63 Recent Developments in the Theory of the Shapley Value

1 Introduction

The Shapley value is an a priori measure of a game's utility to its players; it measures what each player can expect to obtain, "on the average," by playing the game. Other concepts of cooperative game theory, such as the Core, Bargaining Set [6], and N–M Solution [26] predict outcomes (or sets of outcomes) that are in themselves stable, that cannot be successfully challenged or upset in some appropriate sense. Almost invariably, they fail to define a unique result; and in a significant proportion of the cases, they do not define any result at all.[1] The Shapley value, although it is not in any formal sense defined as an average of such "stable" outcomes, nevertheless can be considered a mean, which takes into account the various power relationships and possible outcomes.

It follows from this that the Shapley value may also be thought of as a reasonable compromise, the outcome of an arbitration procedure. A player should be willing to settle for a compromise that yields with certainty what he otherwise would only have expected in the mean. For example, the symmetric N–M solution of the 3-person majority game predicts one of the three payoff vectors (1/2, 1/2, 0), (1/2, 0, 1/2), and (0, 1/2, 1/2), corresponding to the three possible 2-person majorities. Before the beginning of bargaining, each player may figure that his chances of getting into a ruling coalition are 2/3, and conditional on this, his payoff is 1/2; the "expected outcome" would then be (1/3, 1/3, 1/3), and this is also the Shapley value. It would, therefore, also be a reasonable compromise; but it is not in itself stable, since it can be easily improved upon by any two-person coalition.

Mathematically, the Shapley value is perhaps the most tractable of all the concepts of cooperative game theory. This has led to the growth of a considerable theory, which in turn has enabled a wide range of applications to Economics and Political Science. Here we survey some of the more recent of these developments.

2 General Definition in the Transferable Utility Case

We begin by recalling that a *coalitional game*, or simply *game* for short, is a real-valued function v on the σ-field $\mathscr{C}$ of a measurable space $(I, \mathscr{C})$,

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1. The Bargaining Set is the only one of these three covered by a general existence theorem.

with $v(\emptyset) = 0$. Here I is the *player space*, the members of $\mathscr{C}$ are *coalitions*, and $v(S)$ is the *worth* of a coalition S. A game is called *monotonic* if $S \supset T$ implies $v(S) \geqslant v(T)$.

Fix $(I, \mathscr{C})$. An *outcome* (or *payoff vector*) is a finitely additive game.[2] For each game v and automorphism (one-one bimeasurable function) Θ of $(I, \mathscr{C})$, define the game $\Theta_* v$ by $(\Theta_* v)(S) = v(\Theta S)$ for all S.

Let us be given a linear space Q of games, which is *symmetric* in the sense that $\Theta_* Q = Q$ for all Θ. An operator φ from Q to outcomes is called *symmetric* if $\varphi(\Theta_* v) = \Theta_*(\varphi v)$ for all v in Q and all automorphisms Θ; *monotonic* if φv is monotonic whenever v is; and *efficient* if $(\varphi v)(I) = v(I)$ for all v in Q. A *value* on Q is an operator from Q to outcomes that is linear, monotonic, symmetric, and efficient.

3 Finite Games

A game v is called *finite* if there is a finite subset N of I (a *support* of v) such that $v(S) = v(S \cap N)$ for all S. The finite games form a linear space on which there is a unique value; it is given by

$$(\psi v)(\{i\}) = E(v(\boldsymbol{S}_i \cup \{i\}) - v(\boldsymbol{S}_i)), \tag{3.1}$$

where $\boldsymbol{S}_i$ is the set of players (members of N) preceding i in a random order on N, and E is the expectation operator when each order on N has probability $1/|N|!$ [36]. It is easy to check that (3.1) does indeed define a value; as for uniqueness, perhaps the simplest proof is that of Dubey [7], who uses an induction on $|N|$ to show that every finite game is a linear combination of unanimity games (games for which $v(S) = 1$ or 0 according as $S \supset N$ or $S \not\supset N$).

4 Nonatomic Games, Partition Values, and the Diagonal Property

Diametrically opposed to the finite games are the *nonatomic games*, which model situations in which no individual player has any significance [2]. Examples are games of the form $f \circ \mu$, where μ is a nonatomic vector measure, and f is a real-valued function on the range of μ vanishing at 0. One approach to defining a value for a nonatomic game v is via approximations by finite games. Specifically, if Π is a measurable partition of I—i.e. a finite subfield of $\mathscr{C}$—we may define a finite game v_Π, whose support consists of the atoms of Π, by $v_\Pi = v|\Pi$; then v_Π is a kind of

2. Intuitively, the sharing of proceeds in an additive game involves no difficulties, so that by associating an additive game to a non-additive game, we have essentially specified an outcome.

finite approximant to v. Given a coalition S in $\mathscr{C}$, an increasing sequence $\{\Pi_1, \Pi_2, \ldots\}$ of such partitions is called *S-admissible* if $S \in \Pi_1$ and $\bigcup_i \Pi_i$ generates $\mathscr{C}$. A value φ on a space Q is called a *partition value* [30] if for each game v in Q and each coalition S, there is an S-admissible sequence $\{\Pi_1, \Pi_2, \ldots\}$ such that

$$\lim_{n\to\infty} (\psi v_{\Pi_n})(S) \to (\varphi v)(S), \tag{4.1}$$

where ψ is the value for finite games. If for a specific game v and outcome φv, (4.1) holds for all S and all S-admissible sequences, then we write $v \in$ ASYMP and call φv the *asymptotic value* [16] of v. Whereas the partition value is defined in terms of the imbedding space Q, the definition of asymptotic value is independent of any imbedding space; its existence depends on the game v only.

A partition value of a non-atomic game is a limit of values of large finite approximants. The asymptotic value is the strongest possible partition value; if it exists, then no matter how the player space is cut up,[3] in the limit the result is the same.

Are there values that are not partition values? This leads us to the *diagonal property* of values. Let ν be a nonatomic nonnegative measure on $\mathscr{C}$ with $\nu(I) = 1$ ($\nu \in \mathrm{NA}^1$ for short); Π a partition of I into many—say n—"small" sets; and $\boldsymbol{Q}_h$ the union of the first h atoms of Π in a random order on the atoms. For a fixed h, we will have $\nu(\boldsymbol{Q}_h) \approx h/n$ with high probability; moreover, for fixed ε, if Π is sufficiently far out in some S-admissible sequence, then the probability is $> 1 - \varepsilon$ that $|\nu(\boldsymbol{Q}_h) - (h/n)| < \varepsilon$ *simultaneously* for all h. Thus if $\mu \in (\mathrm{NA}^1)^m$ (i.e. μ is an m-tuple of NA^1 measures), almost all the coalitions occurring in Formula (3.1) as applied to v_Π will have μ-measures very near the "diagonal" $D^m = \{(t, \ldots, t) \colon t \in [0,1]\}$ of the m-cube. In particular, let φ be a partition value; then

if φ is defined for two games v_1 and v_2 that agree on all coalitions S with $\mu(S)$ in some ε-neighborhood of D^m, then $\varphi v_1 = \varphi v_2$. (4.2)

Any value φ satisfying (4.2) for all vectors μ of NA^1 measures is called a *diagonal value*.

All the values treated in [2] were diagonal, and for a long time it was not known whether *all* values are diagonal. Finally, Neyman and Tauman [29] and Tauman [40][4] found examples of nondiagonal values. In particular, not all values are partition values.

3. E.g. into n intervals of "length" $1/n$, or into n of length $1/2n$ and n^2 of length $1/2n^2$.

4. [40] avoids a certain undesirable pathology in [29].

What, then, accounts for the diagonality of all previously considered values? In [27], Neyman answered this question by showing that all *continuous* values are diagonal; here continuity is w.r.t. (with respect to) the *variation* norm, defined by

$$\|v\| = \sup\left\{\sum_{i=1}^{k} |v(S_i) - v(S_{i-1})| : \varnothing = S_0 \subset S_1 \subset \cdots \subset S_k = I\right\}.$$

This norm plays a crucial role in the theory, and all previously considered values had been continuous w.r.t. it.

Closely related to the diagonal property is the *diagonal formula* for values. Let pNA denote the smallest variation-closed linear space containing all games $f \circ v$, where $v \in \mathrm{NA}^1$ and f is absolutely continuous. There is a unique value on pNA, and pNA $\subset$ ASYMP [16]. Suppose now that $\mu \in (\mathrm{NA}^1)^m$ and $f \in C^1(R^m)$. Then $f \circ \mu \in$ pNA, and

$$\varphi(f \circ \mu) = \left\langle \mu, \int_0^1 \nabla f(t, \ldots, t) dt \right\rangle \tag{4.3}$$

[2, Theorem B]. To understand (4.3), note that it follows from Lyapunov's theorem that for each t in [0, 1] there is a coalition tI with $\mu(tI) = (t, \ldots, t)$: the tI are called *diagonal coalitions*, and may be considered "perfect samples" of I as far as $f \circ \mu$ is concerned. Let us now think of a "player" in a non-atomic game as an infinitesimal coalition ds; the marginal contribution of ds when added to tI is

$$(f \circ \mu)(tI \cup ds) - (f \circ \mu)(tI) = \langle \mu(ds), \nabla f(t, \ldots, t) \rangle.$$

Thus (4.3) says that *the value of a player is his average contribution to a diagonal coalition*.

This principle, which is of fundamental importance in the theory of nonatomic games and its applications, has been extended far beyond the space pNA for which it was originally established. The deepest and furthest-reaching work on this subject is due to J.-F. Mertens [20], who has established the existence of a value obeying a suitable analogue of (4.3) on a very large space of games, which even contains games not in ASYMP.

5 Political Applications

A *weighted majority* (WM) game is one of the form $f_q \circ v$, where v is a non-negative measure with $v(I) = 1$ (the *vote* measure), $0 < q < 1$ and

$f_q(x) = 0$ or 1 according as $x \leqslant q$ or $x > q$. Finite WM games appear already in [26]. Values of finite WM games were first studied by Shapley and Shubik [38], who interpreted them as measures of political power. They have since been applied to many voting situations, such as the UN security council, the US electoral college, state legislatures, multi-party parliaments, etc.; [18] is a good survey. Shapiro and Shapley [35], Milnor and Shapley [21], and Hart [11] studied values of *oceanic* games, i.e. WM games in which v contains a nonatomic part (the "ocean" of small voters) as well as some atoms (large voters); [21] contains an application to corporations with several large stockholders. An interesting qualitative conclusion is that when $q = 1/2$, a *single* atom has value larger than his vote, as might be expected; but this is often reversed when there are several atoms. For example, when v has 2 atoms and an ocean of measure $1/3$ each, then the atoms get only $1/4$ of the value each.

The above are asymptotic results on the values of the atoms when the largest "small" vote tends to 0. Calculating the values of the small voters themselves, even approximately, is much more difficult, and even when there are no atoms, the problem was open for many years. Only recently did A. Neyman [28] prove, in a remarkable tour-de-force of combinatorial reasoning, that $f_q \circ v \in$ ASYMP when $v \in \mathrm{NA}^1$. Intuitively, his result says that the value of a coalition depends only on its total vote, not on the relative sizes of the voters. It can be used to prove that oceanic games are in ASYMP, and also that $f \circ v \in$ ASYMP when f is monotonic and continuous, and $v \in \mathrm{NA}^1$. Also, there are close connections to renewal theory.

More complex political structures can also often be described by using WM games. A bicameral legislature is the product of 2 WM games, and the electoral college when the players are the individual citizens is a polynomial in WM games. Such games need not be in ASYMP; thus if $\mu, v \in \mathrm{NA}^1$ and $\mu \neq v$, then $(f_{2/3} \circ \mu)(f_{2/3} \circ v) \notin$ ASYMP; however, it is a member of a space with a partition value [30]. Whether there is a partition value on the algebra generated by all nonatomic WM games is an open question.

See [31] for an application using a non-symmetric variant of the value.

A variant of the Shapley value called the *Banzhaf value* has achieved some prominence in connection with political models. For finite games it is defined by (3.1), with the sole difference that now S_i varies over the set of all subsets of $N \setminus \{i\}$, each such coalition receiving probability $1/2^{|N|-1}$. In general, it is not efficient. An account of the theory and a very extensive bibliography may be found in [8].

6 Economic Applications

Games arising in economics often have a property called "homogeneity of degree 1;" roughly, this means that two coalitions differing from each other in their size only, but not in their composition, have worths proportional to their sizes. Examples are games $f \circ \mu$, where $\mu \in (\mathrm{NA}^1)^m$ and f is a function of m variables that is homogeneous of degree 1. Suppose now that φ is a partition value. A principle that is basic to many of the economic applications asserts that

$$\text{if } \varphi \text{ is defined for a superadditive}^5 \text{ game } v \text{ that is homogeneous of degree } 1, \text{ then } \varphi v \text{ is in the core of } v. \tag{6.1}$$

(Recall that the *core* of a game v is the set of outcomes ν such that $\nu(I) = v(I)$ and $\nu(S) \geqslant v(S)$ for all S.)

Let's demonstrate this in the particular case in which $v = f \circ \mu$, where $\mu \in (\mathrm{NA}^1)^m$ and f is a superadditive[6] function defined and homogeneous of degree 1 on the nonnegative orthant of R^m, and C^1 in its interior. Although $f \notin C^1(R^m)$, it can be shown that nevertheless $v \in \mathrm{pNA}$ and the diagonal formula (4.3) holds. Moreover the homogeneity of degree 1 and the superadditivity together yield the concavity of f. Since f is homogeneous of degree 1, $\nabla f(t, \ldots, t)$ is a constant, so (4.3) yields $\varphi v = \langle \mu, \nabla f(1, \ldots, 1) \rangle$. This means that φv is a function h of $\mu(S)$, i.e. $(\varphi v)(S) = h(\mu(S))$; and in fact h is the linear function with coefficients $\nabla f(1, \ldots, 1)$. By the efficiency of the value, $h(1, \ldots, 1) = (\varphi v)(I) = v(I) = f(1, \ldots, 1)$, and hence it follows that the graph of h is tangent to that of f at $(1, \ldots, 1)$. Since f is concave and h is linear, it follows that the graph of h always lies above that of f; but this implies that $(\varphi v)(S) \geqslant v(S)$ for all S, which together with the efficiency $(\varphi v)(I) = v(I)$ means that φv is in the core.

In this case a small additional argument, which depends on the actual tangency (i.e. the differentiability of f), yields that v is the *only* member of the core. This is true whenever $v \in \mathrm{pNA}$; pNA expresses a kind of differentiability property of a game. In general, though, the core will contain more than just the value. For example, when v is the minimum of two NA^1 measures, then the core consists of a non-degenerate interval (i.e. the set of all convex combinations of two different outcomes); in this case the asymptotic value exists and is the midpoint of the core. More generally, Hart [12] has proved that if a superadditive game v that is

5. $v(S \cup T) \geqslant v(S) + v(T)$ whenever $S \cap T = \varnothing$.

6. $f(x + y) \geqslant f(x) + f(y)$.

homogeneous of degree 1 has an asymptotic value φv, then φv is the center of symmetry of the core of v.

If the core has no center of symmetry,[7] there will be no asymptotic value; but not all is lost. If ν is an NA^1 measure, an outcome φv is called a ν-*value* if for all S, (4.1) holds for all S-admissible sequences of partitions whose atoms have equal (or in an appropriate sense almost equal) ν-measures. Suppose now that μ in $(NA^1)^m$ is absolutely continuous w.r.t. ν, with Radon–Nikodym derivative $d\mu/d\nu$ in $(L^2(\nu))^m$. Let f be superadditive and homogeneous of degree 1; then Hart [14] has shown that $v = f \circ \mu$ has a ν-value, which has an interesting expression in terms of the core of v and the m-dimensional normal distribution whose covariance matrix is the same as that of $d\mu/d\nu$.

We come now to the applications. An important model in economic theory is that of the *exchange economy*. Like many economic models, it cannot be expressed as a transferable utility (TU) game as in §2; a more general concept—the *nontransferable utility* (NTU) game—is required. The most commonly used adaptation of the value to NTU games is that introduced[8] in [37], which culminated a long development to which many contributed; see in particular [24], [9]. We will not define the NTU value here; a brief treatment is in [1, §4]. It is enough for our purposes to note that the analysis involves the values of certain TU games auxiliary to the given NTU game.

In an exchange economy, the law of supply and demand defines *competitive prices* and, correspondingly, *competitive allocations* of goods and services. The TU games to which we are led from exchange economies are precisely the superadditive homogeneous games, and their cores are closely related to the cores of the "parent" NTU economies. The relationship between the value and the core expressed by (6.1), and the subsequent discussion, thus imply a close relationship between values and competitive allocations. More precisely, it can be proved that all allocations associated with an NTU value of a non-atomic exchange economy—i.e. all *value allocations*—are competitive. When the utility functions of the agents in the economy are sufficiently differentiable, we can assert the converse as well; in that case, therefore, the value allocations are the same as the competitive allocations.

Again, many people contributed to this development; see in particular [39], [5], [2], [4], [12], [13], [19], [14]. An excellent survey up to 1976 is in [13].

7. For example, the core of the minimum of 3 linearly independent measures is a triangle.
8. For an alternative approach, see Owen [32].

Models containing both political and economic elements, including in particular problems of taxation and redistribution, have been considered recently [1]. The TU games to which these models lead are products of pNA games with nonatomic WM games; the methods of Neyman [28] show that they have asymptotic values, and they are also amenable to the diagonal methods of Mertens [20].

Conceptually, these models differ from exchange economies in that threats play an important role. Games of this kind were treated by Nash [25], and much more generally by Harsanyi [9]. The worth $v(S)$ of a coalition S in an auxiliary TU game is now based as much on the harm that S *could* do to the players outside it as the good that it could do for itself. The value is of course efficient, so that it assumes that destructive threats are not actually carried out; this fits well our interpretation of the value as a reasonable compromise.[9] None of the pie gets thrown out, but how it gets cut up may depend on threats.

7 Cost Sharing

An interesting practical application of the Shapley value is to problems of cost sharing. For example, Littlechild and Owen [17] have considered the problem of airport landing fees. Runways (and other airport components) must be built large enough to accommodate the largest aircraft that will use them; but obviously it makes no sense to share the cost equally among all users, i.e. to charge the same landing fees to a jumbo jet and a private 4-seater. Here one defines a game v by considering the players to be individual aircraft landings, with $v(S)$ the hypothetical cost of building a facility that will accommodate the set S of landings. Each landing is then charged a fee precisely equal to its Shapley value. The efficiency condition assures that the fees will exactly cover the cost, the symmetry condition assures that similar users are charged the same fee, and the linearity condition assures that the cost of using two different and independent facilities is the sum of the costs of using each one separately. Monotonicity, of course, only says that you don't get paid for landing at an airport.

A spectacular recent application of this type is to telephone billing at large institutions. See Billera, Heath, and Raanan [3]; the system proposed by them has been adopted for internal telephone billing at Cornell University.

9. Value models in which threats do sometimes get carried out involve incomplete information; see [10], [23].

8 Other Contributions

A complete review of recent developments in the theory of the Shapley value is impossible in the space allotted to this paper. The quantifier "some", not "all", should be understood in the title; there have been many important contributions not covered here. We close by mentioning two conceptually innovative recent works: In [34], A. Roth formalized the idea that the value measures a game's utility to its players; and in [22], R. Myerson characterized the value in terms of communication networks connecting the players.

9 Conclusion

Much of the analysis in political and economic science has traditionally proceeded on an ad hoc basis, often using different methods and principles for each model under consideration. A unified approach to these disciplines is provided by game theory. Among the tools it provides, the Shapley value is particularly broadly and systematically applicable, and appears able to account for theoretical principles in widely diverse areas.

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