69 An Elementary Proof that Integration Preserves Uppersemicontinuity

The purpose of this 'classroom note' is to prove, using comparatively elementary tools, that under appropriate boundedness conditions, the uppersemicontinuity of a relation[1] $F_p(t)$ in a parameter p is preserved by integration over t. Most of the previous proofs of this result [Aumann (1964), Schmeidler (1970), Hildenbrand (1974)] are far deeper, and all are more involved. In addition to the elementary facts of Lebesgue integration – such as Fatou's lemma and the Dominated Convergence Theorem – we will use only the Convexity Theorem of Lyapunov (1940), and a measurable selection theorem for measurable compact-valued correspondences.[2] It should be noted that Lyapunov's theorem has been proved in an entirely elementary fashion by Halmos (1948). As for the selection theorem, this depends on the measurability of analytic sets, and so is perhaps our most advanced tool. Nevertheless, it too can be proved from 'scratch' in a couple of pages, and certainly it is significantly simpler than the general selection theorem of von Neumann (1949).

The result proved here is the deepest of the lemmas needed for the existence of competitive equilibria in markets with a non-atomic continuum of traders [see e.g. Schmeidler (1969) or Hildenbrand (1974)]. Thus the Existence Theorem is brought within the reach of relatively unsophisticated audiences.

Let $(T, \mathscr{C}, \mu)$ be a complete non-atomic measure space, and let P be a metric space. The integral $\int \boldsymbol{x}(t)\mu(\mathrm{d}t)$ of a function $\boldsymbol{x}$ on T will be denoted $\int \boldsymbol{x}$. A *selection* from a correspondence $\boldsymbol{F}$ on T is a function $\boldsymbol{x}$ on T such that $\boldsymbol{x}(t) \in \boldsymbol{F}(t)$ for all t. The *integral* of such a correspondence $\boldsymbol{F}$ is the set of all integrals of its selections; it is denoted $\int \boldsymbol{F}$. A relation $\boldsymbol{F}$ on T to a Euclidean space E^n is *bounded by* a non-negative real-valued function $\boldsymbol{h}$ if $\|x\| \leqq \boldsymbol{h}(t)$ whenever $x \in \boldsymbol{F}(t)$. A function $\boldsymbol{x}$ on T is *bounded by* $\boldsymbol{h}$ if $\|\boldsymbol{x}(t)\| \leqq \boldsymbol{h}(t)$ for all t. A relation on P to E^n is *uppersemicontinuous* if its graph is closed.

[1] We use the term 'relation' to mean 'set-valued function'.
[2] Relations with non-empty values.

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For each p in P and t in T, let $F_p(t) \subset E^n$. We wish to prove that if all the relations F_p are bounded by the same integrable function h, and if for each fixed t, $F_p(t)$ is uppersemicontinuous in p, then also $\int F_p$ is uppersemicontinuous in p. This may be restated as follows:

Lemma. Let F be a correspondence from T to E^n. Let $\{x_k\}$ be a sequence of measurable functions from T to E^n, all of which are bounded by the same integrable function h. Assume that for each t, each limit point of $\{x_k(t)\}$ belongs to $F(t)$. Then each limit point of $\int x_k$ belongs to $\int F$.

Before proving the lemma, we quote the selection theorem that we shall need. A relation on T to E^n is *measurable* if its graph is measurable in the product σ-field $\mathscr{C} \times \mathscr{B}$, where $\mathscr{B}$ denotes the Borel sets of E^n.

Compact-valued selection theorem. Every compact-valued measurable correspondence on T to E^n has a measurable selection.

This can be proved in a few lines from the 'projection theorem', which in turn is essentially equivalent to the absolute measurability of analytic sets.[3]

The proof of the lemma is by induction on n. For $n = 0$ there is nothing to prove. Suppose the lemma has been proved up to (and including) $n-1$; we will prove it for n. Let x be a limit point of $\int x_k$; we must show $x \in \int F$. W.l.o.g. (without loss of generality) we may assume $x = \lim \int x_k$; otherwise we can restrict attention to a subsequence of the originally given sequence. Let $D(t)$ be the set of all limit points of $x_k(t)$. Then $D(t) \subset F(t)$, and so it is sufficient to prove that $x \in \int D$. Contrariwise, suppose $x \notin \int D$. A well-known theorem of Richter (1963) states that the integral of a correspondence on a non-atomic measure space is convex,[4] and hence $\int D$ is convex. Since $x \notin \int D$, there is a hyperplane through x that supports $\int D$. W.l.o.g. we may assume that the vector $(1, 0, \ldots, 0)$ is orthogonal to this hyperplane; i.e., that

$$x^1 \leqq \inf\{y^1 : y \in \textstyle\int D\}. \tag{1}$$

Define $\delta(t) = \liminf x_k^1(t)$; then δ is measurable. Since the x_k are bounded by h, so is D; in particular $D(t)$ is compact, and hence for each t there is an $x(t)$ in $D(t)$ with $x^1(t) = \delta(t)$. Hence if we define

$$B(t) = D(t) \cap \{y \in E^n : y^1 = \delta(t)\}, \tag{2}$$

then $B(t)$ is compact and non-empty, and B is measurable. Hence from the

[3]See Hildenbrand (1974, Prop. 3 of D.II.3, p. 60; and D.I.11, p. 40).

[4]This theorem can be proved in a few lines from Lyapunov's Convexity Theorem [see Hildenbrand (1974, p. 62)].

Compact-Valued Selection Theorem it follows that there is a measurable function $\boldsymbol{y}$ such that for all t, $\boldsymbol{y}^1(t) = \boldsymbol{\delta}(t)$ and $\boldsymbol{y}(t) \in \boldsymbol{D}(t)$. Hence $\int \boldsymbol{y} \in \int \boldsymbol{D}$, and so from (1) it follows that

$$x^1 \leqq \int \boldsymbol{\delta}. \tag{3}$$

Hence from Fatou's lemma and the definition of $\boldsymbol{\delta}$ we deduce

$$\begin{aligned} 0 &\leqq \limsup \int |\boldsymbol{x}_k^1 - \boldsymbol{\delta}| = \limsup \left[\int (\boldsymbol{x}_k^1 - \boldsymbol{\delta}) + 2 \int \max (0, \boldsymbol{\delta} - \boldsymbol{x}_k^1)\right] \\ &\leqq \limsup \int (\boldsymbol{x}_k^1 - \boldsymbol{\delta}) + 2 \limsup \int \max (0, \boldsymbol{\delta} - \boldsymbol{x}_k^1) \\ &\leqq \int \lim \boldsymbol{x}_k^1 - \int \boldsymbol{\delta} + 2 \int \max(0, \boldsymbol{\delta} - \liminf \boldsymbol{x}_k^1) = x^1 - \int \boldsymbol{\delta} \leqq 0. \end{aligned}$$

Hence $\int |\boldsymbol{x}_k^1 - \boldsymbol{\delta}| \to 0$ as $k \to \infty$, and hence there is a subsequence of $\{\boldsymbol{x}_k^1\}$ that tends a.e. to $\boldsymbol{\delta}$. W.l.o.g. we may assume that it is the entire sequence; that is,

$$\boldsymbol{x}_k^1(t) \to \boldsymbol{\delta}(t) \text{ a.e.} \tag{4}$$

Hence it follows that a.e. $\boldsymbol{D}(t) \subset \{y \in E^n : y^1 = \boldsymbol{\delta}(t)\}$. Now set $H = \{y \in E^n : y^1 = 0\}$, and define

$$\boldsymbol{D}^*(t) = \boldsymbol{D}(t) - (\boldsymbol{\delta}(t), 0, \ldots, 0),$$

$$\boldsymbol{x}_k^*(t) = \boldsymbol{x}_k(t) - (\boldsymbol{x}_k^1(t), 0, \ldots, 0).$$

Then $\boldsymbol{D}^*(t) \subset H$ and $\boldsymbol{x}_k^*(t) \in H$. From (4) it follows that every limit point of $\{\boldsymbol{x}_k^*(t)\}$ belongs to $\boldsymbol{D}^*(t)$. Hence from the induction hypothesis it follows that every limit point of $\int \boldsymbol{x}_k^*$ belongs to $\int \boldsymbol{D}^*$. Hence from Lebesgue's Dominated Convergence Theorem, and from (4), we obtain

$$\begin{aligned} x &= \lim \int \boldsymbol{x}_k = \lim \int \boldsymbol{x}_k^* + \int (\lim \boldsymbol{x}_k^1, 0, \ldots, 0) \\ &= \lim \int \boldsymbol{x}_k^* + \int (\boldsymbol{\delta}, 0, \ldots, 0) \in \int \boldsymbol{D}^* + \int (\boldsymbol{\delta}, 0, \ldots, 0) = \int \boldsymbol{D}. \end{aligned}$$

This completes the proof of the lemma.

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