1. Representation of algebras

All algebras $\mathcal{H}$ we consider are $\mathbb{C}$-algebras of countable or finite dimension. An $\mathcal{H}$-module M is *irreducible* if it does not have proper $\mathcal{H}$-submodules.

LEMMA 0.1 (Schur's Lemma). *If an algebra $\mathcal{H}$ has countable dimension then $End_{\mathcal{H}}(M) = \mathbb{C}$ for any irreducible $M \in \mathcal{M}(\mathcal{H})$.*

PROOF. Since M is irreducible, $\mathrm{A} := \mathrm{End}_{\mathcal{H}} M$ is a skew-field. Since the algebra $\mathcal{H}$ is of countable dimension over $\mathbb{C}$ the space M is also of countable dimension. Since M is irreducible the map $A \to M, a \to am_0$ is an imbedding for any non-zero $m_0 \in M$ and therefore A is also of countable dimension. So the Schur's Lemma would follows from the following result.

CLAIM 0.2. *If* A *is a skew-field of countable dimension over $\mathbb{C}$, then* $\mathrm{A} = \mathbb{C}$.

PROOF. We have to show that $a \in \mathbb{C}$ for any $a \in \mathrm{A}$. Suppose that $a - \lambda \neq 0$ for all $\lambda \in \mathbb{C}$. Since A has countable dimension, the elements $\{(a-\lambda)^{-1}\}, \lambda \in \mathbb{C}$ are linearly dependent. Thus, there exist non-zero complex numbers $c_i, 1 \leq i \leq k$ so that

$$\sum_{i=1}^{k} c_i(a - \lambda_i)^{-1} = 0.$$

Multiplying through by $\prod_{i=1}^{k}(a - \lambda_i)$, we see that there exists a non-zero polynomial $Q(x) \in \mathbb{C}[x]$ such that $Q(a) = 0$. Factoring this polynomial, we see that there are $\mu_j \in \mathbb{C}$ so that

$$\prod_j (a - \mu_j) = 0.$$

Since A is a skew-field one of these factors must be equal to zero because. Hence $a \in \mathbb{C}$. □

□

DEFINITION 0.3. (1) We denote by $Z(\mathcal{H})$ the the center the algebra $\mathcal{H}$.

(2) By the Schur Lemma for an irreducible representation M of an algebra $\mathcal{H}$ of countable dimensionand $z \in Z(\mathcal{H})$ there exists $\chi_M(z) \in \mathbb{C}$ such that $z_M = \chi_M(z) Id_M$ where z_m is the action of z on M. We say that the map $\chi_M : Z(\mathcal{H}) \to \mathbb{C}$ is the *central character* of the module M.

REMARK 0.4. The statement is false if we drop the condition that a skew-field A has countable dimension. Really take $A = M = \mathbb{C}(x)$.

LEMMA 0.5. *Let $\mathcal{H}$ be an algebra of countable dimension over $\mathbb{C}$ with unit e. Then for any non-nilpotent element $a \in \mathcal{H}$ there exists a simple $\mathcal{H}$-module M such that $a|_M \neq 0$.*

Proof. The proof is similar to that of Schur's lemma. By adding the unit we may assume that the algebra $\mathcal{H}$ is unital. First we establish the following result.

Claim 0.6. *There exists $\lambda \in \mathbb{C} \setminus 0$ such that $a - \lambda$ is not invertible in $\mathcal{H}$.*

Proof. If $a \in \mathbb{C}$, this is trivial. Otherwise, by countable-dimensionality of $\mathcal{H}$, the elements the $(a-\mu)^{-1}$ are linearly dependent. Thus there exists $c_i \in \mathbb{C}, 1 \leq i \leq k$ so that

$$\sum_{i=1}^{k} c_i (a - \lambda_i)^{-1} = 0.$$

Multiplying through by $\prod_{i=1}^{k}(a - \lambda_i)$, we get a non-zero polynomial over $\mathbb{C}$ with a as a root. Thus, there are $\lambda_j \in \mathbb{C} \setminus 0$ and $n_j \geq 0$ so that

$$a^{n_0} \prod_j (a - \lambda_j)^{n_j} = 0$$

But this is not possible since $(a - \lambda_j)$ are invertible and a is not nilpotent. $\square$

Choose $\lambda \in \mathbb{C} \setminus 0$ such that $a - \lambda$ is not invertible in $\mathcal{H}$. Then the quotient $\mathcal{H}$-module $N := \mathcal{H}/(a - \lambda)$ is not trivial. Let $\bar{e} \in N$ be the image of e and $N_0 \subset N$ be a maximal submodule which does not contain $\bar{e}$.

Problem 0.7.
The quotient $M := N/N_0$ satisfies the conditions of the Lemma.

$\square$

Definition 0.8.
1. For any $\mathbb{C}$-algebra $\mathcal{H}$ we denote by $\mathcal{H}^{op}$ the opposite algebra.
2. For any $\mathcal{H}$-module M we define the action of $\mathcal{H}^{op}$ on $M^\vee := \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ by
$$h\lambda(m) := \lambda(mh), m \in M, \lambda \in M^\vee, h \in \mathcal{H}$$
and define an action of $\mathcal{H} \otimes \mathcal{H}^{op}$ on $End_{\mathbb{C}}(M)$ by
$$h' \otimes h''(A) := h'Ah''$$
3. We denote by α_M [or simply α] the $\mathcal{H} \otimes \mathcal{H}^{op}$-morphism
$$M \otimes M^\vee \to \mathrm{End}_{\mathbb{C}}(M)$$
given by
$$\alpha(v \otimes \lambda)(w) := \lambda(w)v, v, w \in M, \lambda \in M^\vee$$
4. We say that a simple finite-dimensional $\mathcal{H}$-module M is *compact* if the action map $a_M : \mathcal{H} \to \mathrm{End}_{\mathbb{C}}(M)$ of $\mathcal{H} \otimes \mathcal{H}^{op}$-modules splits. That is there exists an imbedding $\mu_M : \mathrm{End}_{\mathbb{C}}(M) \to \mathcal{H}$ of $\mathcal{H} \otimes \mathcal{H}^{op}$-modules such that $a_M \circ \mu_M = Id$.

LEMMA 0.9. *For any simple finite-dimensional compact $\mathcal{H}$-module M we have a direct sum algebra decomposition*

$$\mathcal{H} = \mathcal{H}_M \oplus \mathcal{H}_M^{\perp}$$

where $\mathcal{H}_M = Im(\mu_M), \mathcal{H}_M^{\perp} = \mathrm{Ker}(a_M)$.

PROOF. Since we have direct sum algebra decomposition $\mathcal{H} = \mathcal{H}_M \oplus \mathcal{H}_M^{\perp}$ of $\mathcal{H}$ as a vector space and by the defintion $\mathcal{H}_M^{\perp} = \mathrm{Ker}(a_M) \subset \mathcal{H}$ is a subalgebra we have to check that

(a) the map μ_M is an algebra homomorphism and

(b) $h_M h_M^{\perp} = 0$ for all $h_M \in \mathcal{H}_M, h_M^{\perp} \in \mathcal{H}_M^{\perp}$.

Proof of (a). Choose any $r', r'' \in \mathrm{End}_{\mathbb{C}}(M)$. Since the map μ_M is an morphism of $\mathcal{H} \otimes \mathcal{H}^{op}$-modules the product $\mu_M(r')\mu_M(r'')$ is equal to $\mu_M(r)$ for some $r \in \mathrm{End}_{\mathbb{C}}(M)$ and $a_M(r) = a_M(r')a_M(r'')$. Since $a_M \circ \mu_M = Id$ we see that $r = r'r''$.

A proof of (b) is completely analogous. □

PROBLEM 0.10.
1. α_M is an imbedding.
2. Let M be a simple finite-dimensional compact $\mathcal{H}$-module. The any $\mathcal{H}$-module N admits unique decomposition $N = N_0 \oplus N_1$ such that N_0 is a multiple of M and N_1 does not have subquotients isomorphic to M.

1.1. Idempotented Algebras.

DEFINITION 0.11.
1. An algebra $\mathcal{H}$ is *idempotented* if for every finite collection $\{f_i\}, i \in I$ of elements of $\mathcal{H}$ there exists an idempotent $e \in \mathcal{H}$ such that $ef_i = f_i e = f_i$ for all i in I.

   Let $\mathcal{H}$ be an idempotented algebra.
2. For any $\mathcal{H}$ -module M we define $M_{sm} = \mathcal{H}M$.
3. A module M of an idempotented algebra $\mathcal{H}$ is called *non-degenerate* if $M = M_{sm}$.
4. We denote by $\mathcal{M}(\mathcal{H})$ the abelian category of non-degenerate $\mathcal{H}$-modules.
5. For any $M \in \mathcal{M}(\mathcal{H})$ we define $\tilde{M} \in \mathcal{M}(\mathcal{H}^{op})$ by $\tilde{M} := (M^{\vee})_{sm}$.
6. For any $M \in Ob(\mathcal{M}(\mathcal{H})), N \in Ob(\mathcal{M}(\mathcal{H}^{op}))$ we denote by $< N, M >$ the space of bilinear maps $\phi : N \times M \to \mathbb{C}$ such that $\phi(nh, m) \equiv \phi(n, hm)$.
7. For any $M \in Ob(\mathcal{M}(\mathcal{H})), N \in Ob(\mathcal{M}(\mathcal{H}^{op}))$ we denote by
   $$\kappa_{M,N} : \mathrm{Hom}_{\mathcal{H}}(M, \tilde{N}) \to < N, M >$$
   the map given by
   $$\kappa_{M,N}(A)(n, m) := A(m)(n), m \in M, n \in N, A \in \mathrm{Hom}_{\mathcal{H}}(M, \tilde{N})$$
   and denote by $\kappa_{N,M} : \mathrm{Hom}_{\mathcal{H}^{op}}(N, \tilde{M}) \to < N, M >$ the map given by
   $$\kappa_{N,M}(B)(n, m) := B(n)(m), m \in M, n \in N, B \in Hom_{\mathcal{H}}(N, \tilde{M}).$$

(8) A non-degenerate $\mathcal{H}$ -module M is *admissible* if $dim_{\mathbb{C}}(eM) < \infty$ for any idempotent $e \in \mathcal{H}$.

LEMMA 0.12. *The functor $M \to M_{sm}$ from the category of $\mathcal{H}$-modules to $\mathcal{M}(\mathcal{H})$ is exact.*

PROOF. It is clear that for any imbedding $M \hookrightarrow N$ of $\mathcal{H}$-modules the induced map $M_{sm} \to N_{sm}$ is an imbedding. So the functor $M \to M_{sm}$ is left exact.

To show that this functor is right exact we have to check that for and surjection $p : M \to N$ and any $n \in N_{sm}$ we can find $m \in M_{sm}$ such that $n = p(m)$.

Since $n \in N_{sm}$ there exists an idempotent $e \in \mathcal{H}$ such that $en = n$. On the other hand since the map $p : M \to N$ is surjective there exists $m' \in M$ such that $n = p(m')$. Take $m := em' \in M_{sm}$. Then

$$p(m) = p(em') = ep(m') = en = n$$

□

PROBLEM 0.13. (1) Let M_∞ be the algebra of matricies $(m_{i,j}), 1 \leq i,j < \infty$ such that $m_{i,j} = 0$ for almost all pairs i, j. Show that the algebra M_∞ is idempotented.

(2) For any totally disconected topological space X the algebra $\mathcal{S}(X)$ of locall constant functions with compact support is idempotented.
(3) For any $M \in Ob(\mathcal{M}(\mathcal{H})), N \in Ob(\mathcal{M}(\mathcal{H}^{op}))$ the maps $\kappa_{M,N}$ and $\kappa_{N,M}$ are bijections.
(4) For any $M \in Ob(\mathcal{M}(\mathcal{H})), N \in Ob(\mathcal{M}(\mathcal{H}^{op}))$ we have a canonical isomorphism $\mathrm{Hom}_{\mathcal{H}}(M, \tilde{N}) \to \mathrm{Hom}^{op}_{\mathcal{H}}(N, \tilde{M})$.
(5) Show that the correspondence $M \to \tilde{M}$ defines an exact contravariant functor from $\mathcal{M}(\mathcal{H})$ to $\mathcal{M}(\mathcal{H}^{op})$.
(6) Construct a canonical morphism $V \hookrightarrow \tilde{\tilde{V}}$ and show that it is an imbedding.

1.2. Projective and Injective modules. Recall that an object P of an abelian category $\mathcal{M}$ is *projective* if the functor

$$\begin{array}{rcll} \mathcal{M} & \to & \mathrm{Ab} & \quad \text{given by} \\ X & \mapsto & \mathrm{Hom}_{\mathcal{M}}(P, X) & \end{array}$$

is exact.

Analogously an object P is *injective* if the functor

$$\begin{array}{rcll} \mathcal{M} & \to & \mathrm{Ab} & \quad \text{given by} \\ X & \mapsto & \mathrm{Hom}_{\mathcal{M}}(X, I) & \end{array}$$

is exact.

LEMMA 0.14. *For any projective object $P \in Ob(\mathcal{M}(\mathcal{H}))$ the object $\tilde{P} \in \mathcal{M}(\mathcal{H}^{op})$ is injective.*

Proof. We must show that the functor $X \mapsto \mathrm{Hom}(X, \tilde{P})$ on $\mathcal{M}(\mathcal{H}^{op})$ is exact. As follows from the previous Problem we have isomorphisms

$$\mathrm{Hom}(X, \tilde{P}) = \mathrm{Hom}(P, \tilde{X})$$

Since P is projective and the functor $X \to \tilde{X}$ is exact we see that $\tilde{P}$ is injective. □

Theorem 0.15. *For any idempotented algebra $\mathcal{H}$ the category $\mathcal{M}(\mathcal{H})$ has enough projective and injective objects.*

Proof. We start with the proof for the existence of a projective cover $P \to M$ for an $\mathcal{H}$-module M. For any idempotent $e \in \mathcal{H}$ the functor $X \to \mathrm{Hom}(P_e, X) = eX$ is exact and therefore the $\mathcal{H}$-module $P_e = \mathcal{H}e$ is projective. Since direct sums of projective objects are projective direct sums of any collection of the moduels of the form P_e are also projective.

If $X \in \mathrm{Ob}\,\mathcal{M}(\mathcal{H})$ and $\xi \in X$, then it follows from non-degeneracy that there exists an idempotent e so that $e\xi = \xi$. Then ξ is in the image of the map $P_e \to X$ given by $he \mapsto h\xi$. Taking the direct sum over all $\xi \in X$ of the associated P_e, we see that X is a quotient of a projective object.

Now we want to construct an imbedding $M \hookrightarrow I$ of an $\mathcal{H}$-module M into an injective object. As we have enough projectives, there is an epimorphism $P \to \tilde{M}$. Now consider the composition

$$M \hookrightarrow \tilde{\tilde{M}} \hookrightarrow \tilde{P}$$

where $\tilde{P}$ is injective by Lemma 0.14. □

For any idempotented algebra $\mathcal{H}$ we define $\hat{\mathcal{H}} := \mathrm{End}_{\mathcal{H}^{op}}(\mathcal{H})$. Since $\mathcal{H}$ is associative we have a natural imbedding $\mathcal{H} \hookrightarrow \hat{\mathcal{H}}$.

Lemma 0.16. *For any idempotented algebras $\mathcal{H}$ and a non-degenerate $\mathcal{H}$-module M the action of $\mathcal{H}$ on M extends uniquely to an action of $\hat{\mathcal{H}}$ on M.*

Proof. To construct a map $\alpha : \hat{\mathcal{H}} \to \mathrm{End}_{\mathbb{C}}(M)$ we have to define $\alpha(\hat{h})(m)$ for $\hat{h} \in \hat{\mathcal{H}}, m \in M$.

We choose an idempotent $e \in \mathcal{H}$ such that $em = m$ [this is possible since M is non-degenerate] and write $\alpha_e(\hat{h})m) := \hat{h}(e)m$.

Let us show that $\alpha_e(\hat{h})m$ does not depend on a choice of an idempotent $e \in \mathcal{H}$ such that $em = m$. Since for any two idempotents $e', e'' \in \mathcal{H}$ there exists an idempotent $e \in \mathcal{H}$ such that $ee'e = e', ee''e = e''$ it is sufficient to show that $\alpha_e(\hat{h})(m) = \alpha_{e'}(\hat{h})(m)$ for idempotents e', e such that $ee' = e'$. But in the case we have

$$\alpha_{e'}(\hat{h})(m) = \hat{h}(e')(m) = \hat{h}(ee')(m) = \hat{h}(e)e'm = \hat{h}(e)m = \alpha_e(\hat{h})(m)$$

Since the element $\alpha_e(\hat{h})m$ does not depend on a choice of an idempotent $e \in \mathcal{H}$ such that $em = m$ we will write $\alpha(\hat{h})(m)$ instead of $\alpha_e(\hat{h})(m)$.

The uniquenes of an extension $\hat{\mathcal{H}} \to \mathrm{End}_{\mathbb{C}}(M)$ of the action map $a : \mathcal{H} \to \mathrm{End}_{\mathbb{C}}(M)$ follows immediately from the following result.

Problem 0.17. For any $\hat{h} \in \hat{\mathcal{H}}, h \in \mathcal{H} \subset \hat{\mathcal{H}}$ we have $\hat{h}h = \hat{(h)} \in \mathcal{H} \subset \hat{\mathcal{H}}$.

□

Problem 0.18. Describe the algebras $\hat{M}_\infty$ and $\hat{\mathcal{S}}(X)$.

Lemma 0.19. *Let M be an irreducible $\mathcal{H}$-module $M, e \in \mathcal{H}$ be an idempotent and $\mathcal{H}_e := e\mathcal{H}e$. Then*

1. *Either $eM = \{0\}$ or eM is an irreducible $\mathcal{H}_e$-module.*
2. *Every irreducible $\mathcal{H}_e$-module has a form eM for some irreducible $M \in \mathcal{M}(\mathcal{H})$.*

Proof. (1) Assume that $eM \neq \{0\}$. To prove the irreducibility of the $\mathcal{H}_e$-module eM it is sufficient to show that for any $w, v \neq 0$ in eM there exists $\bar{h} \in \mathcal{H}_e$ such that $v = \bar{h}w$. Since the $\mathcal{H}$-module M is irreducible there exists $h \in \mathcal{H}$ such that $v = hw$. But now we can take $\bar{h} = ehe \in \mathcal{H}_e$.

(2) Let $\bar{M}$ be an irreducible $e\mathcal{H}e$-module. As follows from (1) it is sufficient to show the existence of an irreducible $\mathcal{H}$-module L such $\bar{M}$ is a submodule of eL. Set $N = \mathcal{H} \otimes_{e\mathcal{H}e} \bar{M}$. The imbedding $\mathcal{H}_e \subset \mathcal{H}$ induces the inbeddings $\bar{M} \hookrightarrow eN \hookrightarrow N$. Consider the partially ordered set X of proper $\mathcal{H}$-submodules $N_x \subset N$ where the ordering is by inclusion. Since the $\mathcal{H}_e$-module $\bar{M}$ is irreducible and $N = \mathcal{H}\bar{N}$ we see that $N_x \cap \bar{M} = \{0\}$ for all $x \in X$. Therefore the partially ordered set satisfies the conditions of the Zorn's lemma and there exists a maximal proper submodule $N_0 \subset N$. Let $L := N/N_0$. Since $N_0 \cap \bar{M} = \{0\}$ the projection $p : N \to L$ defines an imbedding $\bar{M} \hookrightarrow eL$ of $\mathcal{H}_e$-modules.

I claim that the $\mathcal{H}$-module L is irreducible. Really if $L' \subset L$ is a non-zero proper $\mathcal{H}$-submodule of L the preimage $p^{-1}(L') \subset N$ is a proper $\mathcal{H}$-submodule of N strictly bigger then N_0. But this is impossible since N_0 is a maximal proper submodule of N. □

We introduce a notion of *admissible* and *compact* modules which will be central for our analysis of representations of groups over local non-archimedian fields.

Definition 0.20. Let $\mathcal{H}$ be an idempotented algebra.

1. A non-degenerate $\mathcal{H}$-module M is *admissible* if for every idempotent $e \in \mathcal{H}$ the space eM is finite dimensional.
2. An admissible $\mathcal{H}$-module M is *compact* if for any idempotent $e \in \mathcal{H}$ the finite dimensional $e\mathcal{H}e$-module eM is compact.

Problem 0.21.
(1) M is admissible iff the natural imbedding $M \to \tilde{\tilde{M}}$ is an isomorphism.
(2) For any compact $\mathcal{H}$-module M the direct sum decompositions (see 0.10)

$$\mathcal{H}_e = \mathcal{H}_{M_e} \oplus \mathcal{H}_{M_e}^{\perp}$$

where e run through idempotents $e \in \mathcal{H}$ define a direct sum algebra decomposition

$$\mathcal{H} = \mathcal{H}_M \oplus \mathcal{H}_M^{\perp}$$

such that $\mathcal{H}_M^{\perp}$ acts trivially on M and the restriction of the action map $\mathcal{H} \to \mathrm{End}(M)$ on $\mathcal{H}_M$ is an imbedding.

LEMMA 0.22. *For any admissible $\mathcal{H}$-module M the imbedding [see 0.10] $\alpha_M : M \otimes \tilde{M} \to End(M)_{sm}$ is an isomorphism where we consider $End(M)$ as $\mathcal{H} \otimes \mathcal{H}^{op}$-module.*

PROOF. The statement is clear in the case when $dim(M) < \infty$. We use the admissibility for a reduction to this case.

By 0.10 the map α_M is an imbedding. So it is sufficient to show that for any idempotent $e \in \mathcal{H}$ the map

$$\alpha_e : eM \otimes e\tilde{M} \to eEnd(M)e$$

is onto. As α_e is an imbedding it is sufficient to show that

$$dim(eEnd(M)e) \leq dim(eM) \times dim(e\tilde{M})$$

Since t the restriction $eEnd(M)e \to End(eM)$ is an imbedding this inequality follows from the equalities

$$dim(e\tilde{M}) = dim(eM), dim(End(eM)) = dim^2(e\tilde{M}).$$

□

COROLLARY 0.23. *For any irreducible admissible $\mathcal{H}$-module M we have*

$$dim(\mathrm{Hom}_{\mathcal{H}\otimes\mathcal{H}^{op}}(\mathcal{H}, M \otimes \tilde{M})) = 1$$

PROOF. Since we have a natural non-trivial map $\mathcal{H} \to End(M)_{sm}$ we see that

$$dim(\mathrm{Hom}_{\mathcal{H}\otimes\mathcal{H}^{op}}(\mathcal{H}, M \otimes \tilde{M})) \neq 0$$

So it is sufficient to show that $dim(\mathrm{Hom}_{\mathcal{H}\otimes\mathcal{H}^{op}}(\mathcal{H}, M \otimes \tilde{M})) \leq 1$. For this it is sufficient to show that $dim(\mathrm{Hom}_{\mathcal{H}_e\otimes\mathcal{H}_e^{op}}(\mathcal{H}, eM \otimes e\tilde{M})) \leq 1$ for all idempotents $e \in \mathcal{H}$. But follows immediately from Lemma 0.22. □

1.3. Irreducible modules and the Jordan-Holder Content. Let $\mathcal{H}$ be an idempotented algebra.

DEFINITION 0.24.
(1) We denote by $Irr_{\mathcal{H}}$ be the set of *equivalence classes* of irreducible representations of the algebra $\mathcal{H}$.
(2) If $M \in \mathcal{M}(G)$, then the *Jordan-Holder content* of M, $JH(M)$, is the subset of $Irr_{\mathcal{H}}$ consisting of all irreducible subquotients of M.

LEMMA 0.25. *$JH(M) \neq \emptyset$ for any non-zero $\mathcal{H}$-module M.*

PROOF. Fix any non-zero $m \in M$. By the Zorn's lemma there exists a maximal proper submodule N of the $\mathcal{H}$-module $M_0 := \mathcal{H}m \subset M$. By the construction the quotient M_0/N is irreducible. So $[M_0/N] \in JH(M)$ where $[M_0/N]$ is the class of M_0/N. □

PROBLEM 0.26. (1) If N is a subquotient of M, then $JH(N) \subset JH(M)$.
(2) If $M = \sum_\alpha M_\alpha$ then $JH(M) = \cup_\alpha JH(M_\alpha)$.

1.4. Decomposing Categories.

DEFINITION 0.27. Let $\mathcal{M}$ be an abelian category such that the coproducts exists in $\mathcal{M}$ (i.e. $\mathcal{M}$ is cocomplete) and the coproduct of a family of monomorphisms is a monomorphism. [for example a category of modules over an algebra]. Given full subcategories $\mathcal{M}_1, \mathcal{M}_2$ of $\mathcal{M}$ we write $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ if for any object $M \in \mathcal{M}$, there exist unique subobjects $M_i \in \mathcal{M}_i$ so that $M = M_1 \oplus M_2$.

In such a case any irreducible object either belongs to $\mathcal{M}_1$ or to $\mathcal{M}_2$ and we obtain a decomposition

$$Irr\mathcal{M} = Irr\mathcal{M}_1 \coprod Irr\mathcal{M}_2.$$

where $\coprod$ is the 'disjoint union'. Conversely, such a decomposition on the level of sets uniquely determines the categorical decomposition if it exists. To see this we define for any subset $S \subset Irr\mathcal{M}$ the full subcategory $\mathcal{M}(S)$ of $\mathcal{M}$ consisting of objects M with $JH(M) \subset S$.

LEMMA 0.28. *If subsets $S, S' \subset Irr\mathcal{M}$ do not intersect, then the categories $\mathcal{M}(S)$ and $\mathcal{M}(S')$ are orthogonal, i.e. $M \in \mathcal{M}(S)$ and $M' \in \mathcal{M}(S')$ imply* $\mathrm{Hom}(M, M') = 0$.

PROOF. Suppose $\alpha \in \mathrm{Hom}(M, M')$. Set $N = \alpha(M)$. So, $JH(N) \subset JH(M) \subset S$ and also $JH(N) \subset JH(M') \subset S'$. But $S \cap S' = \emptyset$ so $N = 0$ by 0.25. ☐

If $S \subset Irr\mathcal{M}$, $M \in \mathcal{M}$, we will denote by $M(S)$ the union of all subobjects of M which lie in $\mathcal{M}(S)$. By the lemma, this is the maximal submodule with Jordan-Holder content lying in S.

DEFINITION 0.29. Let $S \subset Irr\mathcal{M}$ and $S' := Irr\mathcal{M} \setminus S$. We say that S is a *splitting* subset if $\mathcal{M} = \mathcal{M}(S) \oplus \mathcal{M}(S')$ [that is, if $M = M(S) \oplus M(S')$ for each $M \in \mathcal{M}$]. In this case we say that S *splits* $\mathcal{M}$.

REMARK 0.30. A decomposition of categories $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ is equivalent to a decomposition of sets $Irr\mathcal{M} = S \coprod S'$ where S is a splitting subset.

PROBLEM 0.31. Let $\mathcal{H}$ be an idempotented algebra and M an admissible compact $\mathcal{H}$-module. Then $[M]$ splits $\mathcal{M}(\mathcal{H})$ where $[M] \in Irr\mathcal{M}(\mathcal{H})$ is the equivalence class of M.

2. l-Groups

2.1. Basic definitions.

DEFINITION 0.32. (1) An l-space is a topological space which is Hausdorff, locally compact and 0-dimensional (i.e. totally disconnected: any point has a basis of open compact neighborhoods).

(2) For any l-space we denote by $\mathcal{S}(X)$ the vector space of locally constant compactly supported $\mathbb{C}$-valued functions on X.
(3) An l-group is a Hausdorff topological group such that the identity e has a basis of neighborhoods which are open compact groups. We will always assume that G is countable at infinity [that is we assume that for any open compact subgroup K of G the quotient G/K is either countable or finite.
(4) For an l-group we denote by $r : G \to Aut(\mathcal{S}(G)), l : G \to Aut(\mathcal{S}(G))$ the right and left regular representations of G on $\mathcal{S}(G)$ given by $(r(x)f)(g) := f(gx), (l(x)f)(g) = f(x^{-1}g), f \in \mathcal{S}(G), x, g \in G$.

Let F be a local non-archimedian field, $\mathcal{O} \subset F$ the ring of integers, $\mathcal{P} = \pi\mathcal{O} \subset \mathcal{O}$ the maximal ideal and $k = \mathcal{O}/\mathcal{P}$ the residue field $k = \mathbb{F}_q$. We denote by $\nu : F^\star \to \mathbb{Z}$ the *valuation* such that $\nu(\pi) = 1$ and define the *norm* by $\|x\| = q^{-\nu(x)}$. The topolgy on F induces a topolgy on F^n and therefore on $X(F)$ for any algebraic F-variety.

Problem 0.33. Show that:
(1) For any F-variety $\underline{X}$ the topological space $X = \underline{X}(F)$ is an l-space.
(2) The space $\mathbb{P}^n(F)$ is compact.
(3) For any $n > 0$ the group $GL(n, F)$ is an l-group and $GL(n, \mathcal{O})$ is an open compact subgroup of $GL(n, F)$. Moreover $GL(n, \mathcal{O})$ is a profinite group.
(4) For any F-group $\underline{G}$ the group $G = \underline{G}(F)$ is an l-group.
(5) For any regular tree T we define a topology on the group G of automorphisms of T in such way that for any finite subset X of T the subgroup $G_X \subset G$ of automorphisms of T fixing all points of X is open and shifts of G_X give a basis of the open sets on G. Show that G is an l-group.
(6) Any compact l-group is a profinite group.

Lemma 0.34. *Let G be an l-group. Then*
(1) *There exists a linear functional $\int_r$ on $\mathcal{S}(G)$ such that $\int_r r(g)f = \int_r f$ for all $f \in \mathcal{S}(G), g \in G$. Such functional is unique up to a multiplication by a scalar. Moreover we can choose a linear functional $\int_r$ to take positive values on non-zero non-negative functions.*
(2) *There exists a character $\Delta : G \to \mathbb{R}_+$ such that*

$$\int_r l(x)f = \Delta(x) \int_r f$$

for any $x \in G$ and any $f \in \mathcal{S}(G)$. In particular $\int_r \chi_{xA} = \Delta(x) \int \chi_A$ for any $x \in G$ and any compact open subset A of G where χ_A is the characteristic function of A.
(3) *The functional $f \to \int_r \Delta^{-1} f$ is left-invariant.*

Proof. To construct a linear functional $\int_r$ we fix an open compact subgroup K_0 and define

$$\int_r f := \frac{1}{|K_0/K|} \sum_{r \in K\backslash G} f(r)$$

where $K \subset K_0$ is any open subgroup such that the function f is invariant under left shifts by $k \in K$. I'll leave for you to show that $\int f$ does not depend on a choice of a subgroup $K \subset K_0$ and the map $f \to \int_r f$ defines a right-invariant linear functional which is positive on non-zero non-negative functions.

Let $K \subset K_0$ be any open subgroup and $\chi_K, \chi_{K_0} \subset \mathcal{S}(G)$ the characteristic functions of K and K_0. Since $\chi_{K_0} = \sum_{x \in K_0/K} r(x)\chi_K$ we see that $\lambda(\chi_K) = \frac{1}{|K_0/K|}\lambda(\chi_{K_0})$ for any right-invariant linear functional λ on $\mathcal{S}(G)$. This proves the uniqueness of $\int_r$.

Let us fix a right-invariant linear functional λ on $\mathcal{S}(G)$ and for any $x \in G$ consider the linear functional λ_x given by $\lambda_x(f) := \lambda(l(x)(f))$. By the construction the functional λ_x on $\mathcal{S}(G)$ is also right-invariant and therefore there exists constant $\Delta(x)$ such that $\lambda_x(f) = \Delta(x)\lambda(f)$ for all $f \in \mathcal{S}(G)$.

Since we may assume that λ is positive on non-zero non-negative functions we see that $\Delta(x) \in \mathbb{R}_+$. It is clear from the definition that the function $\Delta : G \to \mathbb{R}_+$ is a character.

The left-invariantness of the functional $f \to \int_r \Delta^{-1} f$ is clear. $\square$

Problem 0.35. (1) Let $\mathbb{G}_a = F$ be the additive group. Then there exists a Haar measure dg_a on $\mathbb{G}_a$ such that $\int_{\mathcal{P}^r} dg = q^{-r}, r \in \mathbb{Z}$.
(2) Let $\mathbb{G}_m = F^\star$ be the multiplicative group. Then $dg_m := dg_a/\|\|$ is a Haar measure on $\mathbb{G}_m$.
(3) Let $H \subset GL(2, F)$ be the group of upper triangular 2×2 matrices of the form

$$h = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$$

Then the right -invariant Haar measure on H measure is equal to $dg_m(\alpha)dg_m(\beta)dg_a(\gamma)/\|\beta\|$
(4)

$$\Delta(\begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}) = \|\alpha\|/\|\beta\|$$

Definition 0.36. (1) The function Δ is called the *modular character.*
(2) A group is *unimodular* if $\Delta \equiv 1$. In this case right -invariant measures are also left-invariant. We denote such a measure by dg and call a *Haar measure.*
(3) Let U be a unimodular locally compact l-group du a Haar measure on U and σ an automorphism of U. Then $\sigma_\star(du)$ is also a Haar measure on U and we define $mod_U(\sigma) \in \mathbb{R}_+$ by $\sigma_\star(du) = mod_U^{-1}U(\sigma)du$.

(4) If $P = M \ltimes U$ we write $mod_U(m) := mod_U(\sigma_m)$ where $\sigma_m(u) := mum^{-1}$.
(5) If a group G is compact we normalize a Haar measure dg in such a way that $\int_G dg = 1$.

Let G be a unimodular l-group with a Haar measure dg.

PROBLEM 0.37. (1) If $P = M \ltimes U$ where both M and U are unimodular then $\Delta_P(mu) = mod_U(m)$.
(2) Let K be a compact l-group, $K', K'' \subset K$ closed subgroups such that $K'K'' = K$. Then for any $f \in \mathcal{H}(K)$ we have
$\int_K f(k)dk = \int_{K'} f'(k')dk'$ where $f'(k') = \int_{K''} f(k'k'')dk''$.
(3) Find the modular character for the group of $n \times n$-upper triangular matrices with coefficients in $\mathbb{Q}_p$.
(4) Prove that the group $GL(n, F)$ is unimodular and describe the Haar measure dg on $GL(n, F)$ such that $\int_{GL(n,\mathcal{O})} dg = 1$.

Let H be an l-group and λ be a left-invariant linear functional on $\mathcal{S}(H)$. Let X be a principle homogeneous H-space [that is H is acting on an l-space $X, ((h, x) \to hx$ simply transitively]. Then H acts on the space $\mathcal{S}(X), (h, f) \to f_h, f_h(x) := f(h^{-1}x)$ and we denote by $\mathcal{L}$ the space of coinvariants. So $\mathcal{L} = \mathcal{S}(X)/\mathcal{S}_0(X)$ where $\mathcal{S}_0(X) \subset \mathcal{S}(X)$ is the span of $\{f - f_h\}, f \in \mathcal{S}(X), h \in H$.

LEMMA 0.38. (1) *The space $\mathcal{L}$ is one-dimensional*
(2) *The space $\mathcal{L}$ is canonically isomorphic to the space $\mathcal{L}_\Delta$ of functions r on X such that $r(hx) \equiv \Delta(h)r(x), h \in H, x \in X$.*

PROOF. Any point $x \in X$ defines a bijection $\phi_x : H \to X, h \to hx$ which induces an isomorphism $\phi_x^\star : \mathcal{S}(X) \to \mathcal{S}(H)$ and the first claim follows from 0.34.

Consider now the linear map $\kappa : \mathcal{S}(X) \to \mathcal{S}(X)$ given by

$$\kappa(f)(x) := \lambda(\phi_x^\star(f))$$

It is clear that κ is trivial on the subspace $\mathcal{S}_0(X) \subset \mathcal{S}(X)$ and it follows from 0.34 that the image of κ lies in $\mathcal{L}_\Delta$. Therefore κ defines an isomorphism $\mathcal{L} \to \mathcal{L}_\Delta$. □

REMARK 0.39. To give a more conceptual explanation of this result consider the group $Aut_H(X)$ of automorphisms of X which commute with the action of H. It is clear that the right shifts define a canonical isomorphism of the group H with $Aut_H(H)$. Therefore for any point $x \in X$ the isomorphism $\phi_x : H \to X$ defines an isomorphism $\tilde{\phi}_x : H \to Aut_H(X)$ and it is easy to see that for any two points $x, y \in X$ the isomorphisms $\tilde{\phi}_x, \tilde{\phi}_y : H \to Aut_H(X)$ differ by the conjugation by xy^{-1}. Since the charaqcter Δ is invariant under the conjugation we can consider it as a character of the group $Aut_H(X)$. By the definition the group $Aut_H(X)$ acts on the space $\mathcal{L}$ of coinvariants. The second part of the lemma says that the action is given by $\Delta : Aut_H(X) \to \mathbb{R}_+$.

Assume that an l-group H acts freely on an l-space X and denote by $\mathcal{S}_0(X) \subset \mathcal{S}(X)$ be the span of $\{f - f_h\}, f \in \mathcal{S}(X), h \in H$ and by $\mathcal{S}(X)_H := \mathcal{S}(X)/\mathcal{S}_0(X)$ be the space of H-coinvaraints of $\mathcal{S}(X)$. Assume also that a group G acts on $\mathcal{S}(X)$ and this action of G commutes with the natural action of H on $\mathcal{S}(X)$.

Corollary 0.40. (1) *The space $\mathcal{S}(X)_H$ is canonically isomorphic to the space $M(H\backslash X)$ of locally constant functions r on X such that $r(hx) \equiv \Delta(h)r(x), h \in H, x \in X$ and such* $\mathrm{supp}(r) \subset HC$ *for some compact $C \subset X$.*

(2) *For any H-invariant linear functional λ on $\mathcal{S}(X)$ there exists unique linear functional $\bar{\lambda}$ on $\mathcal{S}(X)_H$ such that*

$$\bar{\lambda}(\bar{f}) \equiv \lambda(f), f \in \mathcal{S}(X)$$

where $\bar{f}$ is the image of f in $\bar{\mathcal{S}}(X)$.

(3) *The group G acts naturally on $\bar{\mathcal{S}}(X)$.*

(4) *λ is a G-invariant functional on $\mathcal{S}(X)$ then $\bar{\lambda}$ is a G-invariant functional on $\mathcal{S}(X)_H$.*

Definition 0.41. Let H be a closed subgroup of an unimodulr group G and Δ_H be the modular character of H. Then the restriction of the functional $\int_r$ to the subspace $\mathcal{S}_0(X) \subset \mathcal{S}(X)$ vanishes and therefore the functional $\int_r$ defines a G-invariant linear functional $\int$ on the the space $M(H\backslash G) = \mathcal{S}(X)_H$.

Lemma 0.42. *Assume that K is an open compact subgroup of G such that $G = HK$. Then any K-invariant linear functional on $M(H\backslash G)$ is proportional to $\int$.*

Proof. Since the linear functional $\int$ on $M(H\backslash G)$ is K-invariant it is sufficient to show that all K-invariant linear functionals on $M(H\backslash G)$ are proportional.

Since $G = HK$ the restriction map defines an isomorphism of K-vector spaces $M(H\backslash G)$ and $M(H\cap K\backslash K)$ and the claim follows from the uniquness of the Haar measure. □

Problem 0.43. (1) Show that in the case when $G = SL(2, F)$ and $H \subset G$ is the subgroup of upper triangular matricies. In this case we can identify the space $M(H\backslash G)$ with the space V of locally constant functions f on $F^2 - \{0\}$ such that $f(cx, cy) = 1/\|c\| f(x, y)$ and that the functional $\int$ on V is given by

$$\int(f) = \int_Z f(x, y)dxdy$$

where $Z = \{(x, y) | \max(\|x\|, \|y\|) = 1\}$.

(2) Explain why this result follows from Lemma 0.42 if we take $K = SL(2, \mathcal{O})$.

DEFINITION 0.44. Let G be a unimodular l-group with a Haar measure dg.

(1) We denote by $\mathbb{C}(G)$ the space of $\mathbb{C}$-valued locally constant functions ϕ on G.
(2) For any $f \in \mathcal{S}(G)$ we consider fdg as a linear functional on the space $\mathbb{C}(G)$ given by
$$fdg(\phi) := \int_G f\phi, \phi \in \mathbb{C}(G)$$
(3) We denote by $\mathcal{H}(G)$ the space of linear functionals h the space $\mathbb{C}(G)$ on G of the form $h = fdg, f \in \mathcal{S}(G)$. and say that elements of $\mathcal{H}(G)$ are locally constant measures on G with compact support.
(4) Since G is a unimodular l-group the left and right regular representations $l, r : G \to Aut(\mathcal{S}(G)$ define the left and right regular representations $G \to Aut(\mathcal{H}(G)$ which we also denote by l and r.
(5) For any $h' = f'dg' \in \mathcal{H}(G'), h'' = f''dg'' \in \mathcal{H}(G'')$ we define
$$h' \Box h'' := f'(g')f''(g'')dg'dg'' \in \mathcal{H}(G' \times G'')$$
(6) We denote by $(h', h'') \to h' \star h''$ the *convolution*
$$h' \star h'' := m_\star(h' \Box h'')$$
where $m : G \times G \to G$ is the product map. In other words
$$h' \star h''(\phi) := h' \Box h''(m^\star(\phi)), \phi \in \mathbb{C}(G)$$
(7) The algebra $\mathcal{H}(G)$ is called the *Hecke algebra* of G.
(8) For any compact open subgroup $K \subset G$ we denote by $e_K \subset \mathcal{H}(G)$ the idempotent given by the Haar measure of K and write $\mathcal{H}_K(G) = e_K\mathcal{H}(G)e_K$.
(9) For any compact not necessarily open subgroup K of G we denote by $e_K \in \hat{\mathcal{K}}$ the endomorphism of $\mathcal{H}$ defined by
$$e_K(h) = m_\star(e_K \Box h)$$
In other words if $h = fdg$ then $e_K(h) = f'dg$ where $f'(x) = \int_K f(kx)$.
(10) For any $x \in G$ we denote by $\mathcal{E}_x \in \hat{\mathcal{K}}$ the endomorphism of $\mathcal{H}$ defined by $\mathcal{E}_x(h)(g) := l(x)(h), h \in \mathcal{H}(G)$ and say that $\mathcal{E}_x$ is the *delta function* at x.
(11) We denote by $\int$ the linear functional on $\mathcal{H}(G)$ given by $\int h := h(1)$.

PROBLEM 0.45. (1) If $h' = f'dg, h'' = f''dg$ then $h' \star h'' = f' \star f''dg$ where
$$f' \star f''(g) := \int_{g' \in G} f'(g')f''(g'^{-1}g)dg$$
(2) $\mathcal{H}(G)$ is an idempotented algebra.
(3) $\int f' \star f'' = \int f' \int f''$.

(4) For any $g \in G$ and any compact open subgroup $K \subset G$ there exists unique $a(g) \in \mathcal{H}_K(G)$ such that $\operatorname{supp} a(g) \subset KgK$ and $\int a(g) = 1$.
(5) The element $a(g) \in \mathcal{H}_K(G)$ depends only on the double coset $KgK \in K\backslash G/K$ and the set $\{a(g)\}, gK \in K\backslash G/K$ is a basis of $\mathcal{H}_K(G)$.
(6) For any $n \in N_G(K)$ and any $g \in G$ we have $a(ng) = a(n)a(g), a(gn) = a(g)a(n)$.
(7) Let $K, K', K'' \subset G$ be compact subgroups such that $K'K'' = K$. The $e_{K'}e_{K''} = e_K$.
(8) For any $x \in G$ and any compact subgroup $K \subset G$ we have $e_{xKx^{-1}} = \mathcal{E}_x e_K \mathcal{E}_x x^{-1}$.
(9) $\mathcal{H}(G)$ has countable dimension.

2.2. Representations of l-groups.

DEFINITION 0.46. Let G be a unimodular l-group with a Haar measure dg.

(1) A representation $\pi : G \to Aut(V)$ of the group G on a complex vector space V is *smooth* if for any vector $v \in V$ the stabilizer $St_v \subset G$ of v in G is open.
(2) We denote by $\mathcal{M}(G)$ the category of smooth representations of G.

If (π, V) is a smooth representation of G, we can give V the structure of an $\mathcal{H}(G)$-module as follows. Given $\mathcal{E} \in \mathcal{H}(G)$ and $v \in V$ we choose an open compact subgroup K such that $\mathcal{E}$ is right K-invariant and $Kv = v$. Since $\mathcal{E}$ is right K-invariant and has compact support it is a finite sum of left shifts of the χ_K

$$\mathcal{E} = \sum_{i \in I} c_i g_i \chi_K, c_i \in \mathbb{C}, g_i \in G.$$

Now we define

$$\mathcal{E}(v) := \sum_{i \in I} c_i g_i v.$$

In other words we constructed a functor $\mathcal{M}(G) \to \mathcal{M}(\mathcal{H}(G))$.

PROPOSITION 0.47. *Let G be an l-group. The functor $\mathcal{M}(G) \to \mathcal{M}(\mathcal{H}(G))$ defines an equivalence of categories*

$$\mathcal{M}(G) \equiv \mathcal{M}(\mathcal{H}(G))$$

between smooth representations of G and non-degenerate $\mathcal{H}(G)$-modules.

PROOF. Let M be a smooth $\mathcal{H}$-module and $g \in G$. We want to define a representation $\pi : G \to Aut_{\mathbb{C}}(M)$. To define $\pi(g), g \in G$ consider the automorphism $\mathcal{E}_g$ of $\mathcal{H}(G)$ given by the left shit by g. The automorphism $\mathcal{E}_g$ commutes with the action of $\mathcal{H}^{op}(G)$ and therefore belongs to $\hat{\mathcal{H}}(G)$. Since by Lemma 0.16 the algebra $\hat{\mathcal{H}}(G)$ we can define $\pi(g) := \mathcal{E}_{g_M}$.

PROBLEM 0.48. Show that

(1) The assignment $g \to \pi(g)$ defines a representation of G on M.

(2) The functor $\mathcal{M}(G) \to \mathcal{M}(\mathcal{H}(G))$ defines an equivalence of categories.

□

We will identify categories $\mathcal{M}(G)$ and $\mathcal{M}(\mathcal{H}(G))$ and write $Irr(G) := Irr_{\mathcal{H}(G)}$.

DEFINITION 0.49. Let K be a compact subgroup of G.

(1) We denote by e_K an element of $\mathrm{End}_{\mathcal{H}^{op}(G)}(\mathcal{H}(G))$ given by the left convolution with the Haar measure on K.
(2) For any smooth G-module $\pi : G \to Aut(V)$ we denote by $\pi(e_K) \in \mathrm{End}(V)$ the endomorphism defined as in Lemma 0.16.

PROBLEM 0.50. (1) For any smooth G-module V the associated $\mathcal{H}(G)$-module is non-degenerate.
(2) Let $\mathbb{C}_l(G)$ be the space of $\mathbb{C}$-valued functions on G and $X_(M)$ which are left-sided invariant under some open compact subgroup of G. Construct an isomorphism $\mathbb{C}_l(G) = \widetilde{\mathcal{H}(G)}$ where we consider $\mathcal{H}(G)$ as the left regular representation of G.
(3) Let V_1, V_2 be smooth representations of G and $<,>: V_1 \times V_2 \to \mathbb{C}$ be a G-invariant pairing. There exists unique $G \times G$-equivariant morphism $\kappa_{<,>} : V_1 \otimes V_2 \to \mathbb{C}(G)$ such that
$$\kappa_{<,>} : (v_1 \otimes v_2)(e) =< v_1, v_2 >$$
and the map $<,>\to \kappa_{<,>}$: defines a bijection between G-invariant pairings $V_1 \times V_2 \to \mathbb{C}$ and $G \times G$-equivariant morphisms $V_1 \otimes V_2 \to \mathbb{C}(G)$.
(4) For any smooth representation (π, V) of G and any $v \in V$ the map $h \to \pi(h)v$ defines an element of the space $\psi_v \in \mathrm{Hom}_G(\mathcal{S}(G), V)$. Show that for admissible representations (π, V) of G the map $\psi_V : V \to \mathrm{Hom}_G(\mathcal{S}(G), V), v \to \psi_v$ is a bijection.
(5) Consider the case $G = \mathbb{G}_a(F)$ where F is a local non-archimedian field. Is the map $\psi_{\mathcal{S}(G)}$ a bijection?

The next lemma is a statement that our Hecke algebra resembles a semisimple algebra in a crucial sense.

LEMMA 0.51. *[Separation Lemma]. Suppose that G be an unimodular l-group countable at infinity. Then for any non-zero $h \in \mathcal{H}(G)$ there exists an irreducible representation ρ of G such that $\rho(h) \neq 0$.*

REMARK 0.52. The result is true for all locally compact groups and follows immediately from the theorem of Gelfand-Milman but in the case l-groups one can give a proof which does not use Functional Analysis.

PROOF. Since G is unimodular we choose a Haar measure dg and identify $\mathcal{H}(G)$ with $S(G), f \to fdg$. Consider the map $\mathcal{H}(G) \to \mathcal{H}(G), h \to h^+$

given by $f(g)dg \to \bar{f}(g^{-1})dg$ where $\bar{f}(g)$ is the complex conjugation of $f(g)$ and define $u = h \star h^+$. Then $u = \psi dg$ where

$$\psi(r) = \int_{g \in G} f(g)\bar{f}(r^{-1}g)dg$$

Setting $r = e$, it is obvious that $\psi(e) \neq 0$.

We have shown is that $h \neq 0$ implies $u \neq 0$. So it is enough to find a representation ρ so that $\rho(u) = \rho(h)\rho(h^+) \neq 0$. Note that $u^+ = u$. Thus

$$u^2 = uu^+ = (hh^+)(hh^+)^+ \neq 0$$

and more generally that $u^n \neq 0$. Choose an idempotent e of $\mathcal{H}$ such that $e \star u \star e = u$. As follows from Lemma 0.19 it is sufficient to prove that there exists an irreducible representation ρ of the unital algebra $e\mathcal{H}e$ such that $\rho(u) \neq 0$. Since u is not nilpotent the Separation Lemma follows from Lemma 0.5. $\square$

Remark 0.53. The reduction to the case of a non-nilpotent element $h \in \mathcal{H}(G)$ is not purely algebraic since we use the notion of positivity specific for $\mathbb{R} \subset \mathbb{C}$.

Problem 0.54. Prove the Separation Lemma without the assumption that G is unimodular.

Definition 0.55. Let G be a unimodular l-group. A smooth representation π of G is *compact* if for any $v \in V$ and any open compact subgroup $K \subset G$ there exists a compact C of G such that $\pi(e_K)\pi(g)v = 0$ for $g \notin C$.

Proposition 0.56. *If smooth irreducible representation $\pi : G \to Aut(V)$ is compact then the corresponding representation $\pi : \mathcal{H}(G) \to \mathrm{End}(V)$ of the Hecke algebra $\mathcal{H}(G)$ is also compact.*

Proof. We fix a Haar measure dg on G and will identify the algebra $\mathcal{H}$ with the space $\mathcal{S}(G)$ of functions on G.

Assume that the representation π of G is compact and fix an idempotent $e \in \mathcal{H}(G)$. We want to show that the representation $\pi_e : \mathcal{H}_e \to \mathrm{End}(V_e), \mathcal{H}_e = e\mathcal{H}e, V_e = eV$ is compact. It is easy to see that it is sufficient to analyze the case when $e = e_K$ is a Haar measure of an open compact subgroup $K \subset G$. In this case we can identify the algebra $\mathcal{H}_e$ with the space of functions on $K\backslash G/K$ with finite support.

We first check that the space V_e is finite-dimensional. Fix $v \in V - 0$. Since the representation π is irreducible the space V is spanned by $\pi(g)v$. Thus the space V_e is spanned by vectors $\pi(e_K)\pi(g)v$. But by the definition of compact representation, there exists a finite subset R of $K\backslash G$ such that $\pi(e_K)\pi(g)v = 0$ for $Kg \in K\backslash G - R$. So the the space V_e is finite-dimensional.

Since the representation π of G is compact the function $\kappa(v, \lambda), v \in V_e, \lambda \in V_e^\vee$ on G defined by $\kappa(v, \lambda)(g) := \lambda(\pi(e_K)\pi(g)(v))$ has finite support. So we can define a bilinear form $\kappa : V_e \times V_e^\vee \to \mathcal{H}_e, (v, \lambda) \to \kappa(v, \lambda)$. This bilinear form defines a linear map $\kappa : V_e \otimes V_e^\vee \to \mathcal{H}_e$ and, since the space

V_e is finite-dimensional, a linear map $\kappa : \mathrm{End}(V_e) \to \mathcal{H}_e$. It is clear that the map $\kappa : \mathrm{End}(V_e) \to \mathcal{H}_e$ is a morphism of $\mathcal{H}_e \otimes \mathcal{H}_e^{op}$-modules.

LEMMA 0.57. (1) *The composition* $\pi_e \circ \kappa : \mathrm{End}(V_e) \to \mathrm{End}(V_e)$ *is not equal to* 0.
(2) *There exists a non-zero constant* $d_\pi \in \mathbb{C}$ *such that* $d_\pi(\pi_e \circ \kappa) = Id_{V_e}$.

PROOF. Let $\tilde{V}_e := Im(\kappa)$ and $\rho : \mathcal{H}_e \to \mathrm{End}(W)$ be a irreducible representation of $\mathcal{H}_e$ which is not equivalent to π_e. Then the composition $\rho \circ \kappa : \mathrm{End}(V_e) \to \mathrm{End}(W)$ is a morphism of irreducible $\mathcal{H}_e \otimes \mathcal{H}_e^{op}$-modules. Since ρ is not equivalent to π_e we see that $\rho \circ \kappa = 0$. We see that the restriction of ρ on $\tilde{V}_e$ vanishes for all irreducible representation of $\mathcal{H}_e$ not equivalent to π_e. The Lemma follows now from 0.51.

Since the composition $\pi_e \circ \kappa : \mathrm{End}(V_e) \to \mathrm{End}(V_e)$ is a non-zero endomorphism of an irreducible $\mathcal{H}_e \otimes \mathcal{H}_e^{op}$-module there exists a non-zero constant $d_\pi \in \mathbb{C}$ such that $d_\pi \pi_e \circ \kappa = Id_{V_e}$. □

□

PROBLEM 0.58. (1) Show that the constant d_π does not depend on a choice of an idempotent $e \in \mathcal{H}(G)$ as long as it is defined [that is when $V_e \neq \{0\}$].
(2) Formulate and prove the converse of the Theorem.

DEFINITION 0.59. We say that d_π is the *Formal Dimension* of a compact representation π.

REMARK 0.60. The formal dimension depends on a choice of a Haar measure dg.

PROBLEM 0.61. Assume that the group G is compact and the Haar measure dg is such that $\int_G dg = 1$. Then for any irreducible representation (π, V) of G we have $d_\pi = dim(V)$.

THEOREM 0.62. *Let* (π, V) *be an irreducible compact representation,* $K \subset G$ *an open compact subgroup such that* $V_{e_K} \neq \{0\}$ *and*

$$\mathcal{E}_{W,K} = d_\pi \kappa(Id_{V_{e_K}}) \subset \mathcal{H}_{e_K}$$

Then

(1) $\mathcal{E}_{W,K} \in \mathcal{H}_{e_K}$ *is an idempotent.*
(2) *For any smooth representation* (ρ, W) *of* G *we have a direct sum decomposition* $W = W_0 \oplus W_1$ *where* W_0 *is a multiple of* V *and* W_1 *does not have subquotients isomorphic to* V.

PROOF. The first part follows immediate from the definition of the formal dimension. To construct a decomposition we define

$$W_0 := \cup_{K'} Im(\rho(\mathcal{E}_{W,K'})), W_1 := \cap_{K'} \mathrm{Ker}(\rho(\mathcal{E}_{W,K'}))$$

where K' runs through the set of open coma-ct subgroups of G. As follows from the complete reducibility of the restriction of η on K the subspaces W_0, W_1 are G-invariant and $W = W_0 \oplus W_1$.

As follows from Lemma 0.57 any subquotient of W isomorphic to (ρ, W) is not not killed by $\mathcal{E}_{W,K}$. Hence W_1 cannot have any such subquotients isomorphic to V.

To show that W_0 is a direct sum of copies of V note that by the Lemma 0.57 , $\rho(\mathcal{E}_{W,K}) = 0$ for any irreducible not equivalent to ρ. Therefore, all irreducible subquotients of V_0 are isomorphic to (π, V) and the claim follows from 0.10. □

Definition 0.63. (1) We denote by $G^0 \subset G$ the subgroup generated by $\{K\}$ when K runs through the set of compact subgroups of G.
(2) We write $\Lambda(G) := G/G^0$.
(3) We denote the center of G by $Z(G)$.
(4) A smooth representation (π, V) of a group G is *compact modulo center* if for any $v \in V$ and any open compact subgroup $K \subset G$ there exists a compact C of G such that $\pi(e_K)\pi(g)v = 0$ for $g \notin CZ(G)$.

Problem 0.64. (1) Assume that $Z(G)G^0$ is a subgroup of finite index in G. Then an irreducible representation π of G is compact modulo center iff the restriction of π to G^0 is compact.
(2) Let $G = \mathrm{GL}(n)$. Show that
  (a) $G^0 = \{g \in G | \det g \in \mathcal{O}^*\}$.
  (b) G^0 is an open, normal subgroup of G with $\Lambda(G) = \mathbb{Z} = F^*/\mathcal{O}^*$.
  (c) $Z(G)G^0$ is an open subgroup of finite index in G.

Remark 0.65. Analogous statements are true for an arbitrary reductive group G. For example,$\Lambda(G) = \mathbb{Z}^k$ for $G = \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_k)$.

Example 1. If G is a compact group then every smooth G-module M is completely reducible, that is $M = \bigoplus W_\alpha$ where the W_α are irreducible. Thus, the representation theory is entirely controlled by the the knowledge of irreducible representations and in a simple way.

Example 2. $G = F^*$ (This is "almost" compact.) Let π be a generator for the maximal ideal in the ring of $\mathcal{O} \subset F$. Then we have a decomposition

$$F^* = \mathbb{Z}\pi \oplus \mathcal{O}^*.$$

Here $\mathcal{O}^*$ is compact and $\mathcal{M}(\mathbb{Z}) = \mathcal{M}(\mathbb{C}[t, t^{-1}])$, the category of sheaves on $\mathbb{C}^*$. Thus,

$$Irr(G) = Irr(\mathbb{Z}) \times (\mathcal{O}^*)^\vee$$

where $(\mathcal{O}^*)^\vee$ is the discrete set of characters of the compact group $\mathcal{O}^*$.

The point here is that the structure of the representations is half discreet and half continuous. Specifically, it is a discrete sum of the category of sheaves on some space. We will see that this is a typical situation.

3. The Induction and Jacquet Functors.

REMARK. The way to make an advance in representation theory is to find a way to construct representations. Practically our only tool is the induction.

3.1. Induction.

DEFINITION 0.66. Let H be a closed subgroup of G.

(1) We denote by $Res' = Res'^H_G : \mathcal{M}(G) \to \mathcal{M}(H)$ the restriction functor.
(2) For any smooth representation (ρ, V) of M we denote by $\mathrm{Ind}'(V)$ the functions $f : G \to V$ such that
   (a)
$$\{f : G \to V | f(hg) = \rho(h)f(g)\}$$
   and
   (b) There exists an open subgroup K of G such that $f(gk) \equiv f(g), g \in G, k \in K$.
(3) We define the representation $\mathrm{Ind}'(\rho)$ of G on $\mathrm{Ind}'(V)$ by right shifts
$$\mathrm{Ind}'(\rho)(g)f(x) = f(xg)$$
(4) We consider $\mathrm{Ind}'(V)$ as a functor from $\mathcal{M}(H)$ to $\mathcal{M}(G)$.
(5) We denote by $\mathrm{ind}' = \mathrm{ind}'^G_H(V)$ the subfunctor of Ind'^G_H given by
$$\mathrm{ind}'^G_H(V) = \{f \in \mathrm{Ind}'(V) | f \text{ has compact support modulo } H\}.$$

LEMMA 0.67. (1) *The functor Ind' is the right adjoint of Res' .*
(2) *If H is open then the functor $ind'$is the left adjoint of Res'.*

PROOF. (1) Given (ρ, V) and $(\pi, W) \in \mathcal{M}(G)$ we define a map $\kappa : \mathrm{Hom}_G(W, \mathrm{Ind}'(V)) \to \mathrm{Hom}_M(Res'(W), V)$ by
$$\kappa(\phi)(w) := \phi(w)(e), \phi \in Hom_G(W, \mathrm{Ind}'(V)), w \in W$$
Conversely we define a map
$$\kappa' : \mathrm{Hom}_M(Res'(W), V) \to \mathrm{Hom}_G(W, \mathrm{Ind}'(V))$$
by
$$\kappa'(\psi)(w)(g) := \psi(\pi(g)(w)), \psi \in \mathrm{Hom}_M(Res'(W), V), w \in W$$

(2) Consider a vector $e_{H,v} \in \mathrm{ind}'(V)$ given by $e_{H,v}(g) = \rho(g)v$ if $g \in H$ and $e_{H,v}(g) = 0$ if $g \in G - H$ and define a map
$$\theta : \mathrm{Hom}_G(\mathrm{ind}'(V), W) \to \mathrm{Hom}_M(V, Res'(W))$$
by $\theta(\phi)(v) := \phi(e_{H,v})$.

PROBLEM 0.68. (1) Show that κ and κ' are the inverse maps.
(2) Show that $\theta : \mathrm{Hom}_G(\mathrm{ind}'(V), W) \to \mathrm{Hom}_M(V, Res'(W))$ is a bijection.

□

As we will see it is better to replace the usual induction by the *unitary* induction. Assume that the group G is unimodular.

DEFINITION 0.69. Let H be a closed subgroup of G.

(1) We define the functor $\mathrm{Ind}_H^G : \mathcal{M}(H) \to \mathcal{M}(G)$ of the unitary induction by
$$\mathrm{Ind}(\rho) := \mathrm{Ind}'^G_H(\Delta_H^{1/2} \otimes \rho)$$
(2) We denote by ind_H^G the subfunctor of Ind_H^G of functions compact modulo H.
(3) We de define the functor $\mathrm{Res}_G^H : \mathcal{M}(H) \to \mathcal{M}(G)$ of the unitary restriction by
$$\mathrm{Res}_G^H(\pi) := \mathrm{Res}'^H_G(\pi) \otimes \Delta_H^{-1/2}$$

PROBLEM 0.70. (1) Show that the functor Res_G^H is the left adjoint to Ind_H^G.
(2) Let (ρ, W) be smooth representation of the group H and $<,>$ be an H-invariant bilinear Hermitian form on W. Then for any $f', f'' \in \mathrm{ind}_H^G(W)$ the function $< f'(g), f''(g) >$ on G belongs to the space $M(H\backslash G)$ [see 0.40] and the the bilinear form
$$[f', f''] := \int < f'(g), f''(g) >$$
defines a G-invariant bilinear Hermitian form on the space $\mathrm{ind}_H^G(W)$.

PROPOSITION 0.71. (1) *Both functors Ind_H^G and $ind_H^G$are exact.*
(2) *If $H\backslash G$ is compact, $Ind = ind$.*
(3) *If $H\backslash G$ is compact, induction maps admissible representations to admissible representations.*

PROOF. The parts (1) and (2) are obvious.

(3) Let V be an admissible representation of H and fix $K \subset G$ a compact open subgroup. Let $\{Hg_iK\}$ be a system of coset representatives for $H\backslash G/K$. By our assumption, this is a finite set. It is clear that an element, f, of $L(V)^K$ is determined by its values on the g_i. Moreover, the image of g_i must lie in the subset of V fixed by $H \cap g_iKg_i^{-1}$ which is finite dimensional since we are assuming that V is admissible. □

3.2. Jacquet Functor.

DEFINITION 0.72. . For any smooth representation (ρ, V) of G we denote by V_G the space of *coinvariants* $V/V(G)$ where $V(G)$ is the subspace spanned by vectors $\pi(g)v - v, v \in V, g \in G$ and denote by $J : V \to V_G$ the natural projection.

REMARK 0.73. When G is a finite group, it is often useful to consider the *functor of invariants* $V^G := \mathrm{Hom}_G(\mathbb{C}_G, V)$. It turns out that for non-compact l-groups this notion is not useful. However, the functor J_G of coinvariants is useful since it is often exact.

PROPOSITION 0.74. (1) *The functor J_G from the category $\mathcal{M}(G)$ to the category of vector spaces is right exact.*
(2) *If G is compact then $V(G) = \mathrm{Ker}\, e_G$ and the functor J_G is exact.*
(3) *If $G = \cup_i U_i$ is the union of an increasing family of compact groups G_i, then the functor J_G is exact and*
$V(G) = \{v \in V | \int_{g \in G_i} \rho(g) v dg = 0$ *for some* $i > 0\}$.

PROOF. (1) is obvious.
(2) We have an exact sequence

$$0 \to V(G) \to V \to V_G \to 0.$$

When G is compact, this implies that the composition $V^G \to V \to V_G$ is a bijection. But the functor $V \to V^G$ is left exact.

(3) If $G = \cup_i U_i$ then $J_G(V) = \lim J_{U_i}(V)$. But the direct image of exact functors is exact and the first part of (3) follows from (2).

Since $J_G(V) = \lim J_{U_i}(V)$ the second part of (3) follows from the equality $V(G_i) = \{v \in V | \int_{g \in G_i} \rho(g) v dg = 0 i > 0\}$ which is an immediate consequence of the complete reducibility of smooth representations of compact groups. □

DEFINITION 0.75. Let $P = M \ltimes U$ be an l-group such that both groups M and U are unimodular and $\rho : P \to Aut(V)$ be a smooth representation. Since M normalizes U we have a representation $\tilde{\rho} : M \to \mathrm{Aut}(V_U)$ such that

$$\tilde{\rho}(m) J(v) = J(\rho(m) v), v \in V$$

where $J : V \to V_U$ is the natural projection. If P is a subgroup of a unimodular group G we define the *Jacquet* functor corresponding to $P = M \ltimes U \subset G$ as the composition $r_{M,U} := \tilde{r}_{M,U} \circ Res^P_G : \mathcal{M}(G) \to \mathcal{M}(M)$.

4. Unitary Structure

Let G be an l-group and (ρ, V) a smooth representation of G.

DEFINITION 0.76. (1) A *unitary structure* on a G-module (π, V) is a positive definite, G-invariant Hermitian scalar product $Q : V \otimes V \to \mathbb{C}$.
(2) Let $P = MU$ be a parabolic subgroup of G and $(\rho, W, <, >)$ be a unitary representation of M. We define the unitary structure Q on $(\pi, V) = (i_M(\rho), i_M(W))$ by

$$Q(f_1, f_2) = \int_{x \in P \backslash G} < f_1(x), f_2(x) >$$

where the linear functional $\int$ is as in Definition 0.40.

REMARK 0.77. We do not assume that V is complete with respect to this structure.

The essential uniqueness of an invariant scalar product for irreducible representations follows from the following version of the Schur's lemma.

LEMMA 0.78. *[Schur's Lemma] Let G be a reductive F-group and V be an irreducible G-module. Then any two G-invariant unitary structures on V are proportional.*

PROOF. Let V^+ be anti-linear dual of V, that is the space of anti-linear functionals. A G-invariant Hermitian scalar product $Q : V \otimes V \to \mathbb{C}$ defines a G-equivariant semi-linear map $V \to V^+$. As V is smooth, we obtain a semi-linear map $\kappa : V \to (V^+)_{sm}$. Since V is admissible we see that V^+_{sm} is also admissible and irreducible. Therefore the G-equivariant linear map $\kappa : V \to (V^+)_{sm}$ is a bijection. It follows now from the Schur's lemma that any two non-zero G-equivariant semi-linear maps $V \to (V^+)_{sm}$ are proportional. $\square$

REMARK 0.79. There are not many ways of constructing unitary representations. The only general procedure is to find a space X with a G-action. Then G acts on $C^\infty(X)$ and we can find $V \subset L^2(X, d\mu)$. However, it is not clear how to find such X. The two natural choices are X =point which gives the trivial representation, and $X = G$ which is what we did above.

It is somehow more difficult to classify unitary representations then general ones.

LEMMA 0.80. *Any smooth admissible unitary representation* $(\pi : G\,\mathrm{Aut}(V), Q)$ *of G is completely reducible. [That is, $V = \bigoplus V_i$ where the V_i are irreducible unitary subrepresentations.]*

REMARK 0.81. The assumption of the admissibility is important.

PROOF. We want to show that for any G-invariant subspace $W \subset V$ we can find a G-invariant complement. Consider the orthogonal complement,

$$W^\perp := \{w^\perp \in V | Q(w, w^\perp) = 0, \forall w \in W\}.$$

Since the form Q is G-invariant the subspace $W^\perp \subset V$ is also G-invariant. Since Q is positive definite we have $W \cap W^\perp = 0$. It remains to check that $W + W^\perp = V$. For this it is enough to check that for any compact open subgroup $K \subset G$ we have $W^K + (W^\perp)^K = V^K$. Since [by the admissibility] the space V^K is finitely-dimensional we have $V^K = W^K \oplus (W^K)^\perp \cap V^K$. So it is sufficient to show that $(W^K)^\perp \cap V^K \subset W^\perp$. Since the group K is compact we have $W = W^K \oplus L$ where $L := \{l \in V | \int_K \pi(k)l dk = 0\}$. Then $< l, v >= 0$ for all $v \in V^K, l \in L$ and we see that $< w, v >= 0$ for all $v \in (W^K)^\perp \cap V^K, w \in W$. $\square$

COROLLARY 0.82. *Let V be an admissible unitary representation V of G such that* $\mathrm{End}_G(V) = \mathbb{C}$. *Then V is irreducible.*

This Corollary provides a method for establishing the irreducibility of some representations.

DEFINITION 0.83. (1) For any $v \in V, \lambda \in \tilde{V}$ we define *the matrix coefficient* $m_{\tilde{v},v}(g)$ as a function on G given by $m_{\tilde{v},v}(g) = \tilde{v}(\pi(g)v), v \in V, \tilde{v} \in \tilde{V}$.

(2) Assume now that (π, V) is an irreducible representation of G such that the restriction to the center $Z(G)$ is equal to χId_V where $\chi : Z(G) \to \mathbb{C}^\star$ is a unitary character. We say that V is *square integrable modulo center* if

$$\int_{G/Z} |m_{\xi,\tilde{\xi}}(g)|^2 dg < \infty.$$

PROBLEM 0.84. Let (π, V) be an irreducible representation of G which is essentially square integrable modulo center. Then

(1) (π, V) admits a unitary structure.
(2) $|m_{v,\tilde{v}}(g)|^2 \in L^1(G/Z(G))$ for all $v \in V, \lambda \in \tilde{V}$.

5. Geometry of general linear groups.

5.1. Flags. Let E be a field and V a vector space over E of dimension $n, G := \mathrm{Aut}(V)$.

DEFINITION 0.85. (1) A flag in V is a strictly increasing sequence

$$\mathcal{F} = \{\{0\} \subset V_1 \subset V_2 \subset ... \subset V_k = V\}.$$

(2) A flag $\mathcal{F}$ is *complete* if $k = n$.
(3) Given two complete flags $\mathcal{F} = \{\{0\} \subset V_1 \subset V_2 \subset ... \subset V_n = V\}$ and $\mathcal{F}' = \{\{0\} \subset V'_1 \subset V'_2 \subset ... \subset V'_n = V\}$ we define the *relative position* $w = w(\mathcal{F}', \mathcal{F})$ as a function from $[1, n]$ to $[1, n]$ such that

$$w(i) := min\{j | V_i \subset V_{i-1} + V'_j\}, 1 \leq j \leq n$$

(4) We denote by $\mathcal{B}$ the set of complete flags in V and denote by $(g, \mathcal{F}) \to g\mathcal{F}$ the natural action of the group G on $\mathcal{B}$.
(5) For any flag $\mathcal{F} = \{\{0\} \subset V_1 \subset V_2 \subset ... \subset V_k = V\}$ we denote by $P_{\mathcal{F}} \subset G$ the stabilizer of $\mathcal{F}$ and by $U_{\mathcal{F}} \subset P_{\mathcal{F}}$ the subgroup of transformations acting trivially on quotients $\bar{V}_i := V_i/V_{i-1}$.

LEMMA 0.86. *Let $\mathcal{F}, \mathcal{F}'$ be complete flags and $w := w(\mathcal{F}, \mathcal{F}')$.*

(1) *The function w from $[1, n]$ to $[1, n]$ is a permutation. In other*
(2) *Let $L_i \subset V_i, 1 \leq i \leq n$ be one-dimensional subspaces not contained in $V_{i-1} + V'_{j-1}, j := w(\mathcal{F}, \mathcal{F}')(i)$. Then for all $i, 1 \leq i \leq n$ we have*

$$V_i = \oplus_{k=1}^i L_k, V'_j = \oplus_{k=1}^i L_w^{-1}(k).$$

(3) *For any pair $(\mathcal{F}, \mathcal{F}') \in \mathcal{B}^2, g \in G$ we have*

$$w(g\mathcal{F}, g\mathcal{F}') = w(\mathcal{F}, \mathcal{F}')$$

(4) *For any pair $(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}') \in \mathcal{B}^2$ such that $w(\mathcal{F}, \mathcal{F}') = w(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}')$ there exists $g \in G$ such that $(g\mathcal{F}, g\mathcal{F}') = (\tilde{\mathcal{F}}, \tilde{\mathcal{F}}')$*

PROOF. (1) Assume that $w(\mathcal{F}', \mathcal{F})(a) = w(\mathcal{F}', \mathcal{F})(b) = j$ for some pair $1 \leq a < b \leq n$. By the defintion we can find $v_a \in V_a - V_{a-1}, v_b \in V_b - V_{b-1}$ such that $v_a, v_b \in V'_j - V'_{j-1}$. Let $\bar{v}_a, \bar{v}_b$ be images of v_a, v_b in $\bar{V}'_j := V'_j/V'_{j-1}$.

Since $\bar{v}_a \neq 0$ and $dim(\bar{V}'_j) = 1$ there exists $\lambda \in \mathbb{C}$ such that $\bar{v}_b = \lambda \bar{v}_a$. In other words $v_b - \lambda v_a \in V'_{j-1}$. But this contradicts the assumption that $w(\mathcal{F}, \mathcal{F}')(b) = j$.

(2) We prove the equality $V_i = \oplus_{k=1}^{i} L_k$ be the induction in i. The claim is obviously true for $i = 1$. By inductive assumptions the sum $\oplus_{k=1}^{i} L_k$ is contained in $V_{i-1} + L_i \subset V_i$ and properly containes V_{i-1}. Since $dim(V_i) = dim(V_{i-1}) + 1$ we see that $V_i = \oplus_{k=1}^{i} L_k$.

The proof of the equality $V'_j = \oplus_{k=1}^{i} L_w^{-1}(k)$ is completely analogous.

(3) Follows immediately from the definition.

(4) Follows from the existence of $g \in G$ such that $gL_i = \tilde{L}_i, 1 \leq i \leq n$ where $\tilde{L}_i$ are lines corresponding to the pair $(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}')$. □

Problem 0.87.
Show that $N_G(P) = P$ for any parabolic subgroup P of G where $N_G(P)$ is the normalizer of P in G.

Definition 0.88. (1) For any $w \in W$ we define

$$X(w) := \{(\mathcal{F}, \mathcal{F}') \in \mathcal{B} \times \mathcal{B} | w(\mathcal{F}, \mathcal{F}') = w\}.$$

(2) A *splitting* of V is a choice $\mathcal{SP}$ of a direct sum decomposition $V = \oplus_{i=1}^{k} W_k$.
(3) With any splitting $\mathcal{SP}$ of $V, V = \oplus_{i=1}^{k} W_k$ we associate a flags

$$\mathcal{F}(\mathcal{SP}) := \{\{0\} \subset V_1 \subset V_2 \subset ... \subset V_k = V\}$$

and

$$\bar{\mathcal{F}}(\mathcal{SP}) := \{\{0\} \subset \bar{V}_1 \subset \bar{V}_2 \subset ... \subset \bar{V}_k = V\}$$

where $V_i := \oplus_{j=1}^{i} W_i$ and $\bar{V}_i := \oplus_{j=k-i+1}^{k} W_i$.
(4) A subgroup P of G is a *parabolic* if it is equal to the stabilizer $P_{\mathcal{F}} \subset G$ of a flag $\mathcal{F}$.
(5) A subgroup B of G is a *Borel* if it is equal to the stabilizer of a complete flag $\mathcal{F}$.
(6) We say that $U_{\mathcal{F}}$ is the *unipotent radical* of $P_{\mathcal{F}}$
(7) A subgroup M of G is *Levi* if it is a stabilizer of some splitting $\mathcal{SP}$.
(8) Two parabolic subgroups P, Q of G are *opposite* if the intersection $P \cap Q$ is a Levi subgroup in both P and Q.

We choose a basis $e_1, e_2, ..., e_n$ of V and identify the group G with $GL(n, E)$ and denote by $T \subset G$ the subgroup of diagonal matrices.

Definition 0.89. (1) We denote by $\mathcal{F}_0 = \{\{0\} \subset V_1 \subset V_2 \subset ... \subset V_n = V\}$ the complete flag in V such that V_i is the span of $e_1, e_2, ..., e_i$.
(2) We denote by $\mathcal{F}_0^- = \{\{0\} \subset L_1 \subset L_2 \subset ... \subset L_n = V\}$ the complete flag in V such that L_i is the span of $e_n, e_{n-1}, ..., e_{n-k+1}$.

(3) We denote by B the stabilizer of the flag $\mathcal{F}_0$ and by B^- the stabilizer of the flag $\mathcal{F}_0^-$.

It is clear that B is the group of upper-triangular matrices and B^- is the group of lower-triangular matrices.

(4) A *partition* of n is a decomposition of the interval $[1,n] \subset \mathbb{Z}$ in a disjoint union of subsets $S_1, S_2, ..., S_r$ some of which could be empty.

(5) A partition $[1,n] = S_1 \cup S_2 \cup ... \cup S_r$ is *standard* if $S_i = [n_1 + ... + n_{i-1} + 1, n_1 + ... + n_i]$ for some $n_i \in \mathbb{N}, 1 \leq i \leq r, n = n_1 + ... + n_r$.

(6) For a partition $\theta = (S_1, S_2, ..., S_r)$ of n we denote by $\mathcal{SP}_\theta$ the splitting of $V = \oplus_{i=1}^k W_k$ where W_i is the span of $e_j, j \in S_i$.

(7) We denote by $\mathcal{F}_\theta$ the corresponding flag and by P_θ the stabilizer of $\mathcal{F}_\theta$.

(8) A *standard* parabolic subgroup of G is a subgroup of the form P_θ for some standard partition θ of n.

(9) For any partition $\theta = (S_1, S_2, ..., S_r)$ of n we denote by M_θ the stabilizer of the splitting $V = \oplus_{i=1}^k W_k$ where W_i is the span of $e_j, j \in S_i$. It is clear that is the group $M_\theta = G_1(\theta) \times ... \times G_r(\theta)$ where $G_i(\theta)$ is isomorphic to $GL(n_i, E), n_i := |S_i|$.

(10) For any permutation w we denote by $\mathcal{B}_w \subset \mathcal{B}$ the subspace of flags $\mathcal{F}$ such that $w(\mathcal{F}_0, \mathcal{F}) = w$.

(11) A *standard* Levi subgroup of G is a subgroup of the form M_θ for a standard partition θ of n.

(12) We write $M < G$ if M is a standard Levi subgroup of G.

(13) We denote by $B \subset G$ the subgroup of upper-triangular matrices.

(14) We denote by $\delta : B \to E^\star$ the character which maps $b \in B$ into the product $\prod_{1 \leq i < j \leq n} b_{ii} b_{jj}^{-1}$ where $b_{ii}, 1 \leq i \leq n$ are the diagonal entries of b.

(15) We say that a subgroup P of a product $GL(m_1, E) \times ... \times GL(m_t, E)$ is parabolic if it has a form $P = P_1 \times ... \times P_t$ where all the subgroups $P_i \subset GL(m_i, E), 1 \leq i \leq t$ are parabolic.

(16) For any $w \in W$ we define $U_w := U \cap w B^- w^1 \subset U$.

Problem 0.90. (1) For any $w \in W$ the group U_w acts simply transitively on the set $\mathcal{B}_w$.

(2) Any parabolic subgroup of G is conjugated to unique standard parabolic subgroup.

(3) Any Levi subgroup of G is conjugated to a standard Levi subgroup $M_\theta < G$ but such θ is not necessarily unique.

(4) Any subgroup of G containing B is a standard parabolic subgroup of G.

(5) For any splitting $\mathcal{SP}$ $V = \oplus_{i=1}^k W_k$ we have

$$P_{\mathcal{F}(\mathcal{SP})} = M_{\mathcal{SP}} \ltimes U_{\mathcal{F}(\mathcal{SP})}.$$

(6) Let B, B' be opposite Borel subgroups of $GL(n, E)$. Then there exists $g \in GL(n, E)$ such that gBg^{-1} is the subgroup of upper-triangular matrices and $gB'g^{-1}$ is the subgroup of lower-triangular matrices.
(7)
(8) Let $P = MU, Q = NV$ be parabolic subgroups of G such that $Q \subset P$ and $N \subset M$. Then $Q_M := M \cap Q$ is a parabolic subgroup of M and $Q_M = N \ltimes V_M, V_M := V \cap M$.

Definition 0.91. Let $\theta = (S_1, ...S_r), \tau = (R_1, ...R_t)$ be two partitions of n and fix $i, 1 \leq i \leq r$.

(1) We denote by τ_i the partition $\{S_i \cap R_j\}, 1 \leq j \leq t$ of S_i and denote by by $P_{\tau_i} \subset G_i(\theta)$ the corresponding parabolic subgroup.
(2) We write $P_\tau(\theta) := \prod_{i=1}^r P_{\tau_i} \subset M_\theta$ and write $P_\tau(\theta) = M_\tau(\theta)U_\tau(\theta)$.
(3) Completely analogously we define partitions θ_j of R_j and a subgroup $P_\theta(\tau) = M_\theta(\tau)U_\theta(\tau) \subset M_\tau$.

Problem 0.92. Show that $M_\tau(\theta) = M_\theta(\tau) = M_{\theta \wedge \tau}$ where $\theta \wedge \tau$ the partition

$$\{S_i \cap R_j\}, 1 \leq i \leq r, 1 \leq j \leq t$$

of n.

Remark 0.93. The claim make sense since Levi subgroups $M_{S_1, S_2, ..., S_r} \subset G$ do not depend on the order of the subsets $S_1, S_2, ..., S_r$.

5.2. Symmetric group.

Definition 0.94. (1) Let $W := S_n$ be the symmetric group on n letters.
(2) We consider an imbedding $w \to w_{i,j} := \delta_{w(i),j}$ of S_n into the subgroup of permutation $n \times n$-matrices and identify the symmetric group $W := S_n$ with it image in $GL(n, E)$.
(3) for any diagonal matrix $t \in T, w \in W$ we define $t^w := wtw^{-1}$.
(4) We denote by $w_0 \in S_n$ the permutation $i \leftrightarrow n - i + 1$.
(5) For any $i, 1 \leq i < n$ we denote by $s_i \in S_n, 1 \leq i < n$ the permutation $i \leftrightarrow i + 1$. We say that elements $s_i \in W, 1 \leq i < n$ are *simple reflections*.
(6) For any $w \in W$ we define

$$J_w := \{(i, j), 1 \leq i < j \leq n | w(i) > w(j)\}$$

and write $l(w) := |J_w|$.
(7) For any partition $\theta = (S_1, ..., S_r)$ of n we denote by $W_\theta \subset S_n$ the stabilizer of the partition θ. It is clear that $W_\theta = \prod_{i=1}^r W_i(\theta)$ where $W_i(\theta)$ is the group of permutations of the set S_i.
(8) We now define the Bruhat order on the symmetric group S_n. For any permutations $w : i \to a_i, w' : i \to a'_i, 1 \leq i \leq n$ we say that w' is a reduction of w if the sequence $\{a'_i\}$ can be obtained from the

sequence $\{a_i\}$ by interchanging $a_i \leftrightarrow a_j$ for some $i, 1 \leq i < n, j > i$ $a_i > a_j$. Define $w' < w$ if w' can be obtained from w by a sequence of reductions.

(9) For any $w \in W$ we define $W_{\leq w} := \{w' \in W | w' \leq w\}$.

PROBLEM 0.95. (1) The group S_n is generated by the set of simple reflections $\{s_i\}, 1 \leq i < n$.

(2) For any $w \in S_n$ and any decomposition $w = s_{i_1}...s_{i_l}$ in a product of simple reflections we have $l \geq l(w)$.

(3) For any $w \in S_n$ there exists a decomposition $w = s_{i_1}...s_{i_l}$ in a product of simple reflections such that $l = l(w)$. In this case we say that $w = s_{i_1}...s_{i_l}$ is a *reduced* decomposition of w.

(4) Let $w = s_{i_1}...s_{i_l} \in W$ be a reduced decomposition and w' is obtained by omitting some factors in this decomposition. Then $w' < w$.

(5) Conversely let $w', w \in W$ be such that $w' < w$ and $w = s_{i_1}...s_{i_l}$ be a reduced decomposition. Then we can obtain w' omitting some factors in this decomposition.

(6) The generators $\{s_i\}, 1 \leq i < n$ satisfy the relations
$s_i^2 = e$
$s_i s_j = s_j s_i, |i - j| > 1, 1 \leq i, j < n$ and
$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, , 1 \leq i < n - 1$.

(7) $\star$ The group S_n is defined by this set of relations.

(8) For any $t \in T, w \in W$ we have

$$\delta(t)/\delta(t^w) = \prod_{(i,j)\in J_w} a_i a_j^{-1}$$

where $a_i = t_{ii}$.

(9) If $w' < w$ then $w'w_0 > ww_0$.

DEFINITION 0.96. (1) For any $w \in W$ we define

$$X(w) := \{(\mathcal{F}, \mathcal{F}') \in \mathcal{B} \times \mathcal{B} | w(\mathcal{F}, \mathcal{F}') = w\}.$$

So $X(e) = \Delta_{\mathcal{B}}$ where $\Delta_{\mathcal{B}} \subset \mathcal{B} \times \mathcal{B}$ is the diagonal and

$$\{\mathcal{F}_0\} \times \mathcal{B}_w = \{\mathcal{F}_0\} \times \mathcal{B} \cap X_w.$$

(2) For any $i, 1 \leq i < n$ we define $\bar{X}(s_i) = X(s_i) \cup X(e)$.

(3) For any $w_1, w_2, ..., w_r \in W$ we define

$$Z(w_1, w_2, ..., w_r) = \{(\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_{r+1} \in \mathcal{B}^r | (\mathcal{F}_1, \mathcal{F}_2) \in X(w_1), (\mathcal{F}_2, \mathcal{F}_3) \in X(w_2), ..., (\mathcal{F}_r, \mathcal{F}_{r+1}) \in X($$

(4) For any $l \geq 2$ we denote by $q = q_{1,l+1} : \mathcal{B}^{l+1} \to \mathcal{B}^2$ the projection to the first and the last factors.

(5) For any simple reflection $s_i \in W$ we define $\bar{X}(s_i) := X(s_i) \cup \Delta_{\mathcal{B}}$.

(6) For any reduced decomposition $w = s_{i_1}...s_{i_l}$ we define

$$\bar{Z}(s_{i_1}, s_{i_2}, ..., s_{i_l}) = \{(\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_{l+1} \in \mathcal{B}^r | (\mathcal{F}_1, \mathcal{F}_2) \in \bar{X}(s_1), (\mathcal{F}_2, \mathcal{F}_3) \in \bar{X}(s_2), ..., (\mathcal{F}_l, \mathcal{F}_{l+1}) \in \bar{X}(s_r$$

Problem 0.97. For any $w \in W$ and a simple reflection s_i we have

(1) $Z(w, s_i) = X(ws_i)$ if $l(ws_i) > l(w)$.
(2) $Z(w, s_i) = X(ws_i) \cup X(w)$ if $l(ws_i) < l(w)$.
(3) If $l(ws_i) > l(w)$ then $W_{\leq ws_i} = W_{\leq w} \cup W_{\leq w}s_i$.

Lemma 0.98. *Show that for any reduced decomposition $w = s_{i_1}...s_{i_l}$ the restriction of q on $Z(i_1, ..., i_l)$ defines a bijection between $Z(i_1, ..., i_l)$ and $X(w)$.*

Proof. I'll explain the proof in the case when $G = GL(3), w = w_0 \in S_3$ is the longest elements and $w_0 = s_1s_2s_1$ a reduced decomposition. The general case is quite similar.

In our case

$$X(w_0) = \{\mathcal{F}^0 = (L_1^0 \subset L_2^0), \mathcal{F}' = (L_1' \subset L_2')|L_1 \not\subset L_2', L_1' \not\subset L_2\}$$

and

$$Z(s_1, s_2, s_1) = \{\mathcal{F}^0 = (L_1^0 \subset L_2^0), \mathcal{F}^1 = (L_1^1 \subset L_2^1), \mathcal{F}^2 = (L_1^2 \subset L_2^2), \mathcal{F}' = (L_1' \subset L_2')$$

such that

$$L_1^0 = L_1^1, L_2^1 = L_2^2, L_1^2 = L_1'; L_2^0 \neq L_2^1, L_1^1 \neq L_1^2, L_2^2 \neq L_2'.$$

It is easy to check that these conditions imply that $L_1 \not\subset L_2', L_1' \not\subset L_2$. So $q(Z(s_1, s_2, s_1)) \subset X(w_0)$.

Conversely for any $(\mathcal{F}^0 = (L_1^0 \subset L_2^0), \mathcal{F}' = (L_1' \subset L_2')) \in X(w_0)$ we can define $\mathcal{F}^1 = (L_1^0 \subset L_2^1 := L_1^0 \oplus L_1'$ and $\mathcal{F}^2 = (L_2^1 \cap L_2', L_2')$. Then the sequence $(\mathcal{F}^0, \mathcal{F}^1, \mathcal{F}^2, \mathcal{F}')$ belongs to $Z(s_1, s_2, s_1)$ and it is easy to check the uniqueness of such a sequence for any pair $(\mathcal{F}^0, \mathcal{F}') \in X(w_0)$. □

Proposition 0.99. *For any reduced decomposition $w = s_{i_1}...s_{i_l}$ we have*

$$q(\bar{Z}(s_{i_1}, s_{i_2}, ..., s_{i_l})) = \cup_{w' \leq w} X(w')$$

Proof. The proof is by the induction in l. For $l = 0$ the claim is clear. So assume that $w = \bar{w}s_l, l(\bar{w}) = l - 1$ and that the result is known for the reduced decomposition $\bar{w} = s_{i_1}...s_{i_{l-1}}$. By the definition

$$\bar{Z}(s_{i_1}, s_{i_2}, ..., s_{i_l}) = \{(z', \mathcal{F}) \in \bar{Z}(s_{i_1}, s_{i_2}, ..., s_{i_{l-1}}) \times \mathcal{B}|(q_l(z'), \mathcal{F}) \in \bar{X}(s_l)\}$$

As follows from the inductive assumption we have

$$q(\bar{Z}(s_{i_1}, s_{i_2}, ..., s_{i_l})) = \cup_{\bar{w}' \leq \bar{w}} Y_{\bar{w}'} \cup \cup_{\bar{w}' \leq \bar{w}} Z_{\bar{w}'}$$

where

$$Y_{\bar{w}'} = \{(x, \mathcal{F}) \subset X(\bar{w}') \times \mathcal{B}|(q_l(x), \mathcal{F}) \in X(s_l)\}$$

and

$$Z_{\bar{w}'} = \{(x, \mathcal{F}) \subset X(\bar{w}') \times \mathcal{B}|q_l(x) = \mathcal{F}\}$$

Proposition follows now from the previous Problem.

□

PROBLEM 0.100.
(1) For any standard partition θ of n and any $w \in S_n$ there exists unique $\tilde{w} \in wW_\theta$ such that $\tilde{w}(i) < \tilde{w}(j)$ for any pair $(i,j), i < j$. Moreover $l(\tilde{w}) < l(w')$ for all $w' \in wW_\theta, w' \neq \tilde{w}$.
(2) For any two standard partitions θ, θ' of n and any $w \in S_n$ there exists unique $\tilde{w} \in W_{\theta'} w W_\theta$ such that $\tilde{w}(i) < \tilde{w}(j)$ for any for any pair $(i,j), i < j$ which belong to the same part of the partition θ and

$\tilde{w}^{-1}(i) < \tilde{w}^{-1}(j)$ for any for any pair $(i,j), i < j$ which belong to the same part of the partition θ'. Moreover $l(\tilde{w}) < l(w')$ for all $w' \in W_{\theta'} w W_\theta, w' \neq \tilde{w}$.
(3) We denote by $W^{\theta,\theta'} \subset W$ the set of shortest elements in two-sided classes $W_{\theta'} w W_\theta$.
(4) For any standard Levi subgroup $M < G$ we define $W_M = N_M(T)/T$.

DEFINITION 0.101. Let M, M' be a pair of standard Levi subgroups of G corresponding to standard partitions θ, θ' of n. We define

(1) $W(M,\star) := \{w \in W |$ such that $w(M)$ is a standard Levi subgroup$\}$. It is clear that the subset $W(M,\star)$ of W is right W_M-invariant.
(2) $l_G(M) := |(M,\star)/W_M|$.
(3) If $M' < G$ is another standard Levi subgroups we write $W(M,M') := \{w \in W | w(M) = M'\}$ where $w(M) := w^{-1} M w$ and say that standard Levi subgroups M, M' are *associated* if $W(M,M') \neq \emptyset$. In this case we write $M \sim M'$.
(4) We write $W^{M,N} := W^{\theta,\theta'} \subset W$.

EXAMPLE 0.102. Let $G = GL(3,F)$.

(1) G has four standard parabolic subgroups B, P, Q and G where

$$P = \begin{vmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & 0 & \star \end{vmatrix}$$

$$Q = \begin{vmatrix} \star & \star & x \\ 0 & \star & \star \\ 0 & \star & \star \end{vmatrix}$$

(2) G has four standard Levi subgroups T, M_P, M_Q, G where T is the subgroup of diagonal matricies,

$$M_P = \begin{vmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & \star \end{vmatrix}$$

$$M_Q = \begin{vmatrix} \star & 0 & 0 \\ 0 & \star & \star \\ 0 & \star & \star \end{vmatrix}$$

(3) The Levi subgroups M_P and M_Q are associated.

(4) $W_G = S_3, W_{M_P} = \{e, s_1\}, W_{M_Q} = \{e, s_1\}$ where S_n is the symmetric group on n letters and $s_i \in S_n, 1 \leq i < n$ are the permutations $i \leftrightarrow i+1$.
(5) $W(M_P, M_Q) = W_{M_P} s_2 s_1$.
(6) $W(M_0, \star) = S_3, W(M_P, \star) = W_{M_P} \cup W_{M_P} s_2 s_1$.
(7) $l_G(M_0) = 6, l_G(M_P) = l_G(M_Q) = 2$.

Problem 0.103. (1) The imbedding $W \hookrightarrow GL(n, E)$ induces a bijection $W \to B\backslash GL(n, F)/B$.
(2) For any pair θ, θ' of standard partitions of n the imbedding $W \hookrightarrow GL(n, E)$ induces a bijection

$$W_{\theta'}\backslash W/W_\theta \to P_{\theta'}\backslash G/P_\theta.$$

(3) For any two standard parabolic subgroups P, Q of G s θ, θ' of n the set $W^{\theta,\theta'} \subset W$ is equal to the set of elements $w \in W$ such that $w(M_\theta \cap B) \subset B$ and $w^{-1}(M_{\theta'} \cap B) \subset B$ where as before $w(X) := w^{-1}Xw$ for any subset X of G.
(4) Show that $W^{M_P,M_Q} = \{e, s_2 s_1\}$ where we use notation from the previous example.

Consider now the case when E is a local field. Then $\mathcal{B}$ is a compact topological space.

Definition 0.104. For any $w \in W = S_n$ we denote by $\bar{X}(w)$ the closure of the $X(w)$ in $\mathcal{B} \times \mathcal{B}$.

Problem 0.105. (1) The closure $\bar{X}(s_i)$ of $X(s_i)$ in $\mathcal{B}^2$ is equal to the union $X(s_i) \cup X(e)$ for all $i, 1 \leq i < n$.
(2) Let $q_1, q_2 : \bar{X}(s_i) \to \mathcal{B}$ be the restriction of natural projections $p_1, p_2 : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$. Then q_1, q_2 are fibrations with fibers $\mathbb{P}^1$.
(3) The set $Z(i_1, ..., i_l)$ is compact.
(4) For any $w \in W$ and any reduced decomposition $w = s_{i_1}...s_{i_l}$ we have $\bar{X}(w) = q(Z(i_1, ..., i_l))$.
(5) $\bar{X}(w) = \cup_{w' \leq w} X(w')$.

5.3. Mackey theory. For a finite group G we denote by $\mathcal{M}(G)$ the category of representations of G.

Definition 0.106. Let G be a finite group and P, Q subgroups of G.

(1) For any $w \in G$ we define $w(P) := w^{-1}Pw$ we denote by $R_w(P, Q)$ [or simply R_w] the functor $\mathcal{M}(P) \to \mathcal{M}(Q)$ given by the composition

$$\mathrm{Ind}_{w(P)\cap Q}^{Q} \circ \sigma \circ \mathrm{Res}_P^{P\cap w^{-1}(Q)}$$

where Ind is the functor of induction, Res is the functor of restriction and σ is the isomorphism $\mathcal{M}(w(P) \cap Q) \to \mathcal{M}(P \cap w^{-1}(Q))$ induced by the isomorphism $Ad(w)$.

(2) We denote by $\alpha_w(P,Q)$ [or simply α_w] the functorial morphism $R_w \to res_G^Q \circ ind_P^G$ such that for any representation (ρ, V) of P and $f \in T_w(V)$ we have
$\alpha_w(P,Q)(f)(g) = 0$ if $g \notin PwQ$ and
$\alpha(P,Q)(f)(pwq) = \rho(p)f(q)$
where $f \in R_w(V) = ind_P^G(V)$ is given by a function $f : G \to V$, such that $f(pg) \equiv \rho(p)f(g), g \in G, p \in P$.

LEMMA 0.107. *[Mackey] Let $w_i \in G, 1 \le i \le r$ be representatives of double cosets $P\backslash G/Q$. Then the morphism*

$$\oplus_{i=1}^r \alpha_{w_i} : \oplus_{i=1}^r R_{w_i} \to res_G^Q \circ ind_P^G$$

is an isomorphism.

PROOF. By the definition the space of the representation $res_G^Q \circ ind_P^G(V)$ consists of functions $f : G \to V$, such that $f(pg) \equiv \rho(p)f(g), g \in G, p \in P$. and the group Q acts by right shifts. The decomposition $G = \cup_{i=1}^r P w_i Q$ induces a direct sum decomposition

$$res_G^Q \circ ind_P^G(V) = \oplus_{i=1}^r R^{w_i}(V)$$

of Q-submodules where

$$R^{w_i}(V) := \{f \in res_G^Q \circ ind_P^G(V) | supp(f) \subset P w_i Q\}$$

Now we observe that the map $f \to \hat{f}, \hat{h}(q) := f(w_i q)$ defines an equivalence between Q-modules $R^{w_i}(V)$ and $R_{w_i}(V)$. $\square$

We will need a variant

EXAMPLE 0.108. Let $k = \mathbb{F}_q$ be a finite field, $G = GL(n,k)$ and $P = Q = B$ and $U_0 \subset B$ be the subgroup of unipotent upper triangular matrices. For any representation π of T we denote by $\mathrm{Inf}_T^B(\pi) \in Ob(\mathcal{M}(B))$ the composition $\pi \circ p_T$ where $p_T : B \to T$ is the projection $B \to B/U_0 = T$. Since [see Problem 0.103] $G = \cup_{w \in S_n} BwB$ we have an isomorphism

$$res_G^B \circ ind_B^G(\pi) = \oplus_{w \in S_n} \mathrm{ind}_{B \cap w(B)}^B \pi^w$$

where $\pi^w(b) := \pi(wbw^{-1})$.
Let $c_B^T : \mathcal{M}(B) \to \mathcal{M}(T)$ be the functor of U_0-invariants.

PROBLEM 0.109. For any representation ρ of T and any subgroup H of B containing T we have

$$c_B^T \circ \mathrm{ind}_H^B \circ \mathrm{Inf}_T^H(\rho) = \rho$$

We see that

$$c_B^T(\mathrm{ind}_{B \cap w(B)}^B(\pi^w)) = \pi^w$$

and therefore

$$r_G^T \circ i_T(\pi) = \oplus_{w \in S_n} \pi^w$$

where $r_T := c_B^T \circ res_G^B, i_T := \mathrm{ind}_B^G \circ p$.

Definition 0.110. Let θ be a partition of n.

(1) We denote by i_θ the functor from the category $\mathcal{M}(M_\theta)$ to $\mathcal{M}(G)$ which associates with a representation V of the group M_θ the representation $\mathrm{ind}_{P_\theta}^G(V)$ where we extend the action of the group M_θ on V to the action of P_θ through the homomorphism $P_\theta \to M_\theta$.
(2) We denote by r_θ the functor from the category $\mathcal{M}(G)$ to $\mathcal{M}(M_\theta)$ which associates with a representation V of the group G the action of the group M_θ on the subspace V^{U_θ} of U_θ-invariants.

Let θ' be another partition of n. As we know the partition $\theta' \wedge \theta$ defines a parabolic subgroup $P_{\theta'}(\theta)$ of the group M_θ.

(3) We define the functors $r_{\theta'}(\theta)$ from the category $\mathcal{M}(M_\theta)$ to $\mathcal{M}(\theta' \wedge \theta)$ and $i_{\theta'}(\theta)$ from the category $\mathcal{M}(\theta' \wedge \theta)$ to $\mathcal{M}(M'_\theta)$
in the way we defined the functors i_θ and r_θ.
(4) For any representation (π, V) of the group M_θ we denote by $\Psi_\theta^{\theta'}(V) \subset i_\theta(V)$ the subspace of functions $f : G \to V$ such that $f \in i_\theta(V)$ and $\mathrm{supp}(f) \subset P_\theta P_{\theta'}$.
(5) The group $P_{\theta'}$ acts on the space $\Psi_\theta^{\theta'}(V)$ by right translations and we denote by $i_\theta^{\theta'}(V)$ the representation of the group M_θ on the subspace of $U_{\theta'}$-invariants in $\tilde{\Psi}_\theta^{\theta'}(V)$. By the construction $i_\theta^{\theta'}(V)$ is a subrepresentation of $i_\theta(V)$.

Problem 0.111. For any $f \in i_\theta^{\theta'}(V)$ we denote by $a(f)$ the restriction of f to $M_{\theta'} \subset P_\theta P_{\theta'}$. Show that

(1) $a(f)(qm') = \pi(q)f(m')$ for all $m' \in M_{\theta'}, q \in M_{\theta'} \cap P_\theta = P_\theta(\theta')$. In other words a is a morphism from the representation $i_\theta^{\theta'}(V)$ of the group $M_{\theta'}$ to $i_{\theta'}(\theta) \circ r_{\theta'}(\theta)(V)$.
(2) The morphism $a : \Psi_\theta^{\theta'}(V) \to i_{\theta'}(\theta) \circ r_{\theta'}(\theta)(V)$ is an isomorphism.

Proposition 0.112. *We denote by $X = X(\theta, \theta')$ the set of double cosets $W_\theta \backslash W /_{\theta'}$ and choose representatives $w_x \in W$ for $x \in X$. Then we have a functorial isomorphism*

$$r_{\theta'} \circ i_\theta = \oplus_{x \in X} i_{\theta'}(\theta^{w_x}) \circ \sigma_x \circ r_{\theta' w_x^{-1}}(\theta)$$

where σ_w is the isomorphism between categories $\mathcal{M}(M(\theta^w \wedge \theta')$ and $\mathcal{M}(M(\theta \wedge \theta'^{w^{-1}})$ defined by the conjugation by $w \in W$.

Proof. The proof is completely analogous to the proof of the Mackey's lemma. $\square$

5.4. Mackey theory for local fields.

6. Representation of $GL(n, F)$.

Let F be a local non-archimedean field, $\mathcal{O} \subset F$ the ring of integers, $\mathcal{P} \subset \mathcal{O}$ the maximal ideal, t a generator of $\mathcal{P}$ and $k = \mathcal{O}/\mathcal{P}$ the residue field $k = \mathbb{F}_q$. We denote by $\nu : F^\star \to \mathbb{Z}$ the *valuation* such that $\nu(t) = 1$ and define the *norm* by $|x| = q^{-\nu(x)}$. For any $x \in \mathcal{O}$ we denote by $\bar{x}$ the image of x in k.

Let $\underline{G}$ be a reductive F-group and $G = \underline{G}(F)$. Then G is an l-group. Here is the main result.

THEOREM 0.113. *For any compact open subgroup $K \subset G$ the algebra $\mathcal{H}_K(G)$ is finitely generated as a module over it center.*

COROLLARY 0.114. *All irreducible smooth representations of G are admissible.*

PROOF. Let $\pi : G \to Aut(V)$ be a smooth irreducible representation. We want to show that for any compact open subgroup $K \subset G$ the space V^K is finite-dimensional. As follows from Proposition 0.47 and Lemma 0.19 the space V^K is an irreducible $\mathcal{H}_K(G)$-module. By the Schur's lemma the center Z_K of $\mathcal{H}_K(G)$ acts by scalars. Since the algebra $\mathcal{H}_K(G)$ is finitely generated as a module over Z_K we see that the space V^K is finite-dimensional. □

We concentrate on the case when $G = GL(n, F)$. Although, arguments in the general case are not much different from the case when $G = GL(n, F)$ they require a thorough knowledge of the structure theory of reductive groups over local fields.

6.1. Lattices.

DEFINITION 0.115. (1) Let V be a F-vector space of dimension $d < \infty$. A *lattice* of V is an $\mathcal{O}$-submodule L of V such that $L \otimes_{\mathcal{O}} F = V$.
(2) A *basis* of a lattice L is a set of vectors $l_1, ..., l_d \in L$ such that the map
$$\mathcal{O}^d \to L, (c_1, ..., c_d) \to c_1 l_1 + ... + c_d l_d$$
is a bijection.

PROBLEM 0.116. Let $L \subset V$ be a lattice, $\bar{L} := L/\mathcal{P}L$. Then
(1) $dim_k(\bar{L}) = d$.
(2) Let $l_1, ..., l_d \in L$ be elements such that the set $\bar{l}_1, ..., \bar{l}_d \in \bar{L}, \bar{l}_i := l_i + \mathcal{P}L$ is a basis of the k-vector space $\bar{L}$. Then the set $l_1, ..., l_d$ is a basis of L.
(3) Let $M \subset L$ be an $\mathcal{O}$-submodule and $W \subset V$ be the F-subspace generated by M. Then M is a lattice of W.
(4) For any $l \in L - \mathcal{P}L$ there exists a subspace W of V such that $l \notin W + \mathcal{P}L$.
(5) Let M be another lattice an $m \in M - \mathcal{P}M$. Then there exists a subspace W of V of codimension 1 such that $l \notin W + \mathcal{P}L$ and $m \notin W + \mathcal{P}M$.

LEMMA 0.117. *Let $L \subset V$ be a lattice.*
(1) *For any $l \in L - \mathcal{P}L$ and any subspace W of V of codimension 1 such that $l \notin W + \mathcal{P}L$ the map $\kappa : \mathcal{O} \oplus L \cap W \to L, (a, x) \to av + x, a \in \mathcal{O}, x \in L_W$ is a bijection.*

(2) *For any other lattice M of V there exists a basis $l_1, ..., l_d$ of L and non-zero elements $c_1, ..., c_d \in F$ such that the set $c_1 l_1, ..., c_d l_d$ is a basis of M.*

PROOF. (1) Since $l \notin W$ the map κ is an imbedding. To show the surjectivity we observe that any $x \in L$ can be uniquely written in the form $x = y + al, a \in F, y \in W$. I claim that $a \in \mathcal{O}$.

Assume that $a \notin \mathcal{O}$. Then $a^{-1} \in \mathcal{P}$ and we have $l = a^{-1}x - a^{-1}y$. But such a decomposition contradicts the assumption that $l \notin W + \mathcal{P}L$.

Since $a \in \mathcal{O}$ we see that $y = x - al \in L$. So $y \in L \cap W$.

We prove (2) by the induction in $d = dim_F(V)$. Choose $c_1 \in F$ such that $c_1 L$ contains $\mathcal{P}M$ but does not contain M. Choose now a vector $l \in L$ such that $c_1 l \notin \mathcal{P} c_1 M$. As was shown in the last problem there exists a subspace W of V of codimension 1 such that $c_1 l \notin W + \mathcal{P}M$ and $l \notin W + \mathcal{P}L$. By the inductive assumptions we can find a basis $l_2, ..., l_d$ of $L \cap W$ and elements $c'_2, ..., c'_d \in F$ such that the set $c'_2 l_2, ..., c'_d l_d$ is a basis of $M \cap W$. But it follows now from the part (1) that the set $c_1 l_1, ..., c_d l_d, c_i = c_1^{-1} c'_i, 1 < i \leq d$ is a basis of M. □

PROBLEM 0.118. For any lattice L and a complete flag $W_1 \subset W_2 \subset ... \subset W_d = V$ of V there exists a basis $l_1, ..., l_d$ of L such that $l_i \in L \cap W_i, 1 \leq i \leq d$.

6.2. The Geometry of $GL(n, F)$. Let $G = GL(n, F)$.

To describe the geometry of G we introduce a number of definitions. These definitions can [be extended to the case of an arbitrary reductive group.

DEFINITION 0.119. (1) let $K_0 := \mathrm{GL}(n, \mathcal{O}) \subset GL(n, F)$.

(2) We denote by T the diagonal group, by $B \subset G$ the subgroup of upper-triangular matrices, by $U \subset B_0$ the subgroup of unipotent matrices and by $\bar{U} \subset G$ the subgroup of lower-triangular matrices. Then $B = TU$ and $T_0 := T \cap K_0$ is the maximal compact subgroup of T.
(3) We denote by Λ the quotient $\Lambda := T/T_0$. The map from $\mathbb{Z}^n$ to Λ which associates with $(e_1, \ldots, e_n) \in \mathbb{Z}^n$ the class of the diagonal matrix with elements $a_{ii} = t^{e_i}, 1 \leq i \leq n$ provides an identification of the group Λ with $\mathbb{Z}^n$.
(4) We define

$$\Lambda^+ = \{(e_1, \ldots, e_n) | e_1 \geq e_2 \geq \cdots \geq e_n\}.$$

The subset $\Lambda^+ \subset \Lambda$ is called *the Weyl chamber*.
(5) We define

$$\Lambda^{++} = \{(e_1, \ldots, e_n) | e_1 > e_2 > \cdots > e_n\}$$

(6) We fix an imbedding $\Lambda \hookrightarrow T$ by

$$\lambda = (l_1, \dots, l_n) \mapsto \begin{pmatrix} \pi^{l_1} & & \\ & \ddots & \\ & & \pi^{l_n} \end{pmatrix}.$$

(7) For any $r > 0$ we denote by $K_r \subset K_0$ the kernel of the natural projection $p_r : \mathrm{GL}(n, \mathcal{O}) \to \mathrm{GL}(n, \mathcal{O}/\mathcal{P}^r)$. We say that $K_r, r > 0$ are *congruence subgroups* of G.
(8) For any congruence subgroup K of G we denote by $\mathcal{H}_K(K_0) \subset \mathcal{H}_K(G)$ the subset of measures supported on K_0.
(9) We fix a Haar measure dg on $GL(n,F)$ such that $\int_{K_0} dg = 1$.

Remark 0.120. To simplify notations we denote the subgroup of upper-triangular matrices by B and not by B_0 as in the definition 0.86.

Problem 0.121. Show that
(1) $G = K_0\Lambda^+K_0$. [The Cartan decomposition].
(2) $G = K_0B = BK_0$. [The Iwasawa decomposition].
(3) For any congruence subgroup $K \subset G$ and any $k', k'' \in K_0, g \in G$ we have $a(k')a(g)a(k'') = a(k'gk'')$ where $a(k) \in \mathcal{H}_K$ are as in 0.45.
(4) For any set $\{x_1, \dots, x_r\}$ of representatives for $K\backslash K_0 = K_0/K$ $\mathcal{H}_0 = \mathcal{H}_K(K_0) \subset \mathcal{H}_K(G)$ is equal to the span of $a(x_i)$.
(5) The shifts of congruence subgroups form a basis for the topology for G.

A hint. Use the Lemma 0.117 and the Problem 0.118 to prove (1) and (2).

Definition 0.122. Let $\lambda = (m_1, ..., m_n)$ where the sequence m_i decreasing but is not necessarily strictly decreasing

$$m_1 = ... = m_{i_1} > m_{i_1+1} = ... = m_{i_1+i_2} > ... > m_{i_1+i_2+...i_{r-1}+1} = ... = m_n.$$

(1) We denote by $P_\lambda \subset GL(n,F)$ the subgroup of block upper-triangular matricies with blocks of the size $n_j := m_{i_j} - m_{i_{j-1}}, 1 \leq j \leq r$.
(2) We denote by $\bar{P}_\lambda \subset GL(n,F)$ the subgroup of block lower-triangular matricies with blocks of the size $n_j, 1 \leq j \leq r$.
(3) We denote by $P_\lambda \subset GL(n,F)$ the subgroup of block diagonal matricies with blocks of the size $n_j, 1 \leq j \leq r$.
(4) We denote by U_λ the unipotent radical of P_λ and by $\bar{U}_\lambda$ the unipotent radical of $\bar{P}_\lambda$.
(5) We denote by $\Lambda^+_{M_\lambda}$ the subset of $\mu \in \Lambda^+$ such that $M_\mu = M_\lambda$.
(6) For any congruence subgroup K we define

$$K^+_\lambda = K \cap U_\lambda, K^-_\lambda = K \cap \bar{U}_\lambda, K^0_\lambda = K \cap M_\lambda.$$

Problem 0.123. Show that
(1) M_λ is a standard Levi subgroup of P_λ and of $P_{\lambda^{-1}}$ for any $\lambda \in \Lambda^+$. Moreover M_λ is the centralizer of λ in G.

(2) Any Levi subgroup is conjugated to a standard Levi subgroup $M_\lambda < G$ but such standard Levi subgroup is not necessirely unique.
(3) For each semisimple $g \in G$ the subgroups P_g and $P_{g^{-1}}$ form a pair of opposite parabolic subgroups.
(4) For any $\lambda \in \Lambda^+$
   (a) $K = K_\lambda^+ K_\lambda^0 K_\lambda^-$, $\lambda(K \cap P)\lambda^{-1} \subset K \cap P$ and $\lambda^{-1} K_\lambda^- \lambda \subset K_\lambda^-$.
   (b) $(\mathrm{Ad}\,\lambda^n)|_{K_\lambda^+} \to \{e\}$ as $n \to \infty$, and $(\mathrm{Ad}\,\lambda^{-n})|_{K_\lambda^-} \to \{e\}$ as $n \to \infty$.
   (c) $\cup_n \mathrm{Ad}(\lambda^{-n})(K_\lambda^+) = U_\lambda$.

Remark 0.124. This problem has a natural extension to the case of an arbitrary reductive group.

6.3. The structure of Hecke algebras. Let $K \subset K_0$ be a congruence subgroup. Since K_0 normalizes K we have $Kx = xK$ for all $x \in K_0$.

Let C be the span of $\{a(\lambda) | \lambda \in \Lambda^+\}$. The next proposition is key for our analysis of the structure of the algebra $\mathcal{H}_K(G)$.

Proposition 0.125. (1) *$\mathcal{H}_K(G) = \mathcal{H}_0 C \mathcal{H}_0$.*
(2) *C is a commutative, finitely generated algebra.*

Remark. This is saying that that $\mathcal{H}_K(G)$ is somehow of finite type but it is neither generated over C on the left nor on the right but rather "in the middle". It is a question whether one can use this property directly to show that $\mathcal{H}_K(G)$ is of finite over the center.

Proof. (1). By the Cartan decomposition, $G = \cup_{\lambda \in \Lambda^+} K_0 \lambda K_0$.Since $K_0 = \cup_{i=1}^r K x_i = \cup_{i=1}^r x_i K$ we have

$$G = \cup_{\lambda \in \Lambda^+, 1 \le i,j \le r} K x_i \lambda x_j K.$$

This implies that the $a(x_i \lambda x_j)$ form a basis for $\mathcal{H}_K(G)$. But we as we have seen $a(x_i \lambda x_j) = a(x_i) a(\lambda) a(x_j)$. This equality proves (1). □

To prove the part (2) of the Proposition we have to show that

$$a(\lambda)a(\nu) = a(\nu)a(\lambda)$$

for all $\lambda, \nu \in \Lambda^+$. Of course it is sufficient to show that $a(\lambda)a(\nu) = a(\lambda + \nu)$. In other words it is sufficient to show that

$$(K\lambda K)(K\nu K) = K\lambda\nu K$$

for all $\lambda, \nu \in \Lambda^+$. This equality is not trivial since the elements of Λ^+ do not normalize K. The idea is to decompose K into parts that can be moved to the right and to the left.

We will use the following notation. For any congruence subgroup K we define
$K^+ = K \cap U, K^- = K \cap \overline{U}$ and $K^0 = K \cap T$.

Lemma 0.126. (1) *$K = K^+ K^0 K^-$.*
(2) *$K = K^- K^0 K^+$.*

(3) *If* $\lambda \in \Lambda^+$ *then* $\lambda K^+ \lambda^{-1} \subset K^+$ *and* $\lambda^{-1} K^- \lambda \subset K^-$.
(4) *For any* $\lambda \in \Lambda$ *we have* $\int_{K_i \lambda K_i} dg / \int_{K_i} dg = mod_U^{-1}(\lambda)$.

Proof. (1) Since $K_j \subset K_i, j > i$ are normal subgroups and $\cap_i K_i = \{e\}$ it is sufficient to prove that for any $i > 0, k \in K_i$ there exist $k^+ \in K_i^+, k^0 \in K_i^0, k^- \in K_i^-$ such that $k \in k^+ k^0 k^- K_{i+1}$. To prove the existence of such a decomposition consider a map $\tilde{\kappa}_i : K_i \to M_n(k)$ given by $x \to \overline{\frac{x-1}{t^i}}$ where as before $y \to \bar{y}$ is the projection $\mathcal{O} \to \mathcal{O}/\mathcal{P} = k$.

Problem 0.127. Show that the map $\tilde{\kappa}_i : K_i \to M_n(k)$ defines a group isomorphism $\kappa_i : K_i K_{i+1} \to M_n(k)$.

To finish the proof of (1) it is sufficient now to observe that any matrix in $M_n(k)$ is a sum of an upper nilpotent, a lower nilpotent and a diagonal matricies.

The proof of (2) is completely analogous. The part (3) is immediate and (4) follows immediately from (1). $\square$

We can now finish the proof of the part (2) of the Proposition. We have reduced the problem to showing $K\lambda K \nu K = K \lambda \nu K$. It is clear that $K\lambda\nu K \subset K\lambda K\nu K$. On the other hand we have

$$\begin{aligned} K\lambda K \nu K &= K\lambda K^+ K^0 K^- \nu K \\ &= K(\lambda K^+ \lambda^{-1} \lambda K^0 \lambda^{-1} \lambda \nu \nu^{-1} K^- \nu) K \\ &\subset K K^+ K^0 \lambda \nu K^- K = K \lambda \nu K \end{aligned}$$

where the inclusions $\lambda K^+ \lambda^{-1} \subset K^+, \lambda K^0 \lambda^{-1} \subset K^0$ and $\nu^{-1} K^- \nu \subset K^-$ follow from the previous Lemma. $\square$

Remark. This decomposition is true only for congruence subgroups $K_r, r > 0$ but not for the group K_0.

6.4. Modules. We have shown that $\mathcal{H}_K = \mathcal{H}_0 C \mathcal{H}_0$ with C commutative and that $a(\lambda^n) = a(\lambda)^n$ for $\lambda \in \Lambda^+$. We use this information to study $\mathcal{H}_K$-modules. In this subsection we fix a congruence subgroup $K \subset G$ and $\lambda \in \Lambda^+$.

Let (π, V) be a representation of G and π_K be the associated representation of $\mathcal{H}_K$ on V^K. I'll use notations and results of the Problem 0.45.

Lemma 0.128. $\operatorname{Ker} a(\lambda)|_{V^K} = \operatorname{Ker} e_{\lambda^{-1} K_\lambda^+ \lambda}|_{V^K}$.

Proof. By the definition $a(\lambda) = e_K * \mathcal{E}_\lambda * e_K$. Using the decomposition $K = K_\lambda^+ K_\lambda^0 K_\lambda^-$ we see that $e_{K_\lambda} = e_{K_\lambda^+} * e_{K_\lambda^0} * e_{K_\lambda^-}$ and therefore

$$a(\lambda) = \mathcal{E}_\lambda * e_{\lambda^{-1} K_\lambda^+ \lambda} * e_{\lambda^{-1} K_\lambda^0 \lambda} * e_{\lambda^{-1} K_\lambda^- \lambda} e_K$$

Since $\lambda^{-1} K^0 \lambda = K^0$ and $\lambda^{-1} K^- \lambda \subset K^-$ we see that $a(\nu) = \mathcal{E}_\nu * e_{\nu^{-1} K^+ \nu} * e_K$.

Since e_K acts as the identity on V^K and $\mathcal{E}_\nu$, which acts on V as $\pi(\nu)$, is invertible we see that

$$\operatorname{Ker} a(\lambda)|_{V^K} = \operatorname{Ker} e_{\nu^{-1}K_\lambda^+\lambda}|_{V^K}$$

□

PROPOSITION 0.129. *For any* $\lambda \in \Lambda^+$ *we have*

$$\cup_n \operatorname{Ker} a(\lambda^n) \cap V^K = V(U_\lambda) \cap V^K$$

where as before $V(U_\lambda) \subset V$ *is the span of vectors* $v - \pi(u)v, v \in V, u \in U_\lambda$.

PROOF. Let $U_m := \lambda^{-m} K_\lambda^+ \lambda^m$. As we know [see Problem 0.123] $U_1 \subset U_2 \subset ... \subset U_m \subset ...$ and $U = \cup_m U_m$. So

$$\operatorname{Ker} e_{U_1} \subset \operatorname{Ker} e_{U_2} \subset \dots$$

and as follows from the Problem 0.74 we have $V(U_\lambda) = \cup_m \operatorname{Ker} e_{U_m}$. The Proposition follows now from Lemma 0.128 applied to $\nu = \lambda^n$. □

Let (ρ, V) be a representation of $G, \lambda \in \Lambda^+, P = P_\lambda = M_\lambda U_\lambda$ and K a congruence subgroup of G. To simplify notations we will write in the formulation of the next problem M instead of M_λ and U instead of U_λ. As before we denote by $J : V \to V_U$ is the projection onto coinvariants and by $\rho_M : M \to \operatorname{Aut}(V_U)$ the representation as in 0.75. We fix $\mu \in \Lambda^+$.

PROPOSITION 0.130. (1) $J(\rho(a_\mu)(v)) = \Delta^{1/2}(\mu) r_M(\mu) J(v), v \in V_K$

(2) *The image* $J(V^K) \subset V_U$ *is* $r_M(\rho)(\mu)$*-invariant.*

Assume now that the representation (ρ, V) *of* G *is admissible. Then*

(3) *The restriction of* $r_M(\rho)(\mu)$ *on* $J(V^K)$ *is invertible.*

(4) $J(V^K) = r_M(V)^{K^0}$.

PROOF. (1) follows from the defintion the representation $r_M(\rho)$.

(2) We have to show that $r_M(\rho)(\mu)(J(v)) \in J(V^K)$ for any $v \in V^K$. Let $w := \rho(\mu)(v)$. Then by (1) we have $r_M(\rho)(\mu)(J(v)) = cJ(w), c \in \mathbb{C}$. So it is sufficient to show that $J(w) \in J(V^K)$.

Let $\tilde{v} := \int_K \rho(k) w dk$. It is clear that $\tilde{v} \in V^K$. On the other hand as follows from the Problem 0.123 $xw = w$ for all $x \in K^0 K^-$ and therefore $\tilde{v} := \int_{K^+} \rho(k^+) w dk^+$. So

$$J(w) = J(\int_{K^+} \rho(k^+) w) dk^+ = J(\tilde{v}) \in J(V^K).$$

To prove (3) we observe that $r_M(\rho)(\mu)$ is an invertible transformation of the space $r_M(V)$. Therefore the restriction of $r_M(\rho)(\mu)$ to $J(V^K) \subset r_M(V)$ does not annihilate any non-zero vector. Since V is admissible the space $J(V^K)$ is finitely dimensional and therefore the restriction of $r_M(\rho)(\lambda)$ to $J(V^K)$ is invertible.

The inclusion $J(V^K) \subset (r_M(V))^{K^0}$ follows from the defintion. So for a proof of (4) it is sufficient to prove that $(r_M(V))^{K^0} \subset J(V^K)$. As follows from (3) it is sufficient to show that for any $\xi \in r_M(V)^{K^0}$ there exist $\mu \in \Lambda^+$ such that $r_M(\rho)(\mu)(\xi) \in V^K$.

To find such μ we fix a lift $v \in V^{K^0}$ of ξ. Since V is smooth there exists a congruence subgroup K' such that v is K'-invariant. Then as follows from 0.123 the vector $v' := \rho(\lambda^N)v$ is invariant with respect to $\lambda^N K'_- \lambda^{-N}$ as well as respect to K^0.

Since, [see 0.123]

$$\cup_N \lambda^N K'^- \lambda^{-N} = \overline{U}$$

there exists $N > 0$ such that $K^- \subset \lambda^N K'^- \lambda^{-N}$. Set $w := \int_K \rho(k)v'$. It is clear that $w \in V^K$ and that that w is K^--invariant. On the other hand since $\lambda k_0 = k_0 \lambda$ for all $k_0 \in K^0$ and and v' is K^0-invariant we see that w is K^0K^--invariant. Therefore

$$w := \int_K \rho(k)v' = \int_{K^+} \rho(k_+)v'.$$

On the other hand we have

$$J(\rho(k_+v')) = J(v'), k_+ \in K^+.$$

So $J(w) = J(v') = r_M(\rho)(\lambda^N)\xi$ and we see that $r_M(\rho)(\lambda^N)\xi \in J(V^K)$. $\square$

Remark 0.131. One can show that the condition of the admissibility of ρ is not necessary for the validity of the parts (3) and (4) of the Proposition.

6.5. An Application. We start with a result on representations of the group $SL(n,F)$. Define

$$\Lambda_1 := \{(m_1, \dots, m_n) \in \Lambda | m_1 + \dots + m_n = 0\}$$

and $\Lambda_1^+ := \Lambda^+ \cap \Lambda_1$. As follows from the Cartan decomposition for $GL_n(F)$ we have an analogous decomposition

$$SL(n,F) = \cup_{\lambda \in \Lambda_1} SL(n,\mathcal{O}) \lambda SL(n,\mathcal{O})$$

Definition 0.132. A representation (π, V) of a group $GL(n,F)$ or $SL(n,F)$ is *quasi-cuspidal* if $r_{M,U}(V) = \{0\}$ for all parabolic subgroups $P = MU \neq G$.

Remark 0.133.
(1) As all unipotent subgroups of $GL(n,F)$ lie in $SL(n,F)$ a representation of $GL(n,F)$ is quasi-cuspidal iff the restriction onto $SL(n,F)$ is quasi-cuspidal.
(2) Since any parabolic subgroup is conjugate to a standard one we can restate the definition by saying that $r_{M,U}(V) = \{0\}$ for unipotent radicals of standard parabolic subgroups $P \neq G$.

Lemma 0.134. *Let (π, V) be a quasi-cuspidal representation of $SL(n,F)$, $v \in V$ and K be a congruence subgroup. Then $\rho(a(\lambda))(v) = 0$ for almost all $\lambda \in \Lambda_1^+$.*

Proof. For any $i, 1 \le i < n$ we define

$$\nu_i := (n-i, n-i, ..., n-i, -i, -i, ..., -i) \in \Lambda_1^+$$

where $n-i$ appears i times and $-i$ appears $n-i$ times. Let $\Lambda_1^+(0) \subset \Lambda_1^+$ be the subsemigroup generated by $\nu_i, 1 \le i < n$.

Problem 0.135. (1) The set $\nu_i, 1 \le i < n$ generates the semigroup Λ_1^+.
(2) For any $N > 0$ there exists a finite subset $S \subset \Lambda_1^+$ such that $S + N\Lambda_1^+ = \Lambda_1^+$.

Since $a(\lambda)a(\nu) = a(\nu)a(\lambda)$ for $\lambda, \nu \in \Lambda_1^+$ [see Proposition 0.125] for a proof of the Lemma it is sufficient to show that $a(\nu_i)^N v = 0$ for any $i, 1 \le i \le n-1$ and any $v \in V$ if $N >> 0$. In other words we have to show that $\cup_n Ker(a(\nu_i^n)) = V$.

Let $P_i = M_i U_i$ be the parabolic subgroup corresponding to ν_i. Since (π, V) is quasi-cuspidal we have $V = V(U_i)$ and it follows from Proposition 0.130 that $\cup_n Ker(a(\nu_i^n)) = V$.

□

Theorem 0.136. *A representation* (π, V) *of* $SL(n, F)$ *is quasi-cuspidal if and only if it is compact.*

Proof. a) Assume that a representation (π, V) of $SL(n, F)$ is quasi-cuspidal. To prove that (π, V) is compact we have to show that for any congruence subgroup K and any $v \in V$ the function

$$SL(n, F) \to V, g \to \pi(e_K)\pi(g)v$$

has compact support. By changing K we can assume that our vector v is K-invariant. Then,

$$\pi(e_K)\pi(g)v = \pi(e_K)\pi(g)\pi(e_K)v$$

As follows from the Cartan decomposition for $SL_n(F)$ it is sufficient to show that the function

$$\Lambda_1^+ \to V, \lambda \to \pi(e_K)\pi(k_0)v$$

has finite support for any $k_0 \in K^0$. By replacing v by $\pi(k_0)v$ we see that it is sufficient to prove that the function

$$\Lambda_1^+ \to V, \lambda \to \pi(e_K)v$$

on Λ_1^+ has finite support . But this is an immediate consequence of the previous Lemma.

b) Conversely, suppose that the represention (π, V) on $SL(n, F)$ is compact. By reversing the reasoning given above, we see that the function $\lambda \mapsto \pi(a(\lambda))v$ from Λ_1^+ to V has finite support for all $v \in V$. Therefore for any non-zero $\lambda \in \Lambda_1^+, v \in V$ we have $a(\lambda^r)(v) = 0$ for $r >> 0$. it follows then from Proposition 0.130 (1) and Proposition 0.125 that $V(U_\lambda) = V$ for any non-zero $\lambda \in \Lambda_1^+$. Therefore $r_{M_\lambda, U_\lambda}(V) = \{0\}$ for for any non-zero

$\lambda \in \Lambda_1^+, v \in V$. It follows from Problem 0.123 (1) that $r_{M,U}(V) = \{0\}$ for all proper parabolic subgroups P of G. $\square$

6.6. Irreducibility implies Admissibility.

DEFINITION 0.137. A representation of G is *cuspidal* if it is both quasi-cuspidal and finitely generated.

Let $G = GL_n(F)$ and $G^0 := \{g \in G | det(g) \in \mathcal{O}^\star\}$. The same arguments as in the proof of Theorem 0.136 prove the following result.

THEOREM 0.138. *[Harish-Chandra] A representation (π, V) of G^0 is quasi-cuspidal if and only if it is compact.*

COROLLARY 0.139. *Any irreducible cuspidal representation of G is admissible.*

Proof of the Corollary. Let (ρ, W) be an irreducible cuspidal representation of G. Then W is a finitely generated G^0-module. Really since $[G : ZG^0]$ is finite, $W|_{ZG^0}$ is a finitely generated module. On the other hand since (ρ, W) is irreducible it follows from the Schur lemma that Z acts on W by multiplication by scalars. Hence W is finitely generated as a G^0-module.

By Harish-Chandra's theorem, $W|_{G^0}$ is compact. As follows from Proposition 0.56 all finitely generated compact representations are admissible. So $W|_{G^0}$ is admissible. As G^0 contains all compact subgroups, the corollary follows.$\square$

This corollary is the first step toward our goal of proving

THEOREM 0.140. *Any smooth irreducible representation of G is admissible.*

PROOF. We start with a reminder on the normalized induction and of the Jacquet Functors.

Since we want to apply the results of the previous section to the case when G is a Levi component of some larger group we consider the case when G can be a product of groups of the form $GL(m, F)$.

DEFINITION 0.141. Let $P = MU$ be a Parabolic subgroup of G.

(1) We denote by $r_{M,U} : \mathcal{M}(G) \to \mathcal{M}(M)$ the *Jacquet functor* which associates with a representation (π, V) of G the representation $r_{M,U}(\pi)$ of M on $V/V(U)$ such that
$$r_{M,U}(\pi)(m)(J(v)) := \Delta_P^{-1/2}(m)J(\rho(m)v)$$
where $J : V \to r_{M,U}(V)$ is the natural projection. If M is a standard Levi subgroup of G we often write r_M instead of $r_{M,U}$.

(2) We denote by $i_{M,U} : \mathcal{M}(M) \to \mathcal{M}(G)$ the induction functor defined as follows: given a representation $\rho : M \to Aut(V)$ extend it trivially to a representation of P on V and define $i_{M,U}(\rho) := \mathrm{ind}_P^G(\rho)$. If M is a standard Levi subgroup of G we often write i_M instead of $i_{M,U}$.

REMARK 0.142. The Iwasawa decomposition implies that $\mathrm{ind}_P^G = \mathrm{Ind}_P^G$.

Let $P = MU, Q = NV$ be parabolic subgroups of G such that $Q \subset P$ and $N \subset M$ and $Q_M := Q \cap M$. As we know Q_M is a parabolic subgroup of M and $Q_M = N \ltimes V_M, V_M := V \cap M$.

PROPOSITION 0.143. (1) *$r_{M,U}$ is left adjoint to $i_{M,U}$.*
(2) *If N is a Levi subgroup of M, then $r_{N,V_M} \circ r_{M,U} = r_{N,V}$ and $i_{M,U} \circ i_{N,V_M} = i_{N,V}$.*
(3) *$i_{M,U}$ maps admissible representations of M to admissible representations of G.*
(4) *$i_{M,U}$ and $r_{M,U}$ are exact.*
(5) *$r_{M,U}$ maps finitely generated representations of G to finitely generated representations of M.*

PROOF. The proof of (1) is contained in Problem 0.70 which is a a modification of Lemma 0.67. The part (2) is a simple verification and parts (3) and (4) follow from the equality $\mathrm{ind}_P^G = \mathrm{Ind}_P^G$ since as we already know the functor ind_P^G is exact and takes admissible representations to admissible ones.

To prove (5) consider a smooth finitely generated G-module (π, V) . As follows from the Iwasawa decomposition V is finitely generated as a P-module. Since the action of $P = MU$ on V descends to an action of $P/U = M$ on $V/V(U)$ we see $V/V(U)$ is finitely generated as a M-module. □

COROLLARY 0.144. (1) *Let (π, W) be an irreducible representation of G and $M < G$ be a standard Levi subgroup, minimal subject to the condition $r_M(\pi) \neq 0$. Then the representation $r_M(\pi)$ of M is cuspidal.*
(2) *For any irreducible representation (π, W) of G there exists a parabolic $P = MU$ and an irreducible cuspidal representation (τ, R) of M, such that W is a submodule of $i_M(R)$.*

PROOF. Let $\rho := M(\pi)$.

(1) By part (2) of the Proposition and the choice of M we have

$$r_{N,M}(\rho) = r_{N,M} \circ r_{M,G}(\pi) = r_{N,G}(\pi) = 0$$

for any proper parabolic $Q = NV$ of M. So ρ is quasi-cuspidal.

Since W is irreducible it is finitely generated. Thus, by part (4) of the Proposition, ρ is finitely generated. So it is cuspidal.

(2) Let ρ be as in (1) and τ be an irreducible quotient of ρ. Then (τ, R) is an irreducible cuspidal representation of M such that

$$\mathrm{Hom}_M(r_{N,M}(W), R) \neq \{0\}.$$

So [by the Frobenious reciprocity] we get a non-zero map $W \to i_{M,U}(R)$. As W is irreducible it is an embedding. □

Remark 0.145. This use of the adjunction property is typical. It helps to show that something is non-zero. but does not give more detailed information.

Now we can prove Theorem 0.140.

Let (π, V) be an irreducible representation of G. By Corollary we can find a parabolic $P = MU$ and an irreducible cuspidal representation (τ, R) of M, such that there exists an embedding $V \hookrightarrow i_{M,U}(L)$. By part (c) of Proposition and Corollary 0.136 the representation $i_{M,U}(L)$ is admissible. Since $V \hookrightarrow i_{M,U}(L)$ the representation V is also admissible. $\square$

Corollary 0.146. *For any irreducible representation (ρ, V) of G, a congruence subgroup K and a parabolic subgroup $P = MU$ of G we have $J(V^K) = (r_M(V))^{K \cap M}$.*

Definition 0.147. To simplify notations we write i_M instead of $i_{M,U}$ and r_M instead of $r_{M,U}$.

(1) We denote by $\kappa_M : Id_{\mathcal{M}(G)} \to i_M \circ r_M$ and $\tau_M : r_M \circ i_M \to Id_{\mathcal{M}(M)}$ the morphisms of functors coming from the adjunction in Proposition 0.231.

Problem 0.148. The morphism $\kappa_M(i_M(\pi) : i_M(\pi) \to i_M \circ r_M \circ i_M(\pi)$ is a monomorphism for all $\pi \in Ob(\mathcal{M}(M))$.

6.7. Uniform Admissiblity. As follows from Theorem 0.140 and Lemma 0.19 for any open compact subgroup K of G all irreducible representations of the algebra $\mathcal{H}_K$ are finite dimensional. However we did not yet show the existence of a bound $c(K)$ on dimensions of irreducible representations of $\mathcal{H}_K$.

Theorem 0.149. *[Uniform Admissibility] For any open compact subgroup $K \subset G$ there exists an effectively computable constant, $c = c(G, K)$, such that all irreducible representations of the algebra $\mathcal{H}_K(G)$ have dimension bounded by $c(K)$.*

Reformulation. For any irreducible representations V of G we have $\dim V^K \leq c(K)$.

The proof is based on the following result from Linear Algebra.

Proposition 0.150. *Let V be a complex vector space of dimension $m < \infty$ and $C \subset \mathrm{End}(V)$ a commutative subalgebra generated by l elements. Then*

$$\dim C \leq m^{2-\epsilon_l}, \epsilon_l := \frac{1}{2^{l-1}}$$

Proof. Since C is commutative we can decompose V into a direct sum of C-invariant subspaces V_i such that for any $c \in C$ the restriction c_i of c to V_i has a form $\lambda_i(c) Id_{V_i} + a$ where $a \in End(V_i)$ is nilpotent.

PROBLEM 0.151. Show that it is sufficient to prove the Proposition in the case when all $c \in C$ are nilpotent.

From now on we assume that all $c \in C$ are nilpotent. For any $r > 1$ we denote by $d_l(r), l \in \mathbb{Z}_+, r > 1$ the maximal dimension of commutative subalgebras of $M_{[r]}(\mathbb{C})$ generated by l nilpotent elements $c_1, ..., c_l$ where $[r]$ is the integral part of r.

PROBLEM 0.152. Show that the Proposition is implied by the following inductive result.

CLAIM 0.153. $d_l(r) \leq d_l(r - \frac{d_l(r)}{r}) + d_{l-1}(r)$.

Proof of the Claim. We may assume that $C \subset \mathrm{End}\, V, V = \mathbb{C}^n$ where $n := [r]$ and that $dim(C) = d_l(r)$. Let $I \subset \mathrm{End}(V)$ be the ideal generated by $c_i, 1 \leq i \leq l$ and $V^k := I^k(\mathbb{C}^n)$. Then

$$V = V^0 \supset V^1 \supset ... \supset V^n = \{0\}$$

Choose a subspace $L \subset V^0$ complementary to V^1 and define $m := dim(L)$.

PROBLEM 0.154. Use the equality $V^k = I^k(L) + V^{k+1}$ to show that $CL = V$.

Since $CL = V$ any $c \in C$ is determined by it restriction to L. So $dim(C) = d_l(r) \leq nm$ and therefore $m \geq d_l(r)/n$.

Let $C' \subset C$ be the subalgebra generated by $c_i, 1 < i \leq l$ and $R := c_1 C$. Then $C = C' + R$. Since c_1 maps V to V^1 the dimension of R is not greater then the dimension of the restriction of C on V^1. So

$$dim(R) \leq d_l(n - m) \leq d_l(n - d_l(r)/n)$$

On the other hand $dim(C') \leq d_{l-1}(r)$. □

We now prove the theorem.

PROOF. We know already that any irreducible representations of $\mathcal{H}_K(G)$ is finite-dimensional. To prove the Theorem it is sufficient to show that

$$dim(V) \leq |K_0/K|^{2^n}$$

for any irreducible representations (ρ, V) of $\mathcal{H}_K(G)$. In other words we have to show that any N-dimensional representations (ρ, V) of $\mathcal{H}_K(G)$ where $N > |K_0/K|^{2^n}$ is reducible.

Since $\mathcal{H}_K = \mathcal{H}_0 C \mathcal{H}_0, dim_{\mathbb{C}}(\mathcal{H}_0) = |K_0/K|$ follows from Lemma 0.125 that

$$dim(\rho(\mathcal{H}_K)) \leq |K_0/K|^2 dim_{\mathbb{C}}(\tilde{)}$$

where $\tilde{C} := \rho(C) \subset \mathrm{End}(V)$. On the other hand since $\tilde{C}$ is a commutative algebra with n generators it follows from the Propostion that $dim(\tilde{C}) \leq N^{2-1/2^{n-1}}$. Since $N^{1/2^{n-1}} > |K_0/K|^2$ we see that

$$dim_{\mathbb{C}}(\rho(\mathcal{H}_K) < dim(\mathrm{End}(V)$$

Now we appeal to the Burnside's Theorem which says that $\rho : \mathcal{H}_K(G) \to \operatorname{End} V$ is surjective for any finite-dimensional irreducible representations (ρ, V) of $\mathcal{H}_K(G)$ to see that the representation ρ is reducible.

□

Consider the subgroup $G^0 \subset G$ as before, and fix a congruence subgroup $K \subset G^0$. We know that given any irreducible cuspidal representation (ρ, V) of G^0, and $v \in V^K$, the function $g \to \rho(e_K)\rho(g)\rho(e_K)v, g \in G^0$ has compact support. We will now show how the uniform admissibility theorem can strengthen this result.

PROPOSITION 0.155. *Given $K \subset G^0 \subset G$ as above, there exists an open compact subset $\Omega \subset G^0$ such that*

$$\operatorname{supp} \rho(e_K)\rho(g)\rho(e_K)v \subset \Omega$$

for all irreducible cuspidal representations (ρ, V) of G and all $v \in V$.

PROOF. It follows from the proof of Harish-Chandra's theorem that compact representations of G^0 are exactly those for which $\lambda \mapsto \rho(a(\lambda))v$ has finite support in Λ_1^+. This is in turn equivalent to the statement that for any non-zero $\nu \in \Lambda_1^+, v \in V^K$ and any irreducible cuspidal representation (ρ, V) of G^0 the operator $\rho(a(\nu))$ on V^K is nilpotent. Since $dim(V^K) \leq c(K)$ we see that $\rho(a(\nu^{c(K)})) = 0$ for all non-zero $\nu \in \Lambda_1^+$. But there exists a finite subset S of Λ_1^+ such that any $\lambda \in \Lambda_1^+ - S$ has a form $\lambda = \nu^{c(K)}\mu, \nu \in \Lambda_1^+ - \{0\}, \mu \in \Lambda_1^+$. So $\operatorname{supp} \rho(e_K)\rho(g)\rho(e_K)v \subset K_0 S K_0$ for any irreducible cuspidal representation (ρ, V) of G and any $v \in V$. □

COROLLARY 0.156. *For any congruence subgroup K of G there are only finitely many equivalence classes of irreducible cuspidal representations of $\mathcal{H}_K(G^0)$.*

PROOF. Since the support of the matrix coefficients of the irreducible cuspidal representations must lie in $\Omega(G, K)$, the corollary follows from the following general result.

PROBLEM 0.157. The matrix coefficients of any set of pairwise non-isomorphic irreducible representations are linearly independent functions.

□

6.8. Decomposing the Categories.

DEFINITION 0.158.
1. We denote by $\operatorname{Irr}_c(G) \subset \operatorname{Irr}(G)$ be the subset of cuspidal irreducible representations.
2. We denote by $\operatorname{Irr}_i(G) \subset \operatorname{Irr}(G)$ be the set equivaence classes of irreducible representations of G which a subquotients of representations induced from proper parabolic subgroups of G.
3. $\mathcal{M}(G)_c := \{V \in \mathcal{M}(G) | JH(V) \subset Irr_c(G)\}$.
4. $\mathcal{M}(G)_i = \{V \in \mathcal{M}(G) | JH(V) \subset Irr_i(G)\}$.

Lemma 0.159. $\mathrm{Irr}_c(G) \cap \mathrm{Irr}_i(G) = \emptyset$.

Proof. It is sufficient to show that $[D] \notin JH(i_M(\rho)$ for any irreducible cuspidal representation D of G, any proper standard parabolic $M < G$ and any $\rho \in \mathcal{M}(M)$. As follows from the Harish-Chandra theorem the restriction D_0 of D on G^0 is compact. So it is sufficinet to show that $[D^0] \notin JH_{G^0}(i_M(\rho)$ for any irreducible compact representation D^0 of G^0, any proper standard parabolic $M < G$ and any $\rho \in \mathcal{M}(M)$. Since every irreducible compact representation of G^0 splits the category $\mathcal{M}(G^0)$ we have a decomposition

$$i_M(\rho) = i_M(\rho)_{D^0} \oplus i_M(\rho)^{\perp}_{D^0}$$

where $i_M(\rho)_{D^0}$ is a mulitple of D_0 and $[D^0] \notin JH_{G^0}(i_M(\rho)^{\perp}_{D^0}$. So it is sufficient to see that $\mathrm{Hom}_{G^0}(D_0, i_M(\rho)) = \{0\}$. Since $MG^0 = G$ we have

$$\mathrm{Hom}_{G^0}(D_0, i_M(\rho)) = \mathrm{Hom}_{G^0 \cap M}(r_M(D^0), \rho) = 0$$

since D is cuspidal. □

Corollary 0.160. $\mathrm{Irr}(G)$ *is a disjoint union of* $\mathrm{Irr}_c(G)$ *and* $\mathrm{Irr}_i(G)$.

Theorem 0.161.

$$\mathcal{M}(G) = \mathcal{M}(G)_c \oplus \mathcal{M}(G)_i$$

.

Proof. We start with the following result. Let V be a representation of G and $V = V_c \oplus V_i$ a direct sum decomposition into G^0-invariant subspaces the set $\mathrm{JH}_{G^0}(V_c)$ consists only of compact representations and the G^0-representation V_i has no compact subquotients.

Problem 0.162. (1) The subspaces $V_c, V_i \subset V$ are G-invariant.
(2) $V_i \in \mathcal{M}(G)_i$.

It follows now from the Harish-Chandra theorem that it is sufficient to prove that the subset $\mathrm{Irr}_c G^0 \subset \mathrm{Irr}\, G^0$ of irreducible compact representations, splits $\mathcal{M}(G^0)$. Let V be a representation of G^0. As we have seen for any congruence subgroup K there are only a finite number of irreducible cuspidal representations $D_1, ..., D_r$ of G^0 with K-invariant vectors. Since representations $D_1, ..., D_r$ are compact we have

$$V = V_{K,c} \oplus V^{\perp}_{c,K}$$

where $V_{K,c}$ is a direct sum of compact irreducible representations of G which have a non-zero K-invariant vector and $V^{\perp}_{c,K}$ does not have irreducible subquotients of this form.

Consider a decreasing sequence of congruence subgroups, $K_1 \supset K_2 \supset \dots$ such that $\cap_i K_i = \{e\}$ and define

$V_c := \cup_{K_i} V_{c,K_i}, V_i := \bigcap_{K_i} V^{\perp}_{c,K_i}$.

Obviously, $\mathrm{JH}(V_c) \subset \mathrm{Irr}_c\, G$ and $\mathrm{JH}(V_i) \cap \mathrm{Irr}_c\, G = \emptyset$.

It only remains to show that $V = V_c \oplus V_i$. Take $v \in V$ and choose a congruence subgroup K which stabilizes v. The decomposition

$$V = V_{c,K} \oplus V_{c,K}^{\perp}$$

shows that we have a decomposition $v = v_{c,K} + v'$ where $v_{c,K} \in V_{c,K}, v' \in V_{c,K}^{\perp}$. To finish the proof of the Theorem it is sufficient to show that $v' \in V_i$.

Since $v' \in V_i$ it is sufficient to show that the G-submodule $V' \subset V$ generated by v' does not contain any compact irreducible representation D without a non-zero K-invariant vector. Since D splits the category we have $V' = V'_D \oplus {V'_D}^{\perp}$ where V'_D is a multiple of D and $D \notin JH({V'_D}^{\perp})$. Since v' generates V' the projection of v' to V'_D generates V'_D. On the other hand v and therefore v' are K-invariant while D does not have non-zero K-invariant vectors. So $V'_D = \{0\}$. □

Definition 0.163. Let $M < G$ be a standard levi subgroup.

(1) For any $V \in \mathcal{M}(M)$ we denote by V_c the projection of V on the subcategory of quasi-cuspidal representations.
(2) We denote by $\kappa_{M,c}(V) : V \to i_M \circ r_M(V_c)$ the composition of $\kappa_M(V_c)$ [see 0.147] with the projection $V \to V_c$.

Lemma 0.164. *For any $V \in \mathcal{M}(G)$ the map*

$$\oplus_{M<G}\kappa_{M,c}(V) : V \to \oplus_{M<G} i_M \circ r_{M,U}(V)$$

is an imbedding.

Proof. Let $V_0 := \cap_{M<G} \operatorname{Ker} \kappa_{M,c}(V)$. It follows from Problem 0.148 that $r_M(V_0) = \{0\}$ for any standard proper parabolic subgroup M of G. So the representation of G on V_0 is quasi-cuspidal. Since $(V_0)_c = \{0\}$ we see that $V_0 = \{0\}$.

□

7. Examples of cuspidal representations.

Let $G = GL(n, F), Z = F^\star$ be the center of G and $\psi : k \to \mathbb{C}^\star$ be a non-trivial additive character.

Definition 0.165. (1) To any matrix $A \in \mathfrak{g} = M_n(k)$ we can associate a function

$$\nu_A : K_1 \to \mathbb{C}^\star, \nu(k) := \psi(Tr(A(k-1)).$$

It is clear that ν_A is a character of K_1.

(2) A matrix $A \in \mathfrak{g} = M_n(k)$ is *anisotropic* iff the characteristic polinomial $p_A(x) := det(xId - A) \in k[x]$ is irreducible.
(3) A character $\nu_A, A \in M_n(k)$ is *anisotropic* is A is anisotropic.
(4) We denote by by $Z_G(\nu_A) \subset ZK_0$ the stabilizer of $\tilde{\nu}_A$ in $Z(G)K_0$ [which make sense since K_1 is a normal subgroup of $Z(G)K_0$]. Since ν_A is a character of K_1 we have $K_1 \subset Z_G(\nu_A)$.

Problem 0.166. Show that

(1) A matrix $A \in M_n(k)$ is anisotropic iff it does not stabilize any non-trivial flag $\mathcal{F}$ in k^n.
(2) A matrix A is anisotropic iff for any proper parabolic subgroup $P = MU$ of G the restriction of ν_A on $K_1 \cap U$ is not equal to the function 1.
(3) For any $g \in G - Z(G)K_0$ there exists a standard proper parabolic $P = MU$ of G and a subgroup H of K_2 such that $gHg^{-1} \subset K_1$ and $K_1 \cap U \subset gHg^{-1}K_2$.
(4) For any congruence subgroup K_n there exists a compact $C_n \subset G$ such that for any $g \in G - Z(G)C_n$ there exists a proper standard parabolic $P = MU$ of G and a subgroup H of K_n such that $gHg^{-1} \subset K_1$ and $K_1 \cap U \subset gHg^{-1}K_2$.
(5) For any $A \in M_n(k)$ there exists a character $\chi : Z_G(\nu_A) \to \mathbb{C}^\star$ with the restriction on K_1 is equal to ν_A.
(6) For any $A \in M_n(k)$ and any unitary character $\chi : Z_G(\nu_A)K_1 \to \mathbb{C}^\star$ the induced represntation $\mathrm{ind}_{Z_G(\nu_A)}^G \chi$ has a natural G-invariant unitary structure.

Proposition 0.167. *Let $A \in M_n(k)$ be an anisotropic matrix $\nu = \nu_A$ and $\chi : Z_G(\nu_A) \to \mathbb{C}^\star$ be a character with the restriction on K_1 equal to ν_A. Then the induced representation $(\pi_{\nu,\chi}, V) := \mathrm{Ind}_{Z_G(\nu_A)K_1}^G \chi$ is cuspidal and irreducible.*

Proof. We first prove that the representation $(\pi_{\nu,\chi}, V)$ is admissible and equal to $\mathrm{ind}_{Z_G(\nu_A)}^G \chi$. It is sufficient to prove that for any $n > 0$ there exists a compact C_n such that $\mathrm{supp}(f) \in Z(G)C_n$ for all $f \in V^{K_n}$. We take C_n to be the compact set as in Problem 0.166. We have to show that for any $g \in G - Z(G)C_n$ we have $f(g) = 0$ if $f \in V^{K_n}$. Choose subgroups $P = MU$ and H as in Problem 0.166. Then for any $h \in H$ we have

$$f(g) = f(gh) = f((ghg^{-1})g) = \nu(ghg^{-1})f(g).$$

Since ν is anisotropic and $K_1 \cap U \subset gHg^{-1}K_2$ there exists $h \in H$ such that $\nu(ghg^{-1}) \neq 1$. Therefore $f(g) = 0$.

Since the representation $(\pi_{\nu,\chi}, V)$ is unitary and admissible it is sufficient to show that $\mathrm{Hom}_G(V, V) = \mathbb{C}$. [See Corollary 0.251]. As follows from the Frobenuous reciprocity we have

$$\mathrm{Hom}_G(V, V) = \mathrm{Hom}_{Z_G(\nu_A)}(\mathbb{C}_\chi, V).$$

For a proof of the inequality $dim(\mathrm{Hom}_{\tilde{Z}}(\mathbb{C}_\chi, V)) \leq 1$ it is sufficient to show that $\phi(1)(g) = 0$ for any $\phi \in \mathrm{Hom}_{Z_G(\nu_A)}(\mathbb{C}_\chi, V)$ and $g \in G - \tilde{Z}$.

Consider first the case when $g \in G - Z(G)K_0$. As follows from the part (3) of the previous problem there exists a proper parabolic subgroup $P = MU$ and a subgroup H of K_2 such that $gHg^{-1} \subset K_1$ and $gHg^{-1}K_2 \supset U \cap K_1$.

Since $\phi \in \mathrm{Hom}_{\tilde{Z}}(\mathbb{C}_\chi, V)$ we see that

$$\phi(1)(g) = \frac{1}{|K_1 \cap u/K_2 \cap U)|} \sum_{u \in \mathfrak{u}(k)} \psi(\nu(u))\phi(1)(g)$$

Since the functional ν is anisotropic the restriction of $\psi \circ \nu$ on $\mathfrak{u}$ is a non-trivial character of the group $\mathfrak{u}$ and we see that

$$\sum_{u \in \mathfrak{u}} \psi(\nu(u)) = 0$$

So $\phi(1)(g) = 0$.

Consider now the case when $g \in Z(G)K_0 - \tilde{Z}$. Since g does not belong to the stabilizer of $\tilde{\nu}$ there exists $k \in K_1$ such that $\tilde{\nu}(gkg^{-1} \neq \tilde{\nu}(k)$. Now the same arguments show that $\phi(1)(g) = 0$.

Now we prove the cuspidality of the representation $\pi := \pi_{\nu,\tilde{\chi}}$. Let $P = MU$ be a proper parabolic subgroup of G and $q : V \to r_M(V)$ the canonical projection. Consider $v \in V, v : G \to \mathbb{C}$ given by the function supported on $Z(G)K_0$ and such that $v(e) = 1$. Since the representation π is irreducible it is sufficient to show that $q(\pi(g)v) = 0$ for all $g \in G$. Since $G = PK_0$ and the projection q commutes with the action of P it is sufficient to show that $q(\pi(k)v) = 0$ for all $k \in K_0$. Since $q(\pi(u)w) = q(w)$ for all $w \in V, u \in U$ it is sufficient to show that

$$\int_{u \in U \cap K_1} \pi(u)\pi(k)v du = 0.$$

In other words it is sufficient to check that

$$\int_{u \in k^{-1}Uk \cap K_1} \pi(u)v du = 0.$$

Since $\operatorname{supp} \pi(u)v \subset Z(G)K_0$ for any $u \in k^{-1}Uk \cap K_1$ it is sufficient to show that

$$\int_{u \in k^{-1}Uk \cap K_1} \pi(u)v(e) = 0.$$

We have

$$\int_{u \in k^{-1}Uk \cap K_1} \pi(u)v(e) = \sum_{u \in \mathfrak{u}(k)} \psi(\nu^g(u))$$

Since the functional ν is anisotropic the restriction of the charaqcter $\nu^g(u)$ on $U \cap K_1$ is a non-trivial character of the group $U \cap K_1$. So $\sum_{u \in \mathfrak{u}(k)} \psi(\nu^g(u)) = 0$. $\square$

8. Cuspidal components

8.1. Relations between representations of a group and of it subgroups of finite index.

Claim 0.168. *Let G be a group, $H \subset G$ a subgroup of finite index and $\rho : G \to Aut(V)$ an irreducible representation. Then the restriction ρ_H is semisimple and of finite length. [That is ρ_H is a finite direct sum of irreducible representations].*

Proof. Let $L := \cap_{g \in G/H} gHg^{-1}$. It is clear that $L \subset G$ is a normal subgroup of finite index. We show that the restriction ρ_L is semisimple and of finite length and will leave for the reader to prove that the restriction ρ_H is also semisimple.

Since V is irreducible, V is a finitely generated $\mathbb{C}[G]$-module and, since L is of finite index, V is a finitely generated $\mathcal{C}[L]$-module. Hence the Zorn's Lemma implies the existence of an L-irreducible quotient $q : V \to W$.

Let $K \subset V$ be the kernel of q. Since $L \subset G$ is a normal subgroup we see that for every $g \in G$ the subspace $\rho(g)(K) \subset V$ is L-invariant and the quotient $V/\rho(g)(K)$ is an irreducible representation of L. The kernel of the natural map $V \to \oplus_{g \in G/L} V/\rho(g)(K)$ is G invariant, and hence [since V is irreducible] is equal to $\{0\}$. So we see that (ρ_L, V) is a subrepresentation of a finite direct sum of irreducible representations of L. Therefore (ρ_L, V) is a finite direct sum of irreducible representations of L. $\square$

Let G be a group, $H \subset G$ a normal subgroup of finite index, $\rho : G \to Aut(V)$ an irreducible representation of G and (π, W) an irreducible representation of H. Since the the restriction ρ_H is completely reducible we can write V as a direct sum $V = V_W \oplus V_W^{\perp}$ where V_W is a multiple of W and W does not appear as a subquotient of $V_W^{\perp}$. We define $\Phi := \mathrm{Hom}_H(W, V) = \mathrm{Hom}_H(W, V_W)$. It is clear that the map $\phi \otimes w \to \phi(w)$ defines an isomorphism $\Phi \otimes W \to V_W$.

Definition 0.169. (1) For any $g \in G$ we denote by $\pi^g : H \to \mathrm{Aut}(V)$ the representation $\pi^g(h) := \pi(ghg^{-1})$.
(2) We denote by $G_\pi \subset G$ the subgroup of elements $g \in G$ such that the representation π^g of H is equivalent to π.

By the definition for any $g \in G_\pi$ there exists an automorphism $A(g) \in \mathrm{Aut}(W)$ such that $\pi(g^{-1}hg) = A(g)^{-1}\pi(h)A(g), \forall h \in H$. Since W is irreducible an automorphism $A(g)$ is defined uniquely up to a multiplication by a scalar $c \in \mathbb{C}^\star$.

Problem 0.170. (1) There exist $c_{g',g''} \in \mathbb{C}^\star, g', g'' \in G$ such that $A(g'g'') = c_{g',g''}A(g')A(g'')$. In other words the map $g \to A(g)$ is a *projective representation* of the group G_π on W.
(2) The subspace V_W of V is G_π-invariant.
(3) There exist a projective representation ν of the group G_π/H on Φ such that $A(g) = \nu(g)u \otimes \pi(g)$ for all $g \in G_\pi$.
(4) Let $a : \mathrm{ind}_{G_\pi}^G(V_W) \to V$ be the morphism corresponding to the imbedding $V_W \hookrightarrow V$ under the Frobenious reciprocity

$$\mathrm{Hom}_G(\mathrm{ind}_{G_\pi}^G(V_W), V) = \mathrm{Hom}_{G_\pi}(V_W, V).$$

Then a is an imbedding.
(5) The representation τ of G_π on V_W is irreducible and ρ is equivalent to $Ind_{G_\pi}^G \tau$.
(6) The projective representation ν of G_π/H on Φ is irreducible.
(7) If G/H is cyclic then $dim(\Psi) = 1$ and we can assume that $\tau \equiv 1$.

DEFINITION 0.171. Let H be a normal subgroup of G such that the quotient group $\bar{G} := G/H$ is commutative.

(1) We denote by Ψ the group $\mathrm{Hom}(\bar{G}, \mathbb{C}^*)$ of characters of $\bar{G}$.
(2) For any irreducible representation ρ of G we define $\Psi_\rho \subset \Psi$ as the subgroup of characters ψ such that the representation $\psi \otimes \rho$ is equivalent to ρ.
(3) For any $\psi \in \Psi_\rho$ we choose an *intertwining operator* $I_\psi \in Aut_H(V)$ which defines an equivalence between ρ and $\psi \otimes \rho$.

LEMMA 0.172. *Let H be a normal subgroup of G such that the quotient group G/H is cyclic.*

(1) *The restriction ρ_H is a direct sum of distinct irreducible representations.*
(2) *The representation ρ_H of H is irreducible iff $\Psi_\rho = \{1\}$.*
(3) *There exist constants $c_{\psi',\psi''} \in \mathbb{C}^*, \psi', \psi'' \in \Psi_\rho$ such that*

$$I_{\psi'\psi''} = c_{\psi',\psi''} I_{\psi'} I_{\psi''}.$$

(4) *The the set $\{I_\psi \in \mathrm{End}_H(V)\}, \psi \in \Psi_\rho$ is a basis of $\mathrm{End}_H(V)$.*

PROOF. We use notation of Problem 0.170.
(1) follows from the equality $dim(\Psi) = 1$.
(2) Follows from the Frobenious reciprocity.
(3) Follows from the Schur lemma and
(4) is true since charaters of any finite commutative group is a basis in the space of functions on this group. □

PROBLEM 0.173. Generalize the statements of the Lemma to the case when $G = \prod_i G_i, H = \prod_i H_i, i \in I$ where G_i is a finite family of groups and $H_i \subset G_i$ a family of normal groups such that the quotients G_i/H_i is cyclic.

8.2. Relations between representations of a reductive group G and the subgroup $G^0 \subset G$. Let now G be a connected reductive group. As before we denote by $G^0 \subset G$ the subgroup generated by all compact subgroups of G. We assume that $G = \prod_i G_i i \in I$ where all the quotients G_i/G_i^0 are cyclic. Then the quotient $\Lambda(G) := G/G^0$ is isomorphic to $\mathbb{Z}^l, l \geq 0$ the natural map $Z(G)/Z(G) \cap G^0 \to \Lambda$ is an isomorphism on a subgroup $\Lambda_Z(G) \subset \Lambda(G)$ of finite index and the subgroup $Z(G)G^0 \subset G$ is of finite index.

PROBLEM 0.174. If $G = GL(n, F)$ then $G^0 = \{g \in GL(n, F)|det(g) \in \mathcal{O}^\star, \Lambda(G) = \mathbb{Z}$ and $\Lambda_Z(G) = n\mathbb{Z}$.

Definition 0.175. An *unramified character* of G is a character $\psi : G \to \mathbb{C}^*$ which is trivial on G^0. The set of unramified characters is denoted $\Psi(G)$.

Remark 0.176. Since $\Lambda(G) := G/G^0$ is isomorphic to $\mathbb{Z}^l, l \geq 0$ we see that $\Psi(G) = \mathrm{Hom}(\Lambda(G), \mathbb{C}^*) = (\mathbb{C}^*)^l$. In this way, we introduce (complex) algebraic geometry into the study of G.

Lemma 0.177. *For any irreducible representations (ρ', V) and (ρ'', V') of G the space $W := \mathrm{Hom}_{G^0}(\rho', \rho'')$ is finite-dimensional.*

Proof. Let $\chi', \chi'' : Z(G) \to \mathbb{C}^*$ be the central characters of representations (ρ', V') and (ρ'', V''). If the restrictions $\chi'_{Z(G)\cap G^0}, \chi''_{Z(G)\cap G^0} : Z(G \cap G^0) \to \mathbb{C}^*$ differ then $W = \{0\}$. So we can assume that $\chi'_{Z(G)\cap G^0} = \chi''_{Z(G)\cap G^0}$. In this case the ratio χ'/χ'' is a character of the group $Z(G)/Z(G)\cap G^0$. As well known there exists a character $\chi : G/G^0 = \Lambda \to \mathbb{C}^*$ such that the restriction of χ on $Z(G)/Z(G) \cap G^0$ is equal to χ'/χ''. Consider now the representation $\tilde{\rho}'' := \rho'' \otimes \chi$. Since the restriction of χ on G^0 is equal to 1 the restriction of $\tilde{\rho}''$ on G^0 is equal to the restriction of ρ'' on G^0 and $\mathrm{Hom}_{G^0}(\rho', \rho'') = \mathrm{Hom}_{G^0}(\rho', \tilde{\rho}'')$. On the other hand since the central characters of representations (ρ', V') and $(\tilde{\rho}'', V'')$ coinside we have

$$W = \mathrm{Hom}_{G^0}(\rho', \tilde{\rho}'') = \mathrm{Hom}_{Z(G)G^0}(\rho', \tilde{\rho}''))$$

and it follows from Lemma 0.168 that $dim(W) < \infty$. $\square$

Proposition 0.178. *Let (ρ, V) and (ρ', V') be irreducible representations of G. The following conditions are equivalent*

(1) *The representations $\rho|_{G^0}$ and $\rho'|_{G^0}$ of the group G^0 are equivalent.*
(2) $\mathrm{JH}(\rho|_{G^0}) \cap \mathrm{JH}(\rho'|_{G^0}) \neq \emptyset$.
(3) $\rho' = \psi\rho$ *for some unramified character* $\psi \in \Psi$.

Proof. (1)The implications $\Rightarrow$(2) and (3)$\Rightarrow$(1) are obvious. Thus, it is enough to show that (2)$\Rightarrow$(3).

Let $W := \mathrm{Hom}_{G^0}(V, V')$. As follows from the previous Lemma the space W is finite-dimensional and the condition (2) implies that $W \neq \{0\}$. We define a representation τ of G on W by

$$\tau(g)f = \rho'(g)f\rho(g)^{-1}g \in G, f \in W.$$

By the definition of W, the restriction $\tau|_{G^0}$ is the identity. Thus, we may think of τ as a representation of the group $\Lambda(G)$ on W. Since the group Λ is commutative and the space W is finite-dimensional there exists an eigenvector $h \in W, h \neq 0$. In other words $\tau(g)h = \psi(g)h$ for all $g \in G$ for some character ψ of G.

Consider h as a linear map from V to V'. Then h intertwines the $(\psi\rho, V)$ with (ρ', V'). As both representations are irreducible, we see that $\rho' = \psi\rho$. $\square$

DEFINITION 0.179. (1) We define an action of the complex the algebraic group $\Psi(G)$ on the set $\operatorname{Irr} G$ by
$$(\psi, \rho) \to \psi \otimes \rho$$
(2) For any $\rho \in \operatorname{Irr} G$ we define
$\Psi_\rho := \{\psi \in |\Psi(G)| \rho \otimes \psi \text{ is equivalent to } \rho\}$.

LEMMA 0.180. *The subgroup Ψ_ρ is finite for all $\rho \in \operatorname{Irr} G$.*

PROOF. It is clear that for any $\psi \in \Psi_\rho$ the restriction of ψ on the center Z is trivial. So Ψ_ρ is a subgroup of characters of the finite quotient group Λ/Λ_Z. □

LEMMA 0.181. *Let (ρ, V) be a cuspidal representation of G and $(\pi, W) \in JH(\rho)$. Then there exists a G-equivariant surjection $V \to W$ and a G-equivariant injection $W \hookrightarrow V$.*

PROOF. We will only prove the existence of a G-equivariant injection $W \hookrightarrow V$. The proof of the existence of a surjection is completely analogous.

Let $\Phi := \operatorname{Hom}_{G^0}(W, V)$. As we know Φ is a finite-dimensional $\mathbb{C}$-vector space and the commutative group $\Lambda = G/G^0$ acts on Φ by
$$g(\phi)(w) := \rho^{-1}(g)\phi(\pi(g)(w))$$
Since $\pi \in JH(\rho)$ there exists a Λ-invariant subspaces $\Phi' \subsetneq \Phi''$ of Φ such that Λ acts trivially on Φ''/Φ'. Since the space Φ is a finite-dimensional there exists a non-zero Λ-invariant vector $\phi \in \Phi = \operatorname{Hom}_{G^0}(W, V)$. Since the map ϕ is Λ-invariant we see that $\phi \in \operatorname{Hom}_G(W, V)$. □

DEFINITION 0.182. (1) A *cuspidal component* of $\mathcal{M}(G)$ is an orbit of $\Psi(G)$ in the set $\operatorname{Irr}_c G$ of cuspidal irreducible representations of G.
(2) We denote by $X_c(G)$ the set of cuspidal components of $\mathcal{M}(G)$.

REMARK 0.183. It is easy to see that $\psi\rho$ is cuspidal whenever ρ is.

Each cuspidal component D has the form where by Lemma 0.180 the subgroup $\Psi_\rho \subset \Psi(G)$ is finite. Therefore, D has the structure of a connected complex algebraic variety and the action of $\Psi(G)$ on D is algebraic.

THEOREM 0.184. *Let $D \subset \operatorname{Irr} G$ be a cuspidal component. Then D splits the category $\mathcal{M}(G)$.*

PROOF. We have to show that every $V \in \mathcal{M}(G)$ can be written $V = V_D \oplus V_D^\perp$ where $\mathrm{JH}(V_D) \subset D$ and $\mathrm{JH}(V_D^\perp) \cap D = \emptyset$. By Proposition 0.178 the restrictions of irreducible objects of $\mathcal{M}(D)$ on G^0 all coincide and are finite direct sums of irreducible representations $\rho_i, 1 \leq i \leq r$ of G^0. These irreducible representations are cuspidal and therefore compact (Harish-Chandra). Since compact representations split the category $\mathcal{M}(G)$ there exists a decomposition $V = V_D \oplus V_D^\perp$ of G^0-modules, where $JH(V_D|_{G^0}) \subset \rho_1, \dots, \rho_r$ and $JH(V_D^\perp|_{G^0}) \cap \{\rho_1, \dots, \rho_r\} = \emptyset$.

It only remains to observe that this decomposition is preserved by the action of G. But this follows from the fact that G permutes the ρ_i. $\square$

Using arguments analogous to ones used in the proof of Theorem 0.161 we obtain a proof of the following result.

THEOREM 0.185. *The subset of irreducible cuspidal representations,* $\mathrm{Irr}_c\, G \subset \mathrm{Irr}\, G$*, splits* $\mathcal{M}(G)$*. In other words, any* $V \in \mathcal{M}(G)$*, can be uniquely decomposed in the direct sum* $V = V_c \oplus V_i$ *where* $\mathrm{JH}(V_c)$ *consists only of cuspidal representations and* V_i *has no cuspidal subquotients.*

DEFINITION 0.186. For any $\rho \in Ob(\mathcal{M}(G))$ we denote by ρ_{cusp} the projection to the cuspidal summand of $\mathcal{M}(G)$.

8.3. A result from the category theory.

DEFINITION 0.187. Let $\mathcal{M}$ be an abelian category with arbitrary direct sums.

(1) A functor F from $\mathcal{M}$ to the category Ab of abelian groups is *faithful* if $\mathrm{F}(f) \neq 0$ for any non-zero morphism $f \in \mathrm{Hom}_{\mathcal{M}}(X, Y)$.
(2) An object X of $\mathcal{M}$ is *compact* if the functor $\mathrm{Hom}(X, \star)$ from $\mathcal{M}$ to $Sets$ commutes with direct limits.
(3) A projective object Π in $\mathcal{M}$ is a *generator* if the functor

$$\mathrm{F}_{\Pi} : X \to \mathrm{Hom}(\Pi, X)$$

from $\mathcal{M}$ to the category Ab is faithful.

PROBLEM 0.188. (1) Let Π be a projective object in $\mathcal{M}$ such that $\mathrm{F}_{\Pi}(X) \neq 0$ for all non-zero objects X of $\mathcal{M}$. Then Π is a generator.
(2) If P is a generator of $\mathcal{M}$ then any $X \in Ob(\mathcal{M})$ can be presented as a cokernel of a morphism $f : P^S \to P^T$ where S, T are sets where $P^S := \oplus_{s \in S} P$.

LEMMA 0.189. *Let* $\mathcal{M}$ *be an abelian category ,* $\Pi \in \mathcal{M}$ *a compact, projective generator and* $A := \mathrm{End}_{\mathcal{M}}\, \Pi$*. Then the functor*

$$\alpha : \mathcal{M} \to \mathbb{C}(A), X \to Hom_{\mathcal{M}}(P, X)$$

from $\mathcal{M}$ *to the category* $\mathcal{M}(A)$ *of right* A*-modules is an equivalence of categories.*

PROOF. It is sufficient for a proof of the Lemma to construct a functor $\beta : \mathcal{M}(A) \to \mathcal{M}$ and functorial morphisms $a : \beta \circ \alpha \to Id_{\mathcal{M}}$ and $b : \alpha \circ \beta \to Id_{\mathcal{M}(A)}$ such that the morphisms $a(M) : \beta \circ \alpha(M) \to M$ and $b(X) : X \to \alpha \circ \beta(X)$ are isomorphisms for any A-module M and any $X \in Ob(\mathcal{M})$.

We start with a construction of a functor $\beta : \mathcal{M}(A) \to \mathcal{M}$. For any A-module M we define $A^M := \oplus_{m \in M} MA$ and denote by $\pi_M : A^M \to M$ the A-morphism given by $\pi_M(a_m) := \sum_{m \in M} a_m m \in M$. Let $N := \mathrm{Ker}(\pi_M) i : N \hookrightarrow A^M$ be the imbedding and $q_M := i \circ \pi_N : A^N \to A^M$. Such a map is given by a matrix $q_M = (q_{m,n}), m \in M, n \in N, q_{m,n} \in A$ and for any $n \in N$ we have $q_{m,n} = 0$ for almost all $m \in M$.

We define $\Pi^M \in Ob(\mathcal{M})$ as the direct sum $\oplus_{m\in M}\Pi$ which exists since $\mathcal{M}$ is an abelian category with arbitrary direct sums. The matrix q_M defines a morphism $\tilde{q}_M : \Pi^N \to \Pi^M$ and we define $\beta(M)$ as the cokernel of $\tilde{q}_M$.

Problem 0.190. (1) The functor β is right exact.
(2) The functor β commutes with direct sums.
(3) $\beta(A) = P$.

Now we define a functorial morphism $a : \beta \circ \alpha \to Id_{\mathcal{M}}$.

For any $X \in Ob(\mathcal{M})$ we have $\alpha(X) = \mathrm{Hom}_{\mathcal{M}}(\Pi, X)$ and therefore $\beta \circ \alpha(X)$ is the quotient of $\oplus_{\phi\in \mathrm{Hom}_{\mathcal{M}}(\Pi,X)}\Pi$ by the image of $\tilde{q}_{\mathrm{Hom}_{\mathcal{M}}(\Pi,X)}$. We define $\tilde{a} : \oplus_{\phi\in \mathrm{Hom}_{\mathcal{M}}(\Pi,X)}\Pi \to X$ by

$$\tilde{a}(\sum_{\phi\in \mathrm{Hom}_{\mathcal{M}}(\Pi,X)} p_\phi) := \sum_{\phi\in \mathrm{Hom}_{\mathcal{M}}(\Pi,X)} \phi(p_\phi) \in X$$

Problem 0.191. $\tilde{a}(X) \circ \tilde{q}_{\mathrm{Hom}_{\mathcal{M}}(\Pi,X)} = 0$ for all $X \in Ob(\mathcal{M})$.

We see that $\tilde{a}(X)$ defines a morphism $a(X) : \beta \circ \alpha(X) \to X$.

Let's now prove that the morphism $a(X) : \beta\circ\alpha \to X$ is an isomorphism. Consider first the case $X = P$. We have $\alpha(P) = A, \beta \circ \alpha(P) = P$ and I'll leave for you to check that the morphism $a(P) : P \to P$ is the identity.

Consider now the case $X = P^S$ for some set S. Since P is compact we have $\alpha(X) = \oplus_{s\in S}A$ and, since β commutes with direct sums, we have $\beta\circ\alpha(X) = X$ and I'll leave for you to check that the morphism $a(P) : P \to P$ is the identity.

Now let X be an arbitrary object of $\mathcal{M}$. Then we can present X as a cokernel of a morphism $f : P^S \to P^T$. Since the functor α is exact and the functor β is right exact the composition $\beta\circ\alpha$ is also right exact . Now the five homomorphisms lemma implies that $a(X) : \beta \circ \alpha \to X$ is an isomorphism.

Problem 0.192. Show that
(1) Construct a functorial morphisms $b : Id_{\mathcal{M}(A)} \to \alpha \circ \beta$.
(2) Show that the morphism $b(M)$ are isomorphisms for any A-module M.

□

8.4. A description of cuspidal components. In this secion we investigate categories $\mathcal{M}(D)$ of representations corresponding to a cuspidal component

$$D = \{\psi\rho, \psi \in \Psi(G)\}$$

where ρ is a cuspidal irreducible representation.

Let R be the algebra of regular functions on the algebraic variety $\Psi(G)$. R is naturally a G/G^0-module and therefore is also a G-module.

Example 0.193. If $G = GL(n, F)$ then $\Lambda = \mathbb{Z}$ and $R = \mathbb{C}[t, t^{-1}]$.

PROBLEM 0.194. (1) Construct a G-equivariant isomorphism

$$R = \mathbb{C}[\Lambda(G)] = \mathrm{ind}_{G^0}^{G}\,\mathbb{C}.$$

(2) Construct a G-equivariant isomorphisms $R \otimes \rho = \mathrm{ind}_{G^0}^{G}(\mathbb{C}) \otimes \rho = \mathrm{ind}_{G^0}^{G}(\rho|_{G^0})$.

PROPOSITION 0.195. *Let* $\Pi(D) := R \otimes \rho$*. Then*

(1) $\Pi(D) \in \mathcal{M}(D)$.
(2) $\Pi(D)$ *is a projective object in* $\mathcal{M}(D)$*.*
(3) $\Pi(D)$ *is a compact object.*
(4) $\Pi(D)$ *is a generator of the category* $\mathcal{M}(D)$

PROOF. For (1), just observe that $\mathrm{JH}(\Pi(D)|_{G^0}) \subset \mathrm{JH}(\rho|_{G^0})$.

To prove (2), we must show that the functor $X \mapsto \mathrm{Hom}_G(\Pi(D), X)$ from $\mathcal{M}(D)$ to the category of sets is exact. Since $\Pi(D) = \mathrm{ind}_{G^0}^{G}(\rho|_{G^0})$ it follows from the Frobenious adjunction [see Problem 0.67] that

$$\mathrm{Hom}_G(\Pi(D), X) = \mathrm{Hom}_{G^0}(\rho|_{G^0}, X|_{G^0}).$$

Since $\rho|_{G^0}$ is a direct sum of compact representation (see Proposition 0.168 and Theorem 0.138) it follows from Problem 0.21 that the functor $X \mapsto \mathrm{Hom}_G(\Pi(D), X)$ is exact.

(3) Follows from the Frobenious adjunction and the finiteness of the decomposition ρ_{G^0} to a direct sum of irreducible representations.

(4) Follows from the Frobenious adjunction. □

This proposition is a powerful tool for elucidating the structure of $\mathcal{M}(D)$ when combined with the previous lemma which implies that $\mathcal{M}(D) = \mathbb{C}(\mathrm{End}(\Pi(D))$. Our next goal is to describe the ring $\mathrm{End}(\Pi(D))$ explicitly.

For any $\psi \in \Psi_\rho$ we fix an intertwining operator $I_\psi \in Aut(V)$ as in Problem 0.168 and define $\nu_\psi := Id \otimes I_\psi \in Aut_{\mathbb{C}}(\Pi(D))$.

PROBLEM 0.196. a) $\nu_\psi \in A(D) := End_G(\Pi(D))$.

b) For any $\psi \in \Psi_\rho, f \in R$ we have

$$f\nu_\psi = \nu_\psi f_\psi$$

where f_ψ is the shift of f by ψ.

LEMMA 0.197. *Let* D *be a cuspidal component. Then*

(1) $A(D)$ *is a free* R*-module whith generators* $\nu_\psi, \psi \in \Psi_\rho$
(2) *As an algebra* $A(D)$ *is defined by the following relations*
   (a)
$$f\nu_\psi = \nu_\psi f_\psi, f \in R, \psi \in \Psi_\rho$$
   where f_ψ *is the translation of* f *by* ψ*.*
   (b) $\nu_\psi \nu_\phi = c_{\psi\phi}\nu_{\psi\phi}, \psi, \phi \in \Psi_\rho$.
(3) *The algebra* $A(D)$ *is Noetherian.*

PROOF. It is clearly sufficient to prove the part (a).

$$A(D) = Hom_G(\Pi(D), \Pi(D)) = Hom_G(\mathrm{ind}_{G^0}^G(\rho_{G^0}), \Pi(D)) = Hom_{G^0}(\rho, \rho \otimes R)$$

where the second and the third equalities follow from Problem 0.194. So $A(D) = \mathrm{End}_{G^0}(\rho) \otimes R$. The result now follows from Lemma 0.170 and Problem 0.173. we have $\square$

COROLLARY 0.198.

(1) *$ba \neq 0$ for any $b \in B - \{0\}, a \in A - \{0\}$.*
(2) *$A(D) = R$ if $\Psi_\rho = \{e\}$.*

8.4.1. *General Remarks.* It is important to keep in mind that there may be more than one projective generator so that we get different realizations of the category. As an example, we took $\Pi = \mathrm{ind}_{G^0}^G(\rho|_{G^0})$ as our projective generator for $\mathcal{M}(D)$. We could also have taken $\Pi' = \mathrm{ind}_{G^0}^G \tau$ for some $\tau \subset \rho|_{G^0}$.

9. Basic geometric Lemma

In this section we will prove a very important result which allows a reduction of number of representation-theoretical problems to the cuspidal case. We start with a reminder of the Mackey theory for finite groups.

9.1. More on l-spaces.

PROBLEM 0.199.

(1) A locally closed subset (i.e. the intersection of an open and a closed subset) of an l-space is an l-space.
(2) If $K \subset X$ is compact and $K \subset \cup_\alpha U_\alpha$ is an open covering, then there exists *disjoint* open compact $V_i \subset X$, $i = 1 \dots k$ such that $V_i \subset U_\alpha$ for some α and $\cup V_i \supset K$.
(3) Let G be a countable at infinity l-group acting on an l-space X with a finite number of orbits. Then G has an open orbit $X_0 \subset X$.

DEFINITION 0.200. For any l-space X we denote by $\mathcal{S}(X)$ the algebra of locally constant, compactly supported, complex-valued functions on X. $\mathcal{S}(X)$ will serve as the "test functions" for our analysis on X. Thus, the set $\mathcal{S}^*(X)$ of linear functionals on $\mathcal{S}(X)$ are called *distributions*. Note that as $\mathcal{S}(X)$ has no topology, there is obviously no continuity assumed.

LEMMA 0.201 (Exact Sequence of an Open Subset). *Let $U \subset X$ be open and $Z = X \setminus U$. Then*

$$0 \to \mathcal{S}(U) \to \mathcal{S}(X) \to \mathcal{S}(Z) \to 0$$

is exact.

PROOF. For the injection at $\mathcal{S}(U)$ just extend functions on U by zero to all of X. For the surjection at $\mathcal{S}(Z)$ we must explain how to extend functions from a *closed* subset. Since $f \in \mathcal{S}(Z)$ is locally constant and compactly supported, we may assume that Z is compact and has a covering by a finite number of open sets U_α with $f|_{U_\alpha} = c_\alpha$ constant. Let V_i be as in

0.199 (2). Then we can extend f by defining $f(x) = c_\alpha$ if $x \in V_i \subset U_\alpha$ and zero otherwise. □

DEFINITION 0.202. Let G be a countable at infinity l-group acting on an l-space X with a finite set I of orbits.

(1) We say that $i \leq j, i, j \in I$ if Ω_i is in the closure $\bar{\Omega}_j$ of the orbit Ω_j. In this way we define a partial order on I.
(2) For any $i \in I$ we define $\mathcal{S}_i = \{f \in \mathcal{S}(X) | f_{\Omega_i} = 0\}$. It is clear that $\mathcal{S}_i \supset \mathcal{S}_j$ if $i \leq j, i, j \in I$.
(3) For any $i \in I$ we define $\bar{\mathcal{S}}_i = (\sum_{j<i} \mathcal{S}_j)/\mathcal{S}_i$.

PROBLEM 0.203. Define a canonical isomorphism between the space $\bar{\mathcal{S}}_i$ and the space $\mathcal{S}(\Omega_i)$.

9.2. The formulation and the proof of the Basic geometric Lemma.

DEFINITION 0.204. (1) For any semidirect product $P = M \ltimes U$ such that M and U are unimodular l-groups and a smooth representation $\rho : P \to Aut(R)$ we denote by R_U the space of U-coinvariants and by $q_U : R \to R_U$ the natural surjection. Since M normalizes U the restriction of ρ on M induced a representation $(m, \bar{r}) \to m\bar{r}, m \in M, \bar{r} \in R_U$ of M on R_U.
(2) We denote by c_P^U the functor $\mathcal{M}(P) \to \mathcal{M}(M), (\rho, R) \to (c_P^U(\rho), R_U)$ where we define the action of M on R_U by
$$c_P^U(\rho)(m)(\bar{r}) = mod_U^{-1/2}(m)m\bar{r}$$
where the function *mod* is defined in 0.36.
(3) Let G be an l-group, P, Q closed subgroups of $G, P = M \ltimes U, Q = N \ltimes V$. As before we say that the semidirect products decompositions of subgroups P, Q are *compatible* if
$$P \cap Q = (M \cap Q)(U \cap Q) = (N \cap P)(V \cap P)$$
In this case we define $L = M \cap N, V' = M \cap V$ and $U' = N \cap U$.
Let $P = M \ltimes U, Q = N \ltimes V$ be a compatible pair of subgroups of G.
(4) We define functors $i_{M,U} : \mathcal{M}(M) \to \mathcal{M}(G), r_{N,V} : \mathcal{M}(G) \to \mathcal{M}(N)$ by
$$i_{M,U} = \mathrm{Ind}_P^G \circ \mathrm{Inf}_M^P, r_{N,V} := c_Q^N \circ \mathrm{Res}_G^Q$$
As before we often write i_U, r_V of i_M, r_N instead of $i_{M,U}$ and $r_{N,V}$.
(5) Since the pair (P, Q) is compatible we can also define functors $i_{U'} : \mathcal{M}(L) \to \mathcal{M}(N)$ and $r_{V'} : \mathcal{M}(M) \to \mathcal{M}(L)$.

Assume now that G is a reductive F-group and P, Q are parabolic subgroups. We consider the action of $P \times Q$ on G given by $(p \times q; g) \to pgq^{-1}$. As we know the set I of $P \times Q$-orbits is finite moreover can be identify with the set $W^{M,N}$. As follows from 0.202 the set $W^{M,N}$ is partially ordered and we have a filtration Ψ_w on the functor $\Psi := r_{N,V} \circ i_{M,U} : \mathcal{M}(M) \to \mathcal{M}(N)$. In particular for any $w \in W^{M,N}$ we can define the functor

$$\bar{\Psi}_w = (\sum_{w'<w} \Psi_{w'})/\Psi_w$$

Since pairs $(w(P), Q)$ are compatible for all $w \in W^{M,N}$ we can define functors

$$\tilde{\Psi}_w := i_{N\cap w(M),N\cap w(U)} \circ Ad(w) \circ r_{M\cap w^{-1}(N),M\cap w^{-1}(V)}$$

PROPOSITION 0.205. *For any $w \in W^{M,N}$ the functors $\bar{\Psi}_w$ and $\tilde{\Psi}_w$ are isomorphic.*

PROOF. We start with the simplest case when $G = SL(2,F), P = Q = B = TU$.

LEMMA 0.206. *Let $G = SL(2,F), P = Q = B = TU$. Then for any character χ of T we have an exact sequence*

$$\{0\} \to \mathbb{C}_{\chi^w} \to r_{T,U}(V(\chi) \to \mathbb{C}_\chi \to \{0\}$$

PROOF. Let $V(\chi) := i_{T,U}(\mathbb{C}_\chi)$. Then

$$V(\chi) = \{f : G \to \mathbb{C} | f(\hat{t}ug) = \chi(t) mod_U^{1/2}(t) f(g)\}, \hat{t} = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, t \in F^\star$$

and G acts on $V(\chi)$ by right shifts. We have $U\backslash G = F^2 - \{(0,0)\}$ where the map $G \to F^2 - (0,0)$ is given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to (\gamma, \delta)$$

So we have

$V(\chi) = \{f(x,y), (x,y) \neq (0,0) | f(tx,ty) = |t|^{-1}\chi(t)f(x,y)\}, t \in F^\star$ and there exists $r > 0$ such that $i_\chi(k)(f) = k, k \in K_r\}$

where G acts by $i_\chi(g)(f)(x,y) = f(\alpha x + \gamma y, \beta x + \delta y)$ for

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

We write $W = \{e, s\}, Y := \{(0,y) \subset F^2 - (0,0)\}$ and define

$$V(\chi)_s := \{f \in V(\chi) | f_{|Y} = 0\}$$

By the construction we have an exact sequence

$$\{0\} \to V(\chi)_s \to V(\chi) \to V(\chi)/V(\chi)_s \to \{0\}$$

where the subspace $V(\chi)_s \subset V(\chi)$ is B-invariant. Since the functor of U-coinvariants is exact we obtain an exact sequence

$$\{0\} \to r_{T,U}(V(\chi)_s) \to r_{T,U}(V(\chi) \to r_{T,U}(V(\chi)/V(\chi)_s) \to \{0\}$$

By the definition we have

$$\Psi_s(\mathbb{C}_\chi) = r_{T,U}(V(\chi)_s), \Psi_e(\mathbb{C}_\chi) = r_{T,U}(V(\chi)/V(\chi)_s)$$

So for the proof of the Lemma we have to check that $r_{T,U}(V(\chi)_e) = \mathbb{C}_\chi$ and $r_{T,U}(V(\chi)_s) = \mathbb{C}_{\chi^w}$.

The restriction to Y defines an isomorphism of the B-representation $\Psi_e(\chi)$ with the one-dimensional space on which $\hat{t}u \in Q$ acts by multiplication by $|t|^{-1}\chi(t)$. So $\Psi_e(\chi) = \mathbb{C}_\chi$.

Let $\mathbb{C}(F)$ be the space of functions on F and consider the operator $\nu : V(\chi)_s \to \mathbb{C}(F)$ of the restriction $\nu(f)(y) := f(1,y)$ of f to the line $l := \{(1,y)\}, y \in F$.

Claim 0.207. *The operator ν defines an isomorphism $\nu : V(\chi)_s \to \mathcal{S}(F)$.*

Proof. Let us fix $f \in V(\chi)_s$. Since the representation $V(\chi)_s$ is smooth there exists a congruence subgroup $K_r \subset G$ such that $i_\chi(k)(f) = k$ for $k \in K_r$. By the definition $f_{|Y} \equiv 0$. Therefore $f_{|K_rY} \equiv 0$. But it is easy to see that the complement $l - l \cap K_rY$ coincides with the set $\{(1,y), \|y\| \leq \|b\|q^r\}$ which is compact. So we see that $\nu(f) \in \mathcal{S}(F)$. On the other hand for any $\phi \in \mathcal{S}(F)$ we can consider the function $f_\phi : F^2 - \{(0,0)\} \to \mathbb{C}$ given by $f_\phi(x,y) = \chi(x)\|x\|^{-1}\phi(y/x)$. □

By the construction the group B acts on the space $V(\chi)_s$ and therefore the isomorphism ν defines a representation $\tau : B \to \mathrm{Aut}(\mathbb{C}(F))$.

Let us describe $\tau(\hat{t})(\phi), \phi \in \mathbb{C}(F)$. Since $\tau(\hat{t})(\phi) := \nu(i_\chi(\hat{t})(f_\phi)$ we have

$$\tau(\hat{t})(\phi)(y) = i_\chi(\hat{t})(f_\phi)(1,y) = f_\phi(t^{-1}, ty) = \chi(t^{-1})\|t\|\phi(t^2y)$$

Problem 0.208.

$$\tau(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix})(\phi)(y) = \phi(y+u)$$

As we know that space $\mathcal{S}(F)_U = \mathbb{C}$ and the map $q : \mathcal{S}(F) \to \mathcal{S}(F)_U = \mathbb{C}$ is given by $\phi \to \int_F \phi(y)dy$. So we see that $dim(r_{T,U}(V(\chi)) = 1$. Let us describe the action θ of T on the space $r_{T,U}(V(\chi)$.

By the definition we have

$$\theta(\hat{t})q(\phi) = \|t\|q(\tau(\hat{t})(\phi))$$

for any $\phi \in \mathcal{S}(F)$. But

$$q(\tau(\hat{t})(\phi)) = \int_F (\hat{t})(\phi)(y)dy = \int_F \chi(t^{-1})\|t\|\phi(t^2y) = \chi(t^{-1})\|t\|^{-1}\int_F (\phi)(y)dy$$

So $\theta(\hat{t})q(\phi) = \chi(t^{-1})q(\phi)$ and we see that $\theta(\hat{t}) = \chi(t^{-1})$.

□

Now we consider the case when $G = GL(n,F), P = Q = B = TU$. since the proof is completely parallel to the proof of the previous Lemma I will omit details. in the proof we will use notations and results from Section 5.2. In particular we know that $\mathcal{B}$ is the union of B-orbits $\mathcal{B}_w \subset \mathcal{B}, w \in W$ and $\mathcal{B}_{w'}$ is in the closure $\bar{\mathcal{B}}_w$ if and only if $w' \leq w$ in the partial order on W defined in in section 5.2. Let $r : U\backslash G \to B\backslash G = \mathcal{B}$ be the natural projection and $X_w := r^{-1}(\mathcal{B}_w)$. It is clear that $X_{w'}$ is in the closure $\bar{X}_w$ if and only if $w' \leq w$.

Let χ be a character of $T, V(\chi) := i_T(\chi)$. Then $V(\chi)$ is the space of functions f on $U\backslash G$ such that $f(tx) = \Delta^{1/2}(t)\chi(t)f(x), x \in U\backslash G, t \in T$ for which there exists $r > 0$ such that $f(xk) \equiv f(x)$ for all $k \in K_r$. The representation of the group $G = GL(n, F)$ on the space $V(\chi)$ comes from the action of G on $U\backslash G$.

For any $w \in W$ we denote by $V_w(\chi) \subset V(\chi)$ of functions f such that $f_{|X_w} \equiv 0$. It is clear that $V_{w_0}(\chi) = \{0\}$ and $V_{w'}(\chi) \subset V_w(\chi)$ if and only if $w' \geq w$. We define

$$\tilde{V}_w(\chi) := \cap_{w'<w} V_{w'}(\chi), \bar{V}i_w(\chi) := \tilde{V}_w(\chi)/V_w(\chi)$$

Problem 0.209. (1) The restriction to $wU_w \subset X_w$ defines an isomorphism $\nu : \bar{V}_w(\chi) \to \mathcal{S}(U_w)$.

Using the isomorphism ν we define can use the action of $i_T(\chi)$ of B on $\bar{\Psi}_w(\chi)$ to define a representation $\tau_w : B \to \mathrm{Aut}(\mathcal{S}(U_w))$ by

$$\tau(b)(\phi) := \nu(i_T(chi)(b)\nu^{-1}(\phi)$$

(2) $\tau_w(u_0)(\phi)(u) = \phi(uu_0), \phi \in \mathcal{S}(U_w), u, u_0 \in U_w$.

(3)

$$\tau_w(t)(\phi)(u) = \Delta^{1/2}(t^w)\chi(t^w)(\phi)(t^{-1}t^w u(t^{-1}t^w)^{-1})$$

Lemma 0.210. *The space $r_T(i_T(\chi))$ has a filtration $\Psi_w(\chi), w \in W$ by T-invariant subspaces such that the quotient spaces $\bar{\Psi}_w(\chi) := \Psi_w(\chi)/\sum_{w'<w} \Psi_{w'}(\chi)$ are one-dimensional and $t \in T$ acts on $\bar{\Psi}_w(\chi)$ by the multiplication by $\chi(t^w)$.*

Proof. As in the proof of the previous Lemma we can define a filtration of the space $r_T \circ i_T(\chi)$ has a filtration $r_T \circ i_T(\chi)$ by T-invariant subspaces such that the quotient spaces $\bar{\Psi}_w(\chi)$ equal to $r_T(\tau_w)$. Since the group $U_w \subset U$ acts transitively on the space U_w we see that the spaces $\bar{\Psi}_w(\chi)$ as one-dimensional. The lemma follows now from Problem 0.90. ☐

We start the proof of the general case with the following observation.

Claim 0.211. *Let G, P, Q, w be as in the Proposition. We denote by w_G, w_M, w_N the longest elements of W_G, W_M, W_N.*

(1) *There exists a character $\kappa(M, N, w) : M \cap w^{-1}(N) \to \mathbb{R}_+$ such that the functor $\bar{\Psi}_w$ is isomorphic to the functor*

$$i_{N\cap w(M),N\cap w(U)} \circ Ad(w) \circ \otimes\kappa(M, N, w) \circ r_{M\cap w^{-1}(N),M\cap w^{-1}(V)}$$

where we consider $\otimes\kappa(M, N, w)$ as an automorphism of the category $\mathcal{M}(M \cap w^{-1}(N))$

(2) *There exists an algebraic homomorphism*

$$\theta(M, N, w) : \underline{M \cap w^{-1}(N)} \to \mathbb{G}_m$$

such that $\kappa(M, N, w) = |\theta(M, N, w)|^{1/2}$.

(3) *For any $L < M, K < N, w \in W^{M,N}$ we have*
$$\theta(M,N,w) = \theta(L,K,w'ww''), w' := w_G w_M, w'' := w_N w_G$$

I will not give a proof but only observe that (1) is completely analogous to the problem **??**, (2) follows from (1) since the only reasons for the appearance of κ are either our twisting by $mod^{1/2}$ or the characters which come as determinants of changes of variables. The part (3) is a tautology.

The Proposition is equivalent to the equality $\theta(M,N,w) \equiv 1$. Since the claim is purely algebraic we can assume that G is split. Using the part (3) of the Claim we reduce the statement to the case $P = Q = B$.

It is easy to check that that for any $w', w'' \in W$ such that $l(w'w'') = l(w') + l(w'')$ we have $\theta(B,B,w) = \theta(B,B,w')w'(\theta(B,B,w''))$. So the proof reduced to the case $G = SL(2)$ which was analyzed in the previous example. □

Corollary 0.212. *[Basic geometric Lemma] For any standard parabolic subgroups $P = M \ltimes U, Q = N \ltimes V$ of a reductive F-groups G the functor $r_N \circ i_M : \mathcal{M}(M) \to \mathcal{M}(N)$ has a filitration by subfunctors $\Psi_w, w \in W^{M,N}$ such that for any $w \in W^{M,N}$ the quotient $\bar{\Psi}_w$ are isomorphic to the functor $i_{N\cap w(M),N\cap w(U)} \circ Ad(w) \circ r_{M\cap w^{-1}(N),M\cap w^{-1}(V)}$.*

We say that the functor $r_N \circ i_M : \mathcal{M}(M) \to \mathcal{M}(N)$ *is glued* from functors $i_{N\cap w(M),N\cap w(U)} \circ Ad(w) \circ r_{M\cap w^{-1}(N),M\cap w^{-1}(V)}$.

Corollary 0.213. *For any $M, N < G$ and a quasicuspidal representation ρ of M we have*

(1) *If N does not have standard subgroups associated with M then $r_N \circ i_M(\rho) = \{0\}$.*
(2) *If N is not associated with M then $r_N \circ i_M(\rho)$ does not have any non-zero quasicuspidal subquotients.*
(3) *If $N \sim M$ then the representation $r_N \circ i_M(\rho)$ is glued from representations $w(\rho), w \in W(M,N)/W_M$.*
(4) *There exists a proper Zariski closed subset X of Ψ_M such that $\mathrm{End}_G(i_M(\rho \otimes \psi) = \mathbb{C}$ for $\psi \in \Psi_M - X$.*

Proof. The parts (1),(2) and (3) are immediate consequences of the Basic geometric Lemma.

Let $D = \{\rho \otimes \psi\}, \psi \in \Psi_M$ be the cuspidal component containing ρ and $W(M,D) = \{w \in W(M,M)/W_M | D^w = D\}$. For any $w \in W(M,M)/W_M$ we fix $\kappa_w \in \Psi_M$ such that $\rho^w = \rho \otimes \kappa_w$ and define
$$Y_w := \{\psi \in \Psi_M | \psi^w \otimes \psi^{-1} \in \kappa_w \Psi_\rho.$$
Since the subgroup Ψ_ρ is finite [see Lemma 0.180] Y_w is a proper Zariski closed subset of Ψ_M for all $w \neq \{e\}$. Consider $X := \cup_{w \in W(M,N)/W_M - \{e\}} Y_w \subset \Psi_M$. As follows from (3) the representation $r_M \circ i_M(\rho \otimes \psi)$ is glued from representations $w(\rho \otimes \psi), w \in W(M,N)/W_M$. By the construction the representation $w(\rho \otimes \psi)$ is not equivalent to the representation $\rho \otimes \psi$ for all

$w \in W(M,N)/W_M - \{e\}$. Therefore

$$dim(\mathrm{Hom}_M(r_M \circ i_M(\rho \otimes \psi), \rho \otimes \psi)) \leq dim(\mathrm{Hom}_M(\rho \otimes \psi, \rho \otimes \psi)) = 1.$$

But then it follows from the Frobenious reciprocity that

$$dim(\mathrm{End}_G(i_M(\rho \otimes \psi)) \leq 1.$$

□

9.3. Some applications.

DEFINITION 0.214. Let $M, M' < G$ be standard Levi subgroups.

(1) $W(M,M') := \{w \in W | w(M) = M'\}$,where $w(M) := wMw^{-1}$. We write $M' \sim M$ if $W(M,M') \neq \emptyset$.
(2) Given representations $\rho \in \mathcal{M}(M), \rho' \in \mathcal{M}(M')$. We define $W(\rho,\rho') := \{w \in W | w(\rho) = \rho'\}$, where $w(\rho) \in \mathcal{M}(M')$ is defined by $w(\rho)(m') := \rho(w^{-1}m'w), m' \in M'$. We write $\rho' \sim \rho$ if $W(\rho,\rho') \neq \emptyset$.
(3) $W(M,\star) := \cup_{M' \sim M} W(M,M')$.
(4) $l(M)$ i s the cardinality of the set $W(M,\star)/W_M$.
(5) We denote by $l' = l'_M$ the function on $\mathcal{M}(G)$ given by
$$l'(\tau) = \sum_{L \sim M} l(r_L(\tau))$$
where $l(\rho)$ is the length of the representation ρ.

PROBLEM 0.215. (1) $l(M) = 2$ iff M is a maximal Levi subgroup of G.
(2) For any associated pair M, M' of standard Levi subgroups and $w \in W(M,M')$ there exists chains $M = M_0, M_1, ..., M_r = M', L_1, L_2, ..., L_r$ of standard Levi subgroups of G and a decomposition $w = w_r...w_1$ such that
  (a) M_{i-1}, M_i are maximal Levi subgroups of $L_i, 1 \leq i \leq r$.
  (b) $w_i \in W_{L_i}$.
  (c) $w_i(M_{i-1}) = M_i, 1 \leq i \leq r$.

CLAIM 0.216. *Let M be a standard Levi subgroup of $G, \rho \in Irr_c(M), \pi = i_M(\rho)$ and π_0 an irreducible subquotient of π. Then $l'(\pi_0) > 0$.*

PROOF. As follows from Lemma 0.231 there exists a standard Levi subgroup $M' < G$ such that $r_{M'}(\pi_0)$ is a non-zero quasi-cuspidal representation of M'. As follows from Corollary 0.213 we have $M' \sim M$. □

LEMMA 0.217. *Let M, M' be standard Levi subgroups of $G, \rho \in Irr_c(M), \rho' \in Irr_c(M')$ and $\pi = i_M(\rho), \pi' = i_{M'}(\rho')$. Then*

(1)
$$dim(Hom_G(\pi,\pi')) \leq |W(\rho,\rho')/W_M|$$
(2) *The length $l(\pi)$ of the representation π is finite and moreover $l(\pi) \leq l(M)$.*

PROOF. (1) By the Frobenious reciprocity we have

$$\mathrm{Hom}_G(\pi, \pi') = Hom_{M'}(r_{M'} \circ i_M(\rho), \rho')$$

As follows from 0.213 the representation $r_{M'} \circ i_M(\rho)$ has a filtration parameterized by $\bar{w} \in W(M, M')/W_M$ with subquotients equal to $\bar{w}(\rho)$. Therefore

$$dim(Hom_G(\pi, \pi')) \leq \sum_{\bar{w} \in W(M,M')/W_M} dim(Hom_{M'}(\bar{w}(\rho), \rho'))$$

Since $dim(Hom_{M'}(\bar{w}(\rho), \rho') = 0$ for $\bar{w} \notin W(\rho, \rho')$ and $dim(Hom_{M'}(\bar{w}(\rho), \rho') = 1$ for $\bar{w} \in W(\rho, \rho')$ we see that $dim(Hom_G(\pi, \pi')) \leq |W(\rho, \rho')/W_M|$.

(2) As follows from the exactness of the functor r_M we have

$$l'(\tau) = l'(\tau') + l'(\tau/\tau')$$

for any subrepresentation τ' of τ. By the previous Claim $l'(\pi_0) > 0$ for any non-zero subquotient π_0 of π. So $l(\pi) \leq l'(\pi)$. But it follows from 0.213 that $l'(\pi) = |W(M, \star)/W_M| = l(M)$. □

LEMMA 0.218. *Let $M < G$ be a standard Levi subgroup such that $l(M) = 2$ and $\rho \in Irr_c(M)$ a representation such that $dim(\mathrm{Hom}_G(i_M(\rho), i_M(\rho))) > 1$. Then*

(1) *$W(M, M) \neq W_M$.*
(2) *$w(M) = M$ and $w(\rho) = \rho$ for any $w \in W(M, M) - W_M$.*

PROOF. (1) By the Frobenious reciprocity we have

$$\mathrm{Hom}_G(i_M(\rho), i_M(\rho)) = \mathrm{Hom}_M(r_M \circ i_M(\rho), \rho)$$

If $W(M, M) = W_M$ then (see 0.213) $r_M \circ i_M(\rho) = \rho$ and

$$\mathrm{Hom}_M(r_M \circ i_M(\rho), \rho) = \mathbb{C}$$

So $W(M, M) - W_M \neq \emptyset$.

Since $l(M) = 2$ we have $W(M, M) - W_M = wW_M$ for any $w \in W(M, M) - W_M$. As follows from 0.213 we have an exact sequence

$$\{0\} \to w(\rho) \to r_M \circ i_M(\rho) \to \rho \to \{0\}$$

So

$$dim(\mathrm{Hom}_G(i_M(\rho), i_M(\rho))) \leq dim(\mathrm{Hom}_M(\rho, \rho)) + dim(\mathrm{Hom}_M(w(\rho), \rho))$$

By the assumption $dim\, \mathrm{Hom}_G(i_M(\rho), i_M(\rho)) > 1$. So $w(\rho) = \rho$. □

THEOREM 0.219. *Let M, M' be standard Levi subgroups of $G, \rho \in Irr_c(M), \rho' \in Irr_c(M'), \pi = i_M(\rho), \pi' = i_{M'}(\rho')$. The following conditions are equivalent.*

(1) *$M \sim M'$ and $\rho \sim \rho'$.*
(2) *$\mathrm{Hom}_G(\pi, \pi') \neq \{0\}$.*
(3) *$JH(\pi) = JH(\pi')$.*
(4) *$JH(\pi) \cap JH(\pi') \neq \emptyset$.*

PROOF. It is clear that (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4). We start with a proof of the implication (4) $\Rightarrow$ (1). Since $JH(\pi) \cap JH(\pi') \neq \emptyset$ we can find $\tau \in JH(\pi) \cap JH(\pi')$. Let $N < G$ be a standard Levi subgroup standard Levi subgroup such that $r_N(\tau)$ is a non-zero quasi-cuspidal representation of N. As follows immediately from Corollary 0.213 we have

$$r_N(\tau) \subset r_N(\pi) = \{w(\rho)\}, w \in W(M, N)$$

and also

$$r_N(\tau) \subset r_N(\pi') = \{w'(\rho')\}, w' \in W(M', N)$$

But this implies that $\rho \sim \rho'$.

We now show that (1) $\Rightarrow$ (2). Since $\rho \sim \rho'$ it follows from Corollary 0.213 that $\rho' \in JH(r_{M'} \circ i_M(\rho))$. Now Lemma 0.181 implies that $r_{M'} \circ i_M(\rho)$ has a quotient isomorphic to ρ'. In other words

$$\mathrm{Hom}_{M'}(r_{M'} \circ i_M(\rho), \rho') \neq \{0\}$$

and therefore it follows from the Frobenious reciprocity that

$$\mathrm{Hom}_G(\pi, \pi') = Hom_{M'}(r_{M'} \circ i_M(\rho), \rho') \neq \{0\}.$$

It is clear that for a proof of Theorem it is sufficient now to prove the implication (1) $\Rightarrow$ (3). We start a proof with the case when $l(M) = 2$.

LEMMA 0.220. *If $l(M) = 2$ then* (1) $\Rightarrow$ (3) .

PROOF. If $M = M', \rho = \rho'$ then there is nothing to prove. So we assume that $\rho \neq \rho'$ and therefore $l'(\pi) = l'(\pi') = 2$. Since we know already that (1) $\Rightarrow$ (2) we can choose non-zero G-equivariant morphisms $a : \pi \to \pi'$ and $a' : \pi' \to \pi$. Since $l(\pi) \leq l(M) = 2$ we see that either π is irreducible or $l(\pi) = 2$. If the morphism a is an isomorphism then there is nothing to prove. So we assume that a is not an isomorphism.

If π is irreducible then a is an imbedding and

$$l'(\pi'/a(\pi)) = l'(\pi') - l'(\pi) = 0$$

and it follows from Claim 0.216 that a is onto. So a is an isomorphism and $JH(\pi) = JH(\pi')$. The same arguments work if π' is irreducible. So

From now on we assume that both π and π' are reducible. Then $l(\pi) = l(\pi') = 2$ and we have exact sequences

$$\{0\} \to \pi_1 \to \pi \to \pi_2 \to \{0\}$$

$$\{0\} \to \pi'_1 \to \pi' \to \pi'_2 \to \{0\}$$

where $\pi_1, \pi'_1, \pi_2, \pi'_2$ are irreducible representations of G such that $l'(\pi_1) = l'(\pi'_1) = l'(\pi_2) = l'(\pi'_2) = 1$.

CLAIM 0.221. (1) *If $dim\, \mathrm{Hom}_G(\pi, \pi) > 1$ then $M = M'$ and $\rho = \rho'$.*

(2) *If $dim\, \mathrm{Hom}_G(\pi, \pi) = 1$ then $Im(a')$ is the only irreducible submodule of π.*

Proof. (1) If $dim\,\mathrm{Hom}_G(\pi,\pi) > 1$ then it follows from Corollary 0.218 that there exists $w \in W(M,M) - W_M$ such that $w(\rho) = \rho$. On the other hand since Since $l(M) = 2$ and $W(M,M) - W_M \neq \emptyset$ we see that $M' = M$ and $\rho' = \rho$.

(2) Since $l'(\mathrm{Ker}(a')) > 0$ and $l(\pi') = 2$ we have $l'(Im(a')) = 1$ and therefore $l(Im(a')) = 1$. So $Im(a')$ is an irreducible submodule of π. Suppose that there exists another irreducible submodule τ of π. Since $l(\pi) = 2$ we have $\pi = Im(a') \oplus \tau$. But then $dim\,\mathrm{Hom}_G(\pi,\pi) > 1$. $\square$

From now on we assume that $dim\,\mathrm{Hom}_G(\pi,\pi) = 1$.

By the Frobenious reciprocity we have $\mathrm{Hom}(r_M(\pi_1),\rho) = Hom(\pi_1,\pi) \neq \{0\}$. As follows from the definition of the function l' the equality $l'(\pi_1) = 1$ implies that $r_M(\pi_1) = \rho$ and $r_{M'}(\pi_1) = \{0\}$. Therefore [by the Frobenious reciprocity] we have

$$\mathrm{Hom}(\pi_1,\pi') = Hom(r_{M'}(\pi_1),\rho') = \{0\}.$$

So $a(\pi_1) = \{0\}$ and [by the Claim] a defines an isomorphism $\bar{a} : \pi/\pi_1 \to \pi_1'$. Analogous arguments show the existence of an isomorphism $\bar{a}' : \pi'/\pi_1' \to \pi_1$. So $JH(\pi) = JH(\pi')$. $\square$

Now we prove the implication (1) $\Rightarrow$ (3) in the general case. Choose $w \in W(M,M')$ such that $w(\rho) = \rho'$. As follows from Problem 0.215 there exists chains $M = M_0, M_1, ..., M_r = M', L_1, L_2, ..., L_r$ of standard Levi subgroups of G and a decomposition $w = w_r...w_1$ such that

(1) M_{i-1}, M_i are maximal Levi subgroups of $L_i, 1 \leq i \leq r$.
(2) $w_i \in W_{L_i}$.
(3) $w_i(M_{i-1}) = M_i, 1 \leq i \leq r$.

Let $\rho_k := w_k...w_1(\rho) \in \mathcal{M}(M_k), 1 \leq k \leq r$. It is sufficient to show that

$$JH(i_{M_k}(\rho)) = JH(i_{M_{k-1}}(\rho_{k-1}))$$

for all $k, 1 \leq k \leq r$. Since

$$i^G_{M_k} = i^G_{L_k} \circ i^{L_k}_{M_k}, i^G_{M_{k-1}} = i^G_{L_k} \circ i^{L_k}_{M_{k-1}}$$

the equality

$$JH(i_{M_k}(\rho)) = JH(i_{M_{k-1}}(\rho_{k-1}))$$

would follow from the equality

$$JH(i^{L_k}_{M_k}(\rho)) = JH(i^{L_k}_{M_{k-1}}(\rho_{k-1})), 1 \leq k \leq r.$$

Since $M_{k-1}, M_k, 1 \leq k \leq r$ are standard maximal Levi subgroups of L_k the equality

$$JH(i^{L_k}_{M_k}(\rho)) = JH(i^{L_k}_{M_{k-1}}(\rho_{k-1}))$$

follows from 0.220.

$\square$

9.4. Cuspidal datas.

DEFINITION 0.222. (1) A *cuspidal data* of G is a pair (M, ρ) where $M < G$ is a standard Levi subgroup and $\rho \in Irr_c(M)$.

(2) Two cuspidal datas (M, ρ) and (M', ρ') are *associated* if there exists $w \in W$ such that $w(M) = M'$ and $[w(\rho)] = [\rho']$. In this case we write $(M, \rho) \sim (M', \rho')$.
(3) We denote by $X(G)$ the set of cuspidal datas up to associate
(4) A *component* of $\mathcal{M}(G)$ is an equivalence class of pairs (M, D) where $M < G$ is a standard Levi subgroup and $D \in X_c(M)$ is cuspidal component of $\mathcal{M}(M)$ where two pairs $(M, D), (M', D')$ are equivalent if there exists $w \in W$ such that $w(M) = M'$ and $w(D) = D'$.
(5) We denote by $\bar{X}(G)$ the set of components of $\mathcal{M}(G)$.
(6) For any component $\Omega \in \bar{X}(G)$ we denote by $X_\Omega \subset X(G)$ the set of cuspidal datas in Ω.

REMARK 0.223. By the definition for any $\Omega \in \bar{X}(G)$ the set X_Ω is equal to the quotient $D/W_{(M,D)}$ of some cuspidal component $D \in X_c(M), M < G$. Therefore the structure of an $\mathbb{C}$-algebraic variety on D induces a structure of an $\mathbb{C}$-algebraic variety on X_Ω.

LEMMA 0.224. *Let π be an irreducible representation of G.*

(1) *There exists a cuspidal data (M, ρ) of G such that $\rho \in JH(r_M(\pi))$.*
(2) *Let (M, ρ) be a cuspidal data of G. Then $\rho \in JH(r_M(\pi))$ iff $\pi \in JH(i_{M'}(\rho'))$ for any cuspidal data (M', ρ') associated with (M, ρ).*
(3) *Let $(M, \rho), (M', \rho')$ be cuspidal datas of G such that $\rho \in JH(r_M(\pi))$ and $\rho' \in JH(r_{M'}(\pi))$ for some $\pi \in \mathrm{Irr}(G)$. Then (M, ρ) and (M', ρ') are associated.*

PROOF. (1) Choose M to be a minimal standard Levi subgroup such that $r_M(\pi) \neq \{0\}$. Then $r_M(\pi)$ is a non-zero cuspidal representation of M and any $\rho \in JH(r_M(\pi))$ satisfies the condition of (1).

(2) Assume that $\rho \in JH(r_M(\pi))$. As follows from Lemma 0.181 we have $\mathrm{Hom}_M(r_M(\pi), \rho) \neq \{0\}$. Then [by the Frobenious duality] $\mathrm{Hom}_M(\pi, i_M(\rho)) \neq \{0\}$ and therefore $\pi \in JH(i_M(\rho))$. If (M', ρ') is cuspidal data associated with (M, ρ) then by Theorem 0.219 we have $JH(i_M(\rho)) = JH(i_{M'}(\rho'))$.

Conversely assume that $\pi \in JH(i_M(\rho))$. Then as, follows from Corollary 0.213, all the irreducible subquotients of $r_M(\pi)$ are of the form $w(\rho), w \in W(M, M)$.

(3) Assume now that $(M, \rho), (M', \rho')$ are cuspidal datas such that $\rho \in JH(r_M(\pi))$ and $\rho' \in JH(r_{M'}(\pi))$. As follows from (2) we have $\rho' \in JH(r_{M'} \circ i_M(\rho))$. Now the result follows from Corollary 0.213 c).

□

DEFINITION 0.225. (1) For any $\pi \in Irr(G)$ we denote by $pr(\pi) \in X(G)$ the associated class of cuspidal datas (M, ρ) such that $\rho \in JH(r_M(\pi))$. As follows from the last Lemma the map $pr : Irr(G) \to X(G)$ is well defined.

(2) For any component $\Omega \in \bar{X}(G)$ of $\mathcal{M}(G)$ we denote by Irr_Ω the preimage $pr^{-1}(X_\Omega) \subset Irr(G)$ and denote by $\mathcal{M}(\Omega)$ the full subcategory of $\mathcal{M}(G)$ of objects V such that $JH(V) \subset Irr_\Omega$.

(3) For and cuspidal date (M,ρ) we define
$X(\rho) := \{\psi \in \Psi_M |$ the representation π_ψ is not irreducible$\}$.

Corollary 0.226. *The map $pr : Irr(G) \to X(G)$ is finite-to -one and surjective.*

Proof. The surjectivity follows immediately from the definition of the map pr and the finiteness of fibers $pr^{-1}(\pi)$ follows from Lemma 0.217 (2). □

Theorem 0.227. *[Decomposition Theorem]. For any $\Omega \in \bar{X}(G)$ the set Irr_Ω splits the category $\mathcal{M}(G)$.*

Proof. We start with the following definition.

Definition 0.228. (1) Let V be a G-module. For any $\Omega \in \bar{X}(G)$ we define $V(\Omega)$ as the maximal submodule of V such that $\mathrm{JH}(V(\Omega)) \subset \mathrm{Irr}_\Omega$.

(2) We say that V is split if $V = \bigoplus_{\Omega \in \bar{X}(G)} V(\Omega)$.

Lemma 0.229. *Any submodule of a split module is split.*

Proof. Let $V' \subset V$ be a submodule of a split module. We want to show that $V' = \bigoplus_{\Omega \in \bar{X}(G)} V'(\Omega)$. As follows from Lemma 0.25 it is sufficient to show that $\mathrm{JH}(W) = \emptyset$ for

$$W = V' / \bigoplus_{\Omega \in \bar{X}(G)} V'(\Omega).$$

Fix $\Omega_0 \in \bar{X}(G)$ and consider

$$p_{\Omega_0} : V \to \bigoplus_{\Omega \neq \Omega_0} V(\Omega)$$

Since V is split, $\operatorname{Ker} p_{\Omega_0} = V(\Omega_0)$. Consequently, $\operatorname{Ker}(p_{\Omega_0 V'}) = V'(\Omega_0)$. Since $V'(\Omega) = V(\Omega) \cap V'$ for all $\Omega \in \bar{X}(G)$ we have

$$\mathrm{JH}(W) \subset \cup_{\Omega \in \bar{X}(G) - \{\Omega_0\}} \mathrm{Irr}_\Omega .$$

So

$$\mathrm{JH}(W) \subset \cap_{\Omega \in \bar{X}(G)} \mathrm{Irr}_\Omega = \emptyset$$

Since $\mathrm{Irr}(G)$ is a disjoint union of $\mathrm{Irr}_\Omega, \Omega \in \bar{X}(G)$ we see that $\mathrm{JH}(W) = \emptyset$. □

Now we remind some results on the the decomposition of the cuspidal part.

Definition 0.230. (1) We define $\mathcal{M}(cusp) := \prod_{M<G} \mathcal{M}(M)_{cusp}$.

(2) We define the functor $I : \mathcal{M}(cusp) \to \mathcal{M}(G)$ by $I(\{(M,\rho_M)\}) := \oplus_{M<G} i_M(\rho_M)$.

(3) We define the functor $R : \mathcal{M}(G) \to \mathcal{M}(cusp)$ by $R(\pi) := \{r_M(\pi)_c\}, M < G$ where V_c is the cuspidal part of V as in Definition 0.163.

LEMMA 0.231. (1) *Let (π, W) be an irreducible representation of G and*
(2) *For any smooth irreducible representation (π, W) of G there exists a standard parabolic $P = MU$ and an irreducible cuspidal representation (τ, W) of M, such that there exists an embedding $W \hookrightarrow i_M(W)$.*
(3) *R is the left adjoint to I.*
(4) *Functors R and I are exact.*
(5) *Functors R and I are faithful.*
(6) *The adjoint map $\kappa(\pi) : \pi \mapsto IR(\pi)$ is a monomorphism.*

PROOF. The parts (1) and (2) are familiar. It is also clear that the functor I is faithful. Let us first of all show that functor $R(\theta) \neq \{0\}$ for any non-zero maps non-zero $\theta \in \mathcal{M}(G)$. Choose $\pi \in JH(\theta)$. Since the functor R is exact it is sufficient to show that $R(\pi) \neq \{0\}$. Let M be a Levi subgroup, minimal subject to the condition $r_{M,U}(\pi) \neq 0$. Then $r_{M,U}(\pi)$ is cuspidal. So $R(\pi) \neq \{0\}$.

Let us now show that $\kappa(\pi) \neq 0$ for any non-zero $\pi \in \mathcal{M}(G)$. By the adjunction we have a bijection

$$a : \mathrm{Hom}_{\mathcal{M}(G)}(\pi, IR(\pi)) \to \mathrm{Hom}_{\mathcal{M}(cusp)}(R(\pi), R(\pi))$$

such that $a(\kappa) = Id_{R(\pi)}$. Since $R(\pi) \neq \{0\}$ we see that $\kappa(\pi) \neq 0$.

Let us now prove the injectivity of the map $\kappa : \pi \mapsto IR(\pi), \pi \in \mathcal{M}(G)$. Let $\tau := \operatorname{Ker} \kappa$. We want to show that $\tau = \{0\}$. Assume that $\tau \neq \{0\}$. Since the functors R and I are exact the map $RI(\kappa) : RI(\tau) \to RI(\pi)$ is a monomorphism and therefore the composition $\tau \to IR(\tau) \to IR(\pi)$ does not vanish. But it is zero by the definition of τ. So $\tau = \{0\}$.

Since the adjoint map $\kappa(\pi) : \pi \mapsto IR(\pi)$ is a monomorphism we see that R is faithful. $IR(\pi) = \{0\}$. Since the functors R and I are faithful we see that $\tau = \{0\}$. □

We are now ready to prove the theorem. As we have shown it is sufficient to prove that any $M \in \mathcal{M}(G)$ is a submodule of a split module. But as follows from Lemma 0.231 any smooth G-module V is a submodule of $IR(V) = \bigoplus_{\{M<G\}} i_M(\tau_M)$ where τ_M is a cuspidal representation of M. Thus, to prove the theorem, it is enough to prove that the G-modules $i_M(\tau_M)$ are split for cuspidal M-modules τ_M. Since τ_M is cuspidal, it follows from Theorem 0.149 that we may write $\tau_M = \bigoplus_D \tau(D)$ where the D run through the cuspidal components of M. This reduces our problem further; we must prove that $i_M(\tau_M)$ splits when $\tau_M \in \mathcal{M}_D(M)$ for some cuspidal component D. But it follows from Corollary 0.213 that $JH(i_M(\tau_M)) \subset X_\Omega$ where X_Ω is the component of $\mathcal{M}(G)$ corresponding to the pair (M, D). So $i_M(\tau_M) = i_M(\tau_M)(\Omega)$. □

LEMMA 0.232. *Let $(M,(\rho,V))$ be a cuspidal data, $K\subset G$ be a congruence subgroup such that $V^{K\cap M}\neq\{0\}$ and $\Omega\in\bar{X}(G)$ be the corresponding component. Then $L^K\neq\{0\}$ for any irreducible $(\pi,L)\in Irr_\Omega$.*

PROOF. As follows from Corollary 0.213 $r_M(L)\neq\{0\}$ the representation $r_M(\pi)$ of M has a subquotient isomorphic $\rho\otimes\psi$ for some character $\psi : M/M_0\to\mathbb{C}^\star$. Therefore $(r_M(L))^{K\cap M}\neq\{0\}$. But then it follows from Lemma 0.125 that $L^K\neq\{0\}$. □

COROLLARY 0.233. *Let $K\subset G$ be a congruence subgroup.*

(1) *The set*

$$\mathrm{Irr}_K(G):=\{(\pi,V)\in Irr(G)|V^K\neq\{0\}\}$$

splits the category $\mathcal{M}(G)$.

(2) *If a representation (ρ,V) of G is generated by the subspace V^K then the same is true for any subquotient of V.*

PROOF. (1) As follows from the decomposition theorem it is sufficient to show that the set $Irr_K(G)\cap Irr_\Omega$ splits the category $\mathcal{M}(\Omega)$. But by the Lemma 0.232 either $Irr_K(G)\cap Irr_\Omega=Irr_\Omega$ or $Irr_K(G)\cap Irr_\Omega=\emptyset$.

(2) Let $V=V_1\oplus V_2$ be the decomposition such that $JH(V_1)\subset\mathrm{Irr}_K(G)$ and $JH(V_1)\cap\mathrm{Irr}_K(G)=\emptyset$. Since V is generated by the subspace V^K we see that V_2 is generated by the subspace V_2^K. By the definition of V_2 we have $V_2^K=\{0\}$. So $V=V_1$ and therefore $JH(L)\subset Irr_K(G)$ for every irreducible subquotient L of V.

Assume now that the claim (2) is false and $V'\subset V''\subset V$ be submodules such that $(V''/V')^K=\{0\}$. Since $JH(V''/V')subsetJH(V)\subset Irr_K(G)$ and $(V''/V')^K=\{0\}$ we see that $V''/V'=\{0\}$. □

REMARK 0.234. (1) The Corollary is true for all open subgroups K of G such that $K=(K\cap U)(K\cap M)(K\cap\bar{U})$ for any $M<G$. For example you can take K to be the Iwahori subgroup.

(2) The result is not true when $K=K_0$. Consider the case when $G=SL(2,F)$ and V the space of smooth measures on $\mathbb{P}^1=B\backslash G$. The integration defines an exact sequence

$$\{0\}\to V_0\to V\to\mathbb{C}\to\{0\}$$

which [see **??**] does not admit a splitting. On the other hand V_0 does not have K_0-invariant vectors.

9.5. Noetherian properties.

DEFINITION 0.235. We say that an object V of an abelian category $\mathcal{C}$ is *Noetherian* if any increasing chain $V_1\subset V_2\subset ...\subset V_n\subset ...$ of subobjects of V stabilizes.

REMARK 0.236. Let A be a Noetherian ring and V an A-module. Then V is Noetherian as an object of the category of A-modules iff V is finitely generated.

LEMMA 0.237. *Let V be a smooth G-module. The following conditions are equivalent*

(1) *V is Noetherian.*
(2) *V is finitely generated.*
(3) *There exists a congruence subgroup K such that $GV^K = V$ and the $\mathcal{H}_K$-module V^K is finitely generated.*

PROOF. It is clear that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Assume that V is finitely generated, $V = G(W), dim(W) < \infty$. Then there exists a congruence subgroup K such that $W \subset V^K$. Then $\mathcal{H}_K(W) = e_K G(W) = e_K(V) = V^K$.

(3) $\Rightarrow$ (1). Let $V \in \mathcal{M}(G)$ be such that there exists a congruence subgroup K such that $GV^K = V$ and the $\mathcal{H}_K$-module V^K is finitely generated.

CLAIM 0.238. *The module $R(V)$ [see Lemma 0.231] is Noetherian.*

PROOF. By the definitions $R(V) = \prod_{\{M<G, D_M \in X_c(M)\}} R(D_M)(V)$ where for any $M < G$ and a cuspidal component $D_M \in X_c(M)$ we denote by $R(D_M)(V)$ the projection of $r_M(V)$ on the factor $\mathcal{M}(D_M)$ of $\mathcal{M}(M)$. As follows from the previous Corollary and the Uniform Admissibility Theorem 0.149 $R(D_M)(V) = \{0\}$ for almost all pairs (M, D_M). So it is sufficient to show that the representations $R(D_M)(V) \in \mathcal{M}(D_M)$ are Noetherian. As follows from Corollary 0.233 the M-module $R(D_M)(V)$ is generated by the subspace $RK \cap M(D_M)(V) = J(V^K)$ [see Proposition 0.130] is finite dimensional. So the M-module $R(D_M)(V)$ is finitely generated. On the other hand it follows from Lemma 0.197 that the category $\mathcal{M}(D_M)$ is equivalent to the category of right modules over a Noetherean ring $A(D_M)$. So $R(D_M)(V) \in \mathcal{M}(D_M)$ are Noetherian objects. $\square$

To prove that V is Noetherian consider an increasing chain $V_1 \subset V_2 \subset ... \subset V_n \subset ...$ of subobjects of V. Since the object $R(V)$ is Noetherian the chain $R(V_1) \subset R(V_2) \subset ... \subset R(V_n) \subset ...$ stabilizes. Since the functor R is faithful this implies the stabilization of the chain $V_1 \subset V_2 \subset ... \subset V_n \subset ...$ also stabilizes. $\square$

COROLLARY 0.239. (1) *The functors r_M map Noetherian objects into Noetherian objects.*
(2) *The functors i_M map Noetherian objects into Noetherian objects.*

PROOF. (1) Let $V \in \mathcal{M}(G)$ be a Noetherian object. Then by 0.237 V is finitely generated, $V = G(W), dim(W) < \infty$ and we may assume that W is K_0-invariant. Choose a congruence subgroup K such that $W \subset V^K$. Since $K_0 W = W$ the decomposition $G = MUK_0$ implies that $Mq(W) = r_M(V)$ where $q : V \to r_M(V)$ is the natural projection. So the module $r_M(V)$ is finitely generated and therefore it is Noetherian.

(2) Let W be a Noetherian object of $\mathcal{M}(M)$. We want to show that any chain increasing $V_1 \subset V_2 \subset ... \subset V_n \subset ...$ of subobjects of $V := i_M(W)$ stabilizes. Since the functor R is faithful it is sufficient to show that the

chain $R(V_1) \subset R(V_2) \subset ... \subset R(V_n) \subset ...$ stabilizes. In other words we have to show that for any standard subgroup N the chain $r_N(V_1)_c \subset r_N(V_2)_c \subset ... \subset r_N(V_n)_c \subset ...$ stabilizes. But now the stabilization of these chains follow from (1), the inductive assumptions and the Basic Geometric Lemma 0.212 which says that objects $r_N(V_1)_c$ are glued from a finite number of Noetherian objects.

□

10. Irreducibility of induced representations

DEFINITION 0.240. Let $M < G$ be a standard Levi subgroup and $(\rho, V) \in \mathcal{M}(M)$.

(1) We write $(\pi, W) := i_M(\rho, V)$ and for any character $\psi \in\in \Psi_M$ write $(\pi_\psi, W_\psi) := i_M(\rho \otimes \psi, V)$.
(2) We use the Iwasawa decomposition $G = PK_0, P = MU$ to identify the restriction of the representation π_ψ on K_0 with the representation
$$\mathrm{ind}_{K_0 \cap P}^{K_0}(inf_{K_0 \cap M}^{K_0 \cap P}(Res_{K_0 \cap M}(\rho))).$$
(3) Using this isomorphism we identify spaces $W_\psi, \psi \in\in \Psi_M$ with the fixed space W in such a way that the restriction of π_ψ on K_0 does not depend on ψ.

As we know the representation $i_M(\rho)$ is admissible and therefore the space W^K is finite-dimensional for any congruence subgroup K.

LEMMA 0.241. *For any $h \in \mathcal{H}_K$ the operator $\hat{h}(\psi) := \pi_\psi(h)$ is a regular function on Ψ_M with values in $\mathrm{End}\, W^K$.*

PROOF. Since the support of h is compact there exists a congruence subgroup $K' \subset K$ such that $x^{-1}k'x \in K$ for all $x \in \mathrm{supp}(h)$. Then $\pi_\psi(x)(W^{K'}) \subset W^K$ for all $x \in \mathrm{supp}(h)$. It is sufficient to show that for any $x \in \mathrm{supp}(h)$ the operator $\pi_\psi(x) : W^{K'} \to W^K$ is a regular function on Ψ_M with values in $\mathrm{Hom}(W^{K'}, W^K)$. But for any $f \in W_\psi = ind_{K_0 \cap P}^{K_0}(V)$ we have $\pi_\psi(x)(f)(k) = \psi(\theta_M(kx))\pi(x)(f)(k)$. □

LEMMA 0.242. *Let $X(\rho, K) := \{\psi \in \Psi_M |$ the representation $\pi_\psi : \mathcal{H}_K \to \mathrm{End}(W^K)$ is not irreducible$\}$.*
Then $X(\rho, K) \subset \Psi_M$ is a Zariski closed subset.

PROOF. We start with the following general result.

PROBLEM 0.243. Let k be an algebraically closed field, L be a finite-dimensional k-vector space, $\underline{Y}$ an algebraic k-manifold, $f_i, i \in I$ a family of L-valued regular function on $Y = \underline{Y}(k)$ and $Z \subset Y$ the set of $z \in Y$ such that $\{f_i(z)\}, i \in I$ do not generate L. Then there exists a Zariski closed subset $\underline{Z} \subset \underline{Y}$ such that $Z = \underline{Z}(k)$.

The proof of the Lemma is based on the Bernside's theorem which says that there is no proper subalgebras of the algebra $\mathrm{End}(W^K)$ which act irreducibly on W^K. Therefore

$X(\rho, K) := \{\psi \in \Psi_M |$ the span of $\hat{h}(\psi), h \in \mathcal{H}_K$ is equal to $\mathrm{End}(W^K)\}$.

We now apply Problem to the case when

$$k = \mathbb{C}, \underline{Y} = \Psi_M, L = \mathrm{End}_{\mathbb{C}}(W^K), I = G, f_g(\psi) := \pi_\psi(e_K g e_K).$$

□

COROLLARY 0.244. *For any cuspidal data (M, ρ) the subset $X(\rho)$ [see 0.225] is Zariski closed.*

PROOF. Follows from Corollary 0.233. □

The main goal of this section is to prove that $X(\rho) \neq \Psi_M$. The proof is based on the analysis of of the *unitary structure.*

10.1. The Unitary Structure. Let (ρ, V) be a smooth representation of G.

DEFINITION 0.245. For any $v \in V, \lambda \in \tilde{V}$ we define *the matrix coefficient* $m_{\tilde{v},v}(g)$ as a function on G given by $m_{\tilde{v},v}(g) = \tilde{v}(\pi(g)v), v \in V, \tilde{v} \in \tilde{V}$.

(1) Assume now that (π, V) is an irreducible representation of G such that the restriction to the center $Z(G)$ is equal to χId_V where $\chi : Z(G) \to \mathbb{C}^\star$ is a unitary character. We say that V is *square integrable modulo center* if
$$\int_{G/Z} |m_{\xi,\tilde{\xi}}(g)|^2 dg < \infty.$$
(2) A *unitary structure* on a G-module (π, V) is a positive definite, G-invariant Hermitian scalar product $Q : V \otimes V \to \mathbb{C}$.
(3) Let $P = MU$ be a parabolic subgroup of G and $(\rho, W, <,>)$ be a unitary representation of M. We define the unitary structure Q on $(\pi, V) = (i_M(\rho), i_M(W))$ by
$$Q(f_1, f_2) = \int_{x \in P\backslash G} < f_1(x), f_2(x) >$$
where the linear functional $\int$ is as in Definition **??**.

REMARK 0.246. We do not assume that V is complete with respect to this structure.

PROBLEM 0.247. Let (π, V) be an irreducible representation of G which is essentially square integrable modulo center. Then

(1) (π, V) admits a unitary structure.
(2) $|m_{v,\tilde{v}}(g)|^2 \in L^1(G/Z(G))$ for all $v \in V, \lambda \in \tilde{V}$.

The essential uniqueness of an invariant scalar product for irreducible representations follows from the following version of the Schur's lemma.

LEMMA 0.248 (Schur's Lemma). *Let V be an irreducible G-module. Then any two G-invariant unitary structures on V are proportional.*

PROOF. Let V^+ be anti-linear dual of V, that is the space of anti-linear functionals. An invariant scalar product $Q : V \otimes \overline{V} \to \mathbb{C}$ defines a G-equivariant semi-linear map $V \to V^+$. As V is smooth, we obtain a semi-linear map $V \to (V^+)_{sm}$. Since V is admissible we see that V^+_{sm} is also admissible and irreducible. Therefore any non-zero G-equivariant semi-linear map $V \to (V^+)_{sm}$ is a bijection. It follows now from the Schur's lemma that any two non-zero G-equivariant semi-linear maps $V \to (V^+)_{sm}$ are proportional. $\square$

10.2. Applications of the unitarity.

LEMMA 0.249. *Any smooth admissible unitary representation (π, V) of G is completely reducible. [That is, $V = \bigoplus V_i$ where the V_i are irreducible unitary subrepresentations.]*

REMARK 0.250. The assumption of the admissibility is important.

PROOF. Suppose $W \subset V$ is a submodule. Then the orthogonal complement, $W^\perp \subset V$ is also a submodule and $W \cap W^\perp = 0$. It remains to check that $W + W^\perp = V$. For this it is enough to check that for any compact open subgroup $K \subset G$ we have $W^K + (W^\perp)^K = V^K$. Since [by the admissibility] the space V^K is finitely-dimensional we have $V^K = W^K \oplus (W^K)^\perp \cap V^K$. So it is sufficient to show that $(W^K)^\perp \cap V^K \subset W^\perp$. Since the group K is compact we have $W = W^K \oplus L$ where $\int_K \pi(k) l dk = 0$ for $l \in L$. Then $< l, v >= 0$ for all $v \in V^K$ and we see that $< w, v >= 0$ for all $v \in W^K)^\perp \cap V^K, w \in W$. $\square$

COROLLARY 0.251. *Let V be an admissible unitary representation V of G such that* $\mathrm{End}_G(V) = \mathbb{C}$. *Then V is irreducible.*

This Corollary provides a method for establishing the irreducibility of some representations. Here is an important example.

Let ρ be an irreducible cuspidal representation of a Levi subgroup $M < G$. We denote by $X(\rho) \subset \Psi_M$ the set characters ψ such that the representation $\pi_\psi := i_M(\rho \otimes \psi)$ of G is reducible.

THEOREM 0.252. *$X(\rho)$ is a proper Zariski closed subset of Ψ_M.*

PROOF. Since ρ is irreducible there exists a character $\chi : Z(M) \to \mathbb{C}^\star$ such that $\rho(z) = \chi(z) Id$ for $z \in Z(M)$. Since the subgroup $Z(G) \cap M^0 \subset Z(G)$ is compact there exists $\psi \in \Psi_M$ such that $|\chi(z)\psi(z)| = 1$ for all $z \in Z(M)$. Therefore [see Problem 0.84] the representation $\rho \otimes \psi$ admits a unitary structure. By replacing ρ by $\rho \otimes \psi$ we may assume that ρ admits a unitary structure.

As follows from Corollary **??** the subset $X(\rho) \subset \Psi_M$ is Zariski closed. So we only have to show that $X(\rho) \neq \Psi_M$.

As follows from Corollary 0.213 there exists a proper Zariski closed subset Y of Ψ_M such that $\mathrm{End}_G(i_M(\rho \otimes \psi)) = \mathbb{C}$ for $\psi \in \Psi_M - Y$. Since the subset $\Psi^u_M \subset \Psi_M$ of unitary characters is Zariski dense in Ψ_M the set $\Psi^u_M - \Psi^u_M \cap Y$ is not empty. But for any $\psi \in \Psi^u_M - \Psi^u_M \cap Y$ the representation π_ψ is unitary and $\mathrm{End}_G(i_M(\rho \otimes \psi)) = \mathbb{C}$. It follows now from Corollary 0.251 that π_ψ is irreducible. $\square$

11. The second adjointness.

11.1. The comparison of orbits of P and $\bar{P}$. We start this section with some results on the structure of of P orbits on G/Q for parabolic subgroups P, Q of G.

Definition 0.253. For a pair M, N of standard Levi subgroups we define

$$W^{M,N} := \{w \in W | w(M \cap B) \subset B, w^{-1}(N \cap B) \subset B\}$$

Problem 0.254. Let P, Q be standard parabolic subgroups of G.

(1) For any $w \in W$ the intersection $W_M w W_N \cap W^{M,N}$ consists of one element. In other words any double coset $W_M w W_N$ contains unique representative in $W^{M,N}$ and we can identify the sets $W^{M,N}$ and $W_M \sigma W / W_N$.
(2) For any standard parabolic subgroups P, Q of G the imbedding $W \hookrightarrow G$ induces the bijection $W_M \backslash W / W_N \to \bar{P} \backslash G / Q$.
(3) So the set $W^{M,N}$ parametrizes the both double cosets $\bar{P} \backslash G / Q$ and double cosets $P \backslash G / Q$ and we defined a bijection $\kappa : P \backslash G / Q \to \bar{P} \backslash G / Q$.
(4) κ reverses the order on partially ordered sets $P \backslash G / Q$ and $\bar{P} \backslash G / Q$.
(5) The map $w \to w^{-1} w_G$ defines bijection $\kappa' : P \backslash G / Q \to Q \backslash G / \bar{P}$ which reverses the order on partially ordered sets.
(6) There exists unique $w_G \in W_G$ such that $w \leq w_G$ for all $w \in W_G$ and the map $w \to w w_G$ reverses the partial order on W_G.
(7) Assume that $P = MU$ is a maximal parabolic subgroup of G. Then
   (a) $W(M, \star) = W_M \cup w_G W_M$.
   (b) If $W(M, M) = W_M$ then the parabolic subgroups P and $\bar{P}$ of G are not conjugate and $N := w_G(M) < G$ is a standard Levi subgroup of G associated with M. Moreover any standard Levi subgroup of G associated with M is either equal to M or is equal to N. In particular $l_G(M) = 2$.
   (c) If $W(M, M) \neq W_M$ then the parabolic subgroups P and $\bar{P}$ are conjugate, $|W(M, M)/W_M| = 2$ and any standard Levi subgroup of G associated with M is equal to M. So in this case we also have $l_G(M) = 2$.

Remark 0.255. In the case when $G = GL(n, F), P = Q = B$ any double coset BgB has a form BwB where $w \in W = S_n$ where the symmetric group $S_n \subset GL(n, F)$ is realized as the subgroup of permutation matrices. In this case standard parabolic subgroups P correspond to partitions σ_P of n and we denote by $W_P \subset W$ the subgroup of permutations preserving this partition.

PROBLEM 0.256. a) Show that one can identify the sets $P\sigma G/Q$ and $W_P\sigma W/W_Q$.

b) Prove all the statements of the last Claim in the case $G = GL(n)$.

c) Show that any parabolic subgroup of G is is conjugate to a standard parabolic by an element of K_0 and that G/P is compact.

Let $P = MU$ be a parabolic subgroup of G and $\bar{P} = M\bar{U}$ be the opposite parabolic. For any parabolic $Q = NV$ of G the set $W^{M,N}$ parametrizes the both double cosets $\bar{P}\backslash G/Q$ and double cosets $P\backslash G/Q$. So we defined a bijection $\kappa : P\backslash G/Q \to \bar{P}\backslash G/Q$.

DEFINITION 0.257. (1) For any parabolic subgroups P, Q of G we define a partial order on the finite set $P\backslash G/Q$ in such a way that $i \leq j, i, j \in P\backslash G/Q$ if Ω_i is in the closure $\bar{\Omega}_j$ of Ω_j where $\Omega_i, \Omega_j \subset G$ are $P \times Q$-orbits corresponding to $i, j \in P\backslash G/Q$.

(2) We define a partial order on W using the bijection $W \to P_0\backslash G/P_0$.

(3) For any standard parabolic subgroup $P = MU$ we denote by $\bar{P} = M\bar{U}$ be the opposite parabolic.

CLAIM 0.258. (1) *For any standard parabolic subgroups P, Q of G the imbedding $N_G(M_0) \hookrightarrow G$ induces the bijection $W_M\backslash W/W_N \to \bar{P}\backslash G/Q$. So the set $W^{M,N}$ parametrizes the both double cosets $\bar{P}\backslash G/Q$ and double cosets $P\backslash G/Q$ and we defined a bijection $\kappa : P\backslash G/Q \to \bar{P}\backslash G/Q$.*

(2) *κ reverses the order on partially ordered sets $P\backslash G/Q$ and $\bar{P}\backslash G/Q$.*

(3) *The map $w \to w^{-1}w_G$ defines bijection $\kappa' : P\backslash G/Q \to Q\backslash G/\bar{P}$ which reverses the order on partially ordered sets.*

(4) *There exists unique $w_G \in W_G$ such that $w \leq w_G$ for all $w \in W_G$ and the map $w \to ww_G$ reverses the partial order on W_G.*

(5) *Assume that $P = MU$ is a maximal parabolic subgroup of G. Then*
   (a) *$W(M, \star) = W_M \cup w_G W_M$.*
   (b) *If $W(M, M) = W_M$ then the parabolic subgroups P and $\bar{P}$ of G are not conjugate and $N := w_G(M) < G$ is a standard Levi subgroup of G associated with M. Moreover any standard Levi subgroup of G associated with M is either equal to M or is equal to N. In particular $l_G(M) = 2$.*
   (c) *If $W(M, M) \neq W_M$ then the parabolic subgroups P and $\bar{P}$ are conjugate, $|W(M, M)/W_M| = 2$ and any standard Levi subgroup of G associated with M is equal to M. So in this case we also have $l_G(M) = 2$.*

REMARK 0.259. In the case when $G = GL(n, F), P = Q = B$ any double coset BgB has a form BwB where $w \in W = S_n$ where the symmetric group $S_n \subset GL(n, F)$ is realized as the subgroup of permutation matrices. In this case standard parabolic subgroups P correspond to partitions σ_P of n and we denote by $W_P \subset W$ the subgroup of permutations preserving this partition.

PROBLEM 0.260. a) Show that one can identify the sets $P\sigma G/Q$ and $W_P\sigma W/W_Q$.

b) Prove all the statements of the last Claim in the case $G = GL(n)$.

c) Show that any parabolic subgroup of G is is conjugate to a standard parabolic by an element of K_0 and that G/P is compact.

11.2. The construction of second adjointness. Let $P = MU$ be a parabolic subgroup of G. We know that $r_{M,U}$ is the left adjoint to $i_{M,U}$. This implies that for any representation π of M we have a canonical isomorphism

$$\mathrm{Hom}_M(r_{M,U} \circ i_{M,U}(\pi), \pi) = \mathrm{Hom}_G(i_{M,U}(\pi), i_{M,U}(\pi)).$$

In particular we have a canonical morphism $r_{M,U} \circ i_{M,U} \to Id_{\mathcal{M}(M)}$ of functors. In fact, the existence of such a morphism is implies by the existence of the filtration of the functor $r_{M,U} \circ i_{M,U}$ described in the Basic Geometric Lemma. Really subquotients in this filtration correspond to orbits of P acting on $X = P\backslash G$. There is a distinguished orbit of the action of P on $P\backslash G$, namely point P which is the only closed orbit. Since P is closed it corresponds to the quotient of $r_{M,U} \circ i_{M,U}$. It is easy to see that this quotient is equal to $\mathrm{Id}_{\mathcal{M}(M)}$. We see that the adjointness property is related to the existence of the distinguished orbit.

Let $\overline{P} = M\overline{U}$ be the parabolic opposite to P.

As follows from Problem 0.258 there is unique open orbit of the action of $\bar{P}$ on G/P and the functor associated with this orbit is equal to $\mathrm{Id}_{\mathcal{M}(M)}$.

Set, $\overline{r}_M = r_{M,\bar{U}}$. We have shown that for any representation $\tau \in \mathcal{M}(M)$ there is a canonical imbedding $\tau \to \bar{r}_M \circ i_M(\tau)$. Now for any $\varphi \in \mathrm{Hom}_G(i_{M,U}(\tau), \pi)$. we define a morphism $\beta(\varphi) \in \mathrm{Hom}_M(\tau, \overline{r}_M(\pi))$ as the composition

$$\beta(\varphi) : \tau \to \bar{r}_M \circ i_M(\tau) \stackrel{\bar{r}_M(\varphi)}{\to} \overline{r}_M(\pi)$$

In other words, we defined a map

$$\beta_{G,M}(\tau, \pi) : \mathrm{Hom}_G(i_M(\tau), \pi) \to \mathrm{Hom}_M(\tau, \overline{r}_M(\pi))$$

We will often write $\beta(\tau, \pi)$ or simply β instead of $\beta_{G,M}(\tau, \pi)$.

THEOREM 0.261. *β is an isomorphism. In other words, $\overline{r}_M$ is the right adjoint functor to i_M.*

PROOF. We assume that the result is known for all proper Levi subgroups of G.

LEMMA 0.262. (1) *Let*

$$\tau'' \to \tau' \to \tau \to \{0\}$$

be an exact sequence in $\mathcal{M}(M)$ such that the morphisms $\beta(\tau'', \pi)$ and $\beta(\tau', \pi)$ are isomorphisms. Then $\beta(\tau, \pi)$ is also an isomorphism.

(2) *Let*

$$\{0\} \to \pi \to \pi' \to \pi''$$

be an exact sequence in $\mathcal{M}(G)$ such that the morphisms $\beta(\tau, \pi'')$ and $\beta(\tau, \pi')$ are isomorphisms. Then $\beta(\tau, \pi)$ is also an isomorphism.

Proof. The result follows from the exactness of functors r_M and i_M and the five-homomorphisms lemma. $\square$

Corollary 0.263. *It is sufficient to check the validity of the Theorem in the case when τ is a projective object of $\mathcal{M}(M)$ and π is an injective object of $\mathcal{M}(G)$.*

Proposition 0.264. *$\beta(\tau, \pi)$ is an isomorphism if τ is projective and π is equal to $i_N(\rho)$ where $N < G$ is a proper Levi subgroup of G and ρ is an injective object of $\mathcal{M}(N)$.*

Proof. As follows from the Frobenious reciprocity we have equalities

$$\mathrm{Hom}_G(i_M(\tau), \pi) = \mathrm{Hom}_G(i_M(\tau), i_N(\rho)) = \mathrm{Hom}_N(r_N(i_M(\tau)), \rho)$$

By the Basic Geometric Lemma the functor $\mathrm{Hom}_N(r_N(i_M(\tau))$ admits a decreasing filtration parametrized by double cosets $P\backslash G/Q$ and the subquotients are isomorphic to $\tilde{\Psi}_w(\tau), w \in W^{M,N}$ where

$$i_{N \cap w(M), N \cap w(U)} \circ Ad(w) \circ r_{M \cap w^{-1}(N), M \cap w^{-1}(V)}$$

Since the object ρ is injective we see that the space $\mathrm{Hom}_N(r_N(i_M(\tau)), \rho)$ admits an increasing filtration parametrized by double cosets $P\backslash G/Q$ and the subquotients are isomorphic to

$$\mathrm{Hom}_N(i_{N \cap w(M), N \cap w(U)} \circ Ad(w) \circ r_{M \cap w^{-1}(N), M \cap w^{-1}(V)}(\tau), \rho), w \in W^{M,N}$$

On the other hand the functor $\bar{r}_M \circ i_N$ admits a decreasing filtration parametrized by double cosets $Q\backslash G/\bar{P}$. By Problem 0.258 this set coincides with the set $\{w^{-1}w_G\}, w \in W^{M,N}$ and we can consider this filtration as an increasing filtration parametrized by the set $P\backslash G/Q$ and the corresponding subquotients are isomorphic to $\tilde{\Psi}'_w(\tau), w \in W^{M,N}$ where

$$\tilde{\Psi}'_w(\tau) = i_{M \cap w^{-1}(N), M \cap w^{-1}(V)} \circ Ad(w^{-1}) \circ r_{N \cap w(M), N \cap w(\bar{U})}$$

Since τ is a projective object of $\mathcal{M}(M)$ we see that the space $\mathrm{Hom}_N(\tau, \bar{r}_M \circ i_N(\rho))$ admits an increasing filtration parametrized by double cosets $P\backslash G/Q$ and the subquotients are isomorphic to

$$\mathrm{Hom}_M(\tau, i_{M \cap w^{-1}(N), M \cap w^{-1}(V)} \circ Ad(w^{-1}) \circ r_{N \cap w(M), N \cap w(\bar{U})}(\rho)), w \in W^{M,N}$$

Problem 0.265. (1) The functor $\beta(\tau, i_{N,V}(\rho))$ is compatible with the increasing filtrations on $\mathrm{Hom}_G(i_M(\tau), \pi)$ and $\mathrm{Hom}_M(\tau, \overline{r}_M(\pi)), w \in W^{M,N}$.

(2) For all $w \in W^{M,N}$ the induced map from

$$\mathrm{Hom}_N(i_{N\cap w(M),N\cap w(U)} \circ Ad(w) \circ r_{M\cap w^{-1}(N),M\cap w^{-1}(V)}(\tau), \rho)$$

to

$$\mathrm{Hom}_M(\tau, i_{M\cap w^{-1}(N),M\cap w^{-1}(V)} \circ Ad(w^{-1}) \circ r_{N\cap w(M),N\cap w(\bar{U})}(\rho))$$

is given by $\beta_{N,N\cap w(M)} Ad(w) \circ r_{M\cap w^{-1}(N),M\cap w^{-1}(V)}(\tau), \rho)$.

Now Proposition follows from the inductive assumptions. □

Corollary 0.266. (1) *$\beta(\tau, \pi)$ is an isomorphism if π is equal to $i_N(\rho)$ where $N < G$ is a proper Levi subgroup of $G, \rho \in \mathcal{M}(M)$.*
(2) *$\beta(\tau, \pi)$ is an isomorphism if $\pi \in Ob(\mathcal{M}_c^{\perp}(G))$.*

Proof. The part (1) follows arguments used in the proof of Lemma 0.262.

Since $\pi \in \mathcal{M}_c^{\perp}(G)$ the map

$$\oplus_{\{M<G,M\neq G\}} \kappa_M(\pi) : V \to \oplus_{\{M<G,M\neq G\}} i_M \circ r_{M,U}(\pi)$$

is an imbedding. Applying this argument once more we find an exact sequence

$$\{0\} \to \pi \to \pi' \to \pi''$$

such that π' and π'' are direct sums of representations of the form $i_{N,V}(\rho)$ where $Q = NV$ is a proper parabolic subgroup of $G, \rho \in \mathcal{M}(M)$. Now the Claim follows from Lemma 0.262. □

Now we can finish the proof of the Theorem. As follows from the last Corollary and the decomposition theorem saying that $\mathcal{M}(G) = \mathcal{M}_c(G) \oplus \mathcal{M}_c^{\perp}(G)$ it is sufficient to prove the theorem in the case when π is quasi-cuspidal. But it is clear that in this case both sides are equal to $\{0\}$. □

Corollary 0.267. *The functor $i_M : \mathcal{M}(G) \to \mathcal{M}(M)$ maps projective objects into projective.*

11.3. The Bernstein's morphism. Let $P = MU \subset G$ be a standard parabolic, $\bar{P} = M\bar{U}$ be the opposite parabolic , $H \subset P \times \bar{P}$ be the preimage of the diagonal under the projection $P \times \bar{P} \to P/U \times \bar{P}/\bar{U} = M \times M$ and $X_M := H\backslash G \times G$. We use the second adjointness to construct a $G \times G$-equivariant morphism $B_M : \mathcal{S}(X_M) \to \mathcal{S}(G)$ introduced by J.Bernstein.

For any $V \in \mathcal{M}(G \times G)$ we write

$$\bar{r}_M \times r_M(V) := r_{M\times M,\bar{U}\times U}(V) \in \mathcal{M}(M \times M).$$

Let $Y := P\bar{P} \subset G$. Then Y is an open $\bar{P} \times P$ subset of G.

Problem 0.268. We can identify the $M \times M$ representation $\bar{r}_M \times r_M(\mathcal{S}(Y))$ with of the regular representations of $M \times M$ on $\mathcal{S}(M)$.

We denote by j_M the composition $j_M : \mathcal{S}(M) \to \bar{r}_M \times r_M(\mathcal{S}(Y)) \hookrightarrow \bar{r}_M \times r_M(\mathcal{S}(G))$.

Definition 0.269. (1) By the construction

$$\mathcal{S}(X_M) = \mathrm{ind}_H^{G\times G}\mathbb{C} = \mathrm{ind}_{P\times\bar{P}}^{G\times G}\mathcal{S}(M)$$

where the action of $P \times \bar{P}$ on $\mathcal{S}(M)$ is the composition of the regular representations of $M \times M$ on $\mathcal{S}(M)$ and the projection $P \times \bar{P} \to M \times M$. As follows from Theorem 0.261 we have an isomorphism

$$\mathrm{Hom}_{G\times G}(\mathcal{S}(X_M), \mathcal{S}(G)) = \mathrm{Hom}_{M\times M}(\mathcal{S}(M), \bar{r}_M \times r_M(\mathcal{S}(G))).$$

Therefore the canonical imbedding j_M defines a a $G\times G$-equivariant morphism $B_M : \mathcal{S}(X_M) \to \mathcal{S}(G)$.

(2) Let $\mathbb{C}(G), \mathbb{C}(X_M)$ be the space of $\mathbb{C}$-valued functions on G and $X_(M)$ which are two-sided invariant under some open compact subgroup of G. Then $\mathbb{C}(G) = \widetilde{\mathcal{S}(G)}, \mathbb{C}(X_M) = \widetilde{\mathcal{S}(X_M)}$ and we denote by $\widetilde{B_M} : \mathbb{C}(G) \to \mathbb{C}(X_M)$ the morphism dual to B_M.

(3) Let $\mathbb{C}(G), \mathbb{C}(X_M)$ be the space of $\mathbb{C}$-valued functions on G and $X_(M)$ which are two-sided invariant under some open compact subgroup of G. Then $\mathbb{C}(G) = \widetilde{\mathcal{S}(G)}, \mathbb{C}(X_M) = \widetilde{\mathcal{S}(X_M)}$ and we denote by $\widetilde{B_M} : \mathbb{C}(G) \to \mathbb{C}(X_M)$ the morphism dual to B_M.

(4) For any pair W_1, W_2 of smooth representations of M denote by $\mathcal{P}(W_1, W_2)$ the space of M-invariant $\mathbb{C}$-valued bilinear forms on $W_1 \times W_2$.

(5) We denote by $\kappa : \mathrm{Hom}_{G\times G}(V_1 \otimes V_2, \mathbb{C}(X_M)) \to \mathcal{P}(r_M(V_1), r_M(V_2))$ given by

$$\kappa(\phi)(J(v_1), \bar{J}(v_2)) := \phi(v_1 \otimes v_2)(e)$$

where $J : V_1 \to r_M(V_1), \bar{J} : V_2 \to \bar{r}_M(V_2)$ are the canonical projections.

Problem 0.270. (1) Show that the bilinear form $\kappa(\phi)(w_1, w_2), w_1 \in r_M(V_1), w_2 \in \bar{r}_M(V_2)$ is well defined. [That is the number $\phi(v_1 \otimes v_2)(e)$ does not depend on a choice $v_1 \in V_1, v_2 \in V_2$ such that $w_1 = J(v_1), w_2 = \bar{J}(v_2)$.]

(2) The map

$$\mathrm{Hom}_{G\times G}(V_1 \otimes V_2, \mathbb{C}(X_M)) \to \mathcal{P}(r_M(V_1), r_M(V_2)), \phi \to \kappa(\phi)$$

is a bijection.

Definition 0.271. Let V_1, V_2 be smooth representations of G and $<,>: V_1 \times V_2 \to \mathbb{C}$ be a G-invariant pairing. We denote by $<,>_M: r_M(V_1) \times \bar{r}_M(V_2) \to \mathbb{C}$ the M-invariant bilinear form as in the last Problem.

Let $M < G$ be a standard Levi subgroup, (σ, V) be a smooth representation of G and $<,>: V \times \tilde{V} \to \mathbb{C}$ be the canonical G-invariant pairing. We denote by $J : V \to r_M(V), \tilde{J} : \tilde{V} \to \bar{r}_M(\tilde{V})$ the canonical projections.

As follows from the Corollary the canonical pairing $<,>: V \times \tilde{V} \to \mathbb{C}$ induces a pairing $\widetilde{<,>_M}: r_M(V) \times \bar{r}_M(\tilde{V}) \to \mathbb{C}$ and therefore a morphism $\kappa_{M,V} : \bar{r}_M(\tilde{V}) \to \widetilde{r_M(V)}$.

LEMMA 0.272. *The $\kappa_{M,V}$ is an isomorphism.*

PROOF. Consider the morphism $a : \widetilde{r_M(V)} \to \bar{r}_M(\tilde{V})$ which is the image of the identity $Id_{r_M(V)}$ under the composition

$$\mathrm{Hom}_M(r_M(V), r_M(V)) \to \mathrm{Hom}_G(V, i_M \circ r_M(V)) \to \mathrm{Hom}_G(i_M \widetilde{\circ r_M}(V), \tilde{V}) =$$

$$= \mathrm{Hom}_G(i_M(\widetilde{r_M(V)}), \tilde{V}) \to \mathrm{Hom}_M(\widetilde{r_M(V)}, \bar{r}_M(\tilde{V}))$$

where the first map comes from the Frobenious reciprocity, the second is the duality, the third comes from the inverse of the natural morphism $i_M(\widetilde{r_M(V)}) \to i_M \widetilde{\circ r_M}(V)$ and the last map is equal to $\beta(\widetilde{r_M(V)}, \tilde{V})$. It is easy to check that a is the inverse of $\kappa_{M,V}$. $\square$

PROPOSITION 0.273. *For any admissible representation (σ, V) of G the M-invariant pairings $<,>_M$ and $\widetilde{<,>}_M$ [see Lemma **??**] between $r_M(V)$ and $\bar{r}_M(\tilde{V})$ coincide.*

PROOF. Let $b(V) : \widetilde{r_M(V)} \to \bar{r}_M(\tilde{V})$ be the isomorphism coming from the pairing $\widetilde{<,>}_M$ and $\tilde{\beta} : Hom_G(i_M(\widetilde{r_M(V)}, \tilde{V}) \to \mathrm{Hom}_M(\widetilde{r_M(V)}, \bar{r}_M(\tilde{V})$ be the map defined as the composition of the isomorphism $\mathrm{Hom}(i(\tau), \tilde{\sigma}) = \mathrm{Hom}(\tau, \widetilde{r(\sigma)})$ and the morphism $b(V)$. It is sufficient to show that $\tilde{\beta}_{G,M} = \beta_{G,M}$. By the definition $\square$