} + \int_x^\infty \frac{-3t^3}{4t^7} e^{-t^4}\,dt,
\end{aligned}$$

and so on. We can systematize the procedure as follows: define

$$I_n(x) = \int_x^\infty \frac{1}{t^{4n}} e^{-t^4}\,dt.$$

We need $I_0(x)$ and we note that

$$I_n(x) = \int_x^\infty \frac{-4t^3}{-4t^{4n+3}} e^{-t^4}\, dt = \frac{e^{-x^4}}{4x^{4n+3}} - \int_x^\infty \frac{(4n+3)}{4t^{4n+4}} e^{-t^4}\, dt = \frac{e^{-x^4}}{4x^{4n+3}} - \left(n + \frac{3}{4}\right) I_{n+1}(x).$$

Thus,

$$I(x) = \frac{e^{-x^4}}{4x^3}\left(1 - \frac{3}{4x^4} + \frac{3\cdot 7}{4^2 x^8} - \frac{3\cdot 7\cdot 11}{4^3 x^{12}} + \dots\right),$$

which is the asymptotic series,

$$I(x) \sim \frac{e^{-x^4}}{4x^3} \sum_{n=0}^\infty (-1)^n \frac{3\cdot 7\cdot \dots \cdot (4n-1)}{(4x^4)^n}.$$

▲▲▲

Example: Consider now the following example,

$$I(x) = \int_0^x t^{-1/2} e^{-t}\, dt$$

as $x \to \infty$. Naive integration by parts is problematic because

$$-\, t^{-1/2} e^{-t}\Big|_0^x - \frac{1}{2}\int_0^x t^{-3/2} e^{-t}\, dt$$

is a difference of two infinite terms. The way around is to express the integral as a difference,

$$I(x) = \int_0^\infty t^{-1/2} e^{-t}\, dt - \int_x^\infty t^{-1/2} e^{-t}\, dt.$$

The first integral is $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Now we can integrate by parts. ▲▲▲

✎ *Exercise 4.4* Find the leading behavior of

$$\int_x^\infty e^{-at^b}\, dt$$

as $x \to \infty$, with $a, b > 0$.

Laplace integrals A Laplace integral has the form

$$I(x) = \int_a^b f(t)e^{x\phi(t)}\,dt.$$

To obtain the behavior as $x \to \infty$,

$$I(x) = \int_a^b f(t)\frac{x\phi'(t)}{x\phi'(t)}e^{x\phi(t)}\,dt = \left.\frac{f(t)}{x\phi'(t)}e^{x\phi(t)}\right|_a^b - \frac{1}{x}\int_a^b \frac{d}{dt}\left(\frac{f(t)}{\phi'(t)}\right)e^{x\phi(t)}\,dt.$$

It is assumed that this new integral exists. If the second terms is much smaller than the first, then

$$I(x) \sim \frac{f(b)}{x\phi'(b)}e^{x\phi(b)} - \frac{f(a)}{x\phi'(a)}e^{x\phi(a)}.$$

We will see more on Laplace integrals shortly.

Failure of integration by parts Consider the integral

$$I(x) = \int_0^\infty e^{-xt^2}\,dt.$$

We know the exact value $I = \frac{1}{2}\sqrt{\pi/x}$. Since it is not an integral power of $1/x$ we expect an asymptotic expansion to fail. It is a Laplace integral with $\phi(t) = -t^2$ and $f(t) = 1$. It is easy to see that the above procedure will fail. Already in the first step

$$I(x) = \int_0^\infty \frac{-2xt}{-2xt}e^{-xt^2}\,dt = \left.\frac{1}{-2xt}e^{-xt^2}\right|_0^\infty - \int_0^\infty \frac{1}{2xt^2}e^{-xt^2}\,dt.$$

Both terms diverge.

4.4 Laplace's method

Consider the Laplace integral

$$I(x) = \int_a^b f(t)e^{x\phi(t)}\,dt,$$

in the limit where $x \to \infty$. It is assumed that both f and ϕ are continuous. Suppose that $\phi(t)$ attains a unique maximum at a point $c \in [a, b]$. Then, for large x we expect the integral to be dominated by contributions in the vicinity of the point c. Assume momentarily that $c \in (a,b)$ and $f(c) \neq 0$. We claim that for every $\epsilon > 0$ such that

$$(c-\epsilon, c+\epsilon) \subseteq (a,b),$$

we have

$$I(x) \sim I(x;\epsilon),$$

where

$$I(x;\epsilon) = \int_{c-\epsilon}^{c+\epsilon} f(t) e^{x\phi(t)}\, dt,$$

and the asymptotic equivalence is to all orders of $1/x$. I.e., the two functions have the same asymptotic expansion *independently of* ϵ! Why is this true? Because

$$\frac{|I(x) - I(x;\epsilon)|}{|I(x;\epsilon)|} \leq \frac{\int_a^{c-\epsilon} |f(t)| e^{x\phi(t)}\, dt + \int_{c+\epsilon}^{b} |f(t)| e^{x\phi(t)}\, dt}{\left|\int_{c-\epsilon}^{c+\epsilon} f(t) e^{x\phi(t)}\, dt\right|}.$$

Since c is a unique maximum, we can find a $\delta < \epsilon$ and numbers M and $\eta > 0$ such that

$$\begin{aligned} \phi(t) &> M \qquad \text{for } |t-c) < \delta \\ \phi(t) &< M - \eta \qquad \text{for } |t-c| > \epsilon. \end{aligned}$$

If furthermore,

$$\beta = \max_{t\in[a,b]} |f(t)| \qquad \text{and} \qquad \gamma = \min_{|t-c|<\epsilon} |f(t)| \neq 0,$$

then

$$\frac{|I(x) - I(x;\epsilon)|}{|I(x;\epsilon)|} \leq \frac{(b-a)\beta e^{x(M-\eta)}}{\delta\gamma e^{xM}} = \frac{(b-a)\beta}{\delta\gamma} e^{-\eta x}.$$

This indeed tends to zero faster than any power of x, hence the difference between $I(x)$ and $I(x;\epsilon)$ is subdominant.

The question is what did we gain by restricting the range of integration. The answer is that this may allow us to perform a convergent Taylor expansion.

Example: Consider the case

$$I(x) = \int_0^{10} \frac{e^{-xt}}{1+t} dt.$$

We claim that for every $\epsilon > 0$,

$$I(x) \sim I(x;\epsilon) = \int_0^\epsilon \frac{e^{-xt}}{1+t}dt.$$

The reason is that

$$\frac{I(x) - I(x;\epsilon)}{I(x)} = \frac{\int_\epsilon^{10} \frac{e^{-xt}}{1+t}dt}{\int_0^{10} \frac{e^{-xt}}{1+t}dt}.$$

This is a ratio of positive terms, which we can bound by

$$\frac{\int_\epsilon^{10} \frac{e^{-xt}}{1+t}dt}{\int_0^{\epsilon/2} \frac{e^{-xt}}{1+t}dt} \le \frac{\int_\epsilon^{10} \frac{e^{-x\epsilon}}{1+0}dt}{\int_0^{\epsilon/2} \frac{e^{-x\epsilon/2}}{1+10}dt} \le \frac{10e^{-\epsilon x}}{\frac{\epsilon}{22}e^{-\epsilon x/2}},$$

which is subdominant.

What did we gain. By taking $\epsilon < 1$ we can write

$$I(x;\epsilon) = \int_0^\epsilon \sum_{n=0}^\infty (-t)^n e^{-xt}dt.$$

We cannot exchange summation and integration as functions, but we are allowed to do it in the sense of asympotic series:

$$I(x;\epsilon) \sim \sum_{n=0}^\infty \int_0^\epsilon (-t)^n e^{-xt}dt.$$

It remains to evaluate these integrals. This is hard, but we can replace these integrals by integrals that are asymptotically equivalent. As bizarre as this may seem, we claim that

$$\int_0^\epsilon (-t)^n e^{-xt}dt \sim \int_0^\infty (-t)^n e^{-xt}dt$$

to all orders. The latter integral is easily evaluated yielding

$$I(x) \sim \sum_{n=0}^\infty (-1)^n \frac{n!}{x^{n+1}}.$$

▲▲▲

Lemma 4.1 (Watson) Let

$$I(x) = \int_0^b f(t)e^{-xt}\,dt.$$

If f is continuous on $[0, b]$ and

$$f(t) \sim \sum_{n=0}^{\infty} a_n t^{\alpha+\beta n}, \qquad t \to 0^+,$$

with $\alpha > -1$ and $\beta > 0$, then

$$I(x) \sim \sum_{n=0}^{\infty} \frac{a_n\,\Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}}.$$

✎ *Exercise 4.5* Prove Watson's lemma.