REPRESENTATIONS OF REDUCTIVE GROUPS

January 19, 2007
Syllabus.

Complex representations of reductive groups over different fields. [Course 80759 - I changed the topic]

Sundays 11.30-13.15

This course consists of two parts. In the first we will study representations of reductive groups over local non-archimedian fields [such as $\mathbb{Q}_p$ and $\mathbb{F}_q((s))$]. In this part I'll closely follow the notes of the course of J.Bernstein. Moreover I'll often copy big chanks from these notes. In the second the representations of reductive groups over 2-dimensional local fields [such as $\mathbb{Q}_p((s))$].

In the first part we explain the basics of

a) induction from parabolic and parahoric subgroups,

b) Jacquet functors,

c) cuspidal representations

d) the second adjointness and

e) Affine Hecke algebras.

In the second we discuss the generalization these concepts to the case of representations of reductive groups over 2-dimensional local fields.

Prerequisites. The familiarity with the following subjects will be helpful.

a) P-adic numbers, [see first few chapters of the book "p-adic numbers, p-adic analysis, and zeta-functions" by N.Koblitz or sections 4-5 in the book "Number theory" of Borevich and Shafarevich].

b) Basics of the theory of split reductive groups G [Bruhat decomposition, Weyl groups, parabolic and Levi subgroups] of reductive groups, [One who does not know this this theory can restrict oneself to the case when $G = GL(n)$ when Bruhat decomposition= Gauss decomposition.]

c) Basics of the category theory: adjoint functors, Abelian categories. [see the chapter 2 of book "Methods of homological algebra" by Gelfand-Manin or Appendix and the first 2 chapters of "An introduction to homological algebra" by C.Weibel].

Lecture 1.

1. Fourier transform

1.1. **Finite fields.** For any prime number p and a number q of the form $q = p^n, n > 0$ there exist unique [up to an isomorphism] finite field $\mathbb{F}_q$ with q elements. It has characteristic p and contains the field $\mathbb{F}_p$. We denote by $\bar{\psi} : \mathbb{F}_p \to \mathbb{C}^\star$ the additive character given by

$$\bar{\psi}(x) = \exp(2\pi\tilde{x})$$

where $\tilde{x} \in \mathbb{Z}$ is any representative of $x \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

For any finite field $\mathbb{F}_q, q = p^n$ we denote by $\bar{\psi} : \mathbb{F}_q \to \mathbb{C}^\star$ by the additive character given by

$$\bar{\psi}(x) = \bar{\psi}(Tr_{\mathbb{F}_q/\mathbb{F}_p}(x))$$

where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the prime subfield of $\mathbb{F}_q$.

For any $\lambda \in \mathbb{F}_q$ we denote by $\bar{\psi}_\lambda$ the additive character
$x \to \bar{\psi}_\lambda(x) := \bar{\psi}(\lambda x)$.

Definition 1.1. D:F0 Let k be a finite field.

a) We denote by $\mathbb{C}(k)$ the linear space of complex-valued functions on k and denote by $<,>$ the scalar product

$$< f, g >:= \sum_{x\in k} f(x)g(x), f, g \in \mathbb{C}(f)$$

b) For any $f \in \mathcal{S}(F)$ we define $f^- \in \mathcal{S}(F)$ by

$$f^-(a) := f(-a)$$

c) We define the Fourier transform $\mathcal{F} : \mathbb{C}(f) \to \mathbb{C}(f)$ by

$$\mathcal{F}(f)(x) := \sum_{y\in k} \bar{\psi}(xy)f(y)$$

Exercise 1.2. . a) Show that $\mathcal{F}^2(f) = qf^-$ where $f^-(v) := f(-v)$.

b) Let χ be a non-trivial multiplicative character of the group $k^\star$ and $\tilde{\chi}$ the extension of χ to a function on k such that $\tilde{\chi}(0) = 0$. Find $\mathcal{F}(\tilde{\chi})$.

c) $< \mathcal{F}(f), \mathcal{F}(g) >= q < f, g^- > \forall f, g \in \mathbb{C}(k)$.

d) Show that the map $\lambda \to \bar{\psi}_\lambda$ defines a bijection between $\mathbb{F}_q$ and the set additive characters $\mathbb{F}_q^\vee$ of $\mathbb{F}_q$.

Definition 1.3. D:FV0 Let V be a finite-dimensional k vector space. We define by $\mathbb{C}(V)$ the space of complex-valued functions on k and by $<,>$ the scalar product

$$< f, g >:= \sum_{x \in V} f(x)g(x), f, g \in \mathbb{C}(f)$$

b) We define the Fourier transform $\mathcal{F} : \mathbb{C}(V) \to \mathbb{C}(V^\vee)$, where $V^\vee$ is the dual space, by

$$\mathcal{F}(f)(v^\vee) := \sum_{v \in V} \bar{\psi}(v^\vee, v) f(v), v^\vee \in V^\vee$$

where $(,) : V \times V^\vee$ is the natural pairing.

Exercise 1.4. a) Show that $\mathcal{F}^2(f) = q^{dim_k(V)} f^-$.

b) $< \mathcal{F}(f), \mathcal{F}(g) >= q^{dim_k(V)} < f, g^- >$.

1.2. **Local fields.** .

Definition 1.5. D:local *Local fields* are locally compact non-discrete fields. As well known archimedian local fields are isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$. In this course we will only consider *non-archimedian* local fields.

If $\mathbf{K}$ is non-archimedian local field then there exists a *valuation* $v : \mathbf{K} \to \mathbb{Z} \cup \infty$ such that

a) $v^{-1}(\infty) = \{0\}$,

b) the restriction of v on the multiplicative group $\mathbf{K}^\star = \mathbf{K} - \{0\}$ is a group homomorphism,

c) the preimage $\mathcal{O} := v^{-1}(\mathbb{Z}_{\geq 0})$ is the maximal compact subring of $\mathbf{K}$,

d) the preimage $\mathfrak{m} := v^{-1}(\mathbb{Z}_{>0)}$ is the maximal ideal of $\mathcal{O}$. Moreover $\mathfrak{m}$ is a principal ideal of $\mathcal{O}$. Sometimes we will fix a generator of $\mathfrak{m}$.

e) the quotient field $\mathcal{O}/\mathfrak{m}$ is finite and therefore is isomorphic to $\mathbb{F}_q$.

f) For any $a \in \mathbf{K}, n > 0$ we define $U_{a,n} = \{b \in \mathbf{K} | v(a - b) \geq n\}$. It is clear that that subsets $U_{a,n} \subset \mathbf{K}$ are open and closed and they constitute a basis of open sets.

g) For $a \in \mathbf{K}$ we define $\|a\| = q^{-v(a)}$ if $a \neq 0$ and define $\|0\| = 0$.

We will use the letter $\mathbf{K}$ to denote non-archimedian local fields.

Examples 1.6. a) Let $\mathbf{K} = \mathbb{F}_q((t))$ be the field of formal Laurent series over a finite field $\mathbb{F}_q$. So any element a in $\mathbf{K}$ is a formal power series

$a(t) = \sum_{i\geq i_0} a_i t^i$. We can define a topology of $\mathbf{K}$ such that the sets $t^i\mathbb{F}_q[[t]], i > 0$ are a basis of open neighborhoods of 0 in $\mathbf{K}$. In this case we have $v(a(t)) = i_0$ is $a_{i_0} \neq 0$.

b) $\mathbf{K} = \mathbb{Q}_p$. In this case the maximal compact subring is equal to the ring $\mathbb{Z}_p$ and $\|\|$ is the usual norm on $\mathbb{Q}_p$.

c) Any non-archimedian local field of positive characteristic is isomorphic to the field $\mathbb{F}_q((t))$.

d) Any non-archimedian local field of characteristic zero is isomorphic to a finite extension of the field $\mathbb{Q}_p$.

Definition 1.7. Let L be a one-dimensional K-vector space. We denote by $\|L\|$ the one-dimensional $\mathbb{R}$-vector space $L - \{0\} \times \mathbb{R}/K^{star}$ where $a \in K^{star}$ acts on $L - \{0\} \times \mathbb{R}/K^{star}$ by $a(l, r) = (al, \|a^{-1}\|r)$.

Exercise 1.8. E:tor a) Let V be a finite-dimensional $\mathbf{K}$-vector space of dimension d. Show that we can identify the real line $\|\Lambda^d(V^\vee)\|$ with the line of real-valued invariant measures on V.

b) Show that $\|\Lambda^d(V^\vee)\| \otimes \|\Lambda^d(V)\| \equiv \mathbb{R}$

Definition 1.9. D:add An additive character on $\mathbf{K}$ is a continuous complex-valued function ψ on $\mathbf{K}$ such that

$$\psi(a + b) = \psi(a)\psi(b), \forall a, b \in \mathbf{K}$$

Examples 1.10. a) $\mathbf{K} = \mathbb{F}_q((t))$. Then t is a generator of $\mathfrak{m}$ and the function

$\psi(a) := \bar{\psi}(a_{-1})$ is a non-trivial additive character of $\mathbf{K}$.

b) $\mathbf{K} = \mathbb{Q}_p$. Then p is a generator of $\mathfrak{m}$ and the function

$a \to \psi(a) := \exp(2\pi i\tilde{a})$ where $\tilde{a} \in \mathbb{Z}[1/p]$ is such that $a - \tilde{a} \in \mathbb{Z}_p$ is a well defined non-trivial additive character of $\mathbf{K}$.

c) If $\mathbf{K}$ is a finite extension of $\mathbb{Q}_p$, then the function $a \to \psi(Tr_{\mathbf{K}/\mathbb{Q}_p} a)$ is a non-trivial additive character of $\mathbf{K}$ which we will also denote by ψ. For any $\lambda \in \mathbf{K}$ we denote by ψ_λ the additive character

$x \to \psi_\lambda(x) := \psi(\lambda x), x \in \mathbf{K}$.

Exercise 1.11. Show that the map $\lambda \to \bar{\psi}_\lambda$ defines a bijection between $\mathbf{K}$ and the set additive characters of $\mathbf{K}$.

Definition 1.12. D:sp1 a) We denote by $\mathcal{S}(\mathbf{K})$ the space of locally constant complex-valued functions on $\mathbf{K}$ with compact support.

b) For any $f \in \mathcal{S}(\mathbf{K})$ we define $f^- \in \mathcal{S}(\mathbf{K})$ by

$$f^-(a) := f(-a)$$

c) we denote by $\phi_0 \in \mathcal{S}(\mathbf{K})$ the characteristic function of the subset $\mathcal{O} \subset \mathbf{K}$ and by μ_0 be the Haar measure on the additive group of $\mathbf{K}$ such that $\mu_0(\mathcal{O}) = 1$.

Remark 1.13. The integration

$$\int : \mathcal{S}(\mathbf{K}) \to \mathbb{C}, f \to \int_{\mathbf{K}} f(a)\mu_0$$

is reduced to a finite sum and the topology of $\mathbb{C}$ does not play any role.

Definition 1.14. D:F1 a) Using our choice of a Haar measure μ_0 we define the scalar product $<,>$ on $\mathcal{S}(\mathbf{K})$ by

$$< f, g >:= \int_{\mathbf{K}} f(a)g(a)\mu_0, f, g \in \mathcal{S}(\mathbf{K})$$

b) For any function $f \in \mathcal{S}(\mathbf{K})$ we define the Fourier transform $\mathcal{F}(f)$ as a function on $\mathbf{K}$ given by

$$\mathcal{F}(f)(x) := \int_{y\in K} \psi(xy)f(y)\mu_0$$

Exercise 1.15. a) For any $f \in \mathcal{S}(\mathbf{K})$ we have $\mathcal{F}(f) \in \mathcal{S}(\mathbf{K})$,

b)$\mathcal{F}^2(f) = f^-$

c) $< \mathcal{F}(f), \mathcal{F}(g) >=< f, g^- >$.

d) $\mathcal{F}(\phi_0) = \phi_0$.

Definition 1.16. D:F'1 a) We denote by $\mathcal{S}'(\mathbf{K})$ the space of linear functionals on $\mathcal{S}(\mathbf{K})$ and by $M(\mathbf{K}) \subset \mathcal{S}'(\mathbf{K})$ the subspace of locally constant measures on $\mathbf{K}$ with compact support.

b) For any $\mu', \mu'' \in M(\mathbf{K})$ we define a measure $\mu' \star \mu'' := +_\star(\mu'\Box\mu'')$ on $\mathbf{K}$ where $+ : \mathbf{K}^\times\mathbf{K} \to \mathbf{K}$ is the addition.

c) For any invertible linear operator $A \in Aut(\mathcal{S}(\mathbf{K}))$ we define $A^\vee \in Aut(\mathcal{S}'(\mathbf{K}))$ by

$$A^\vee(\alpha)(f) := \alpha(A^{-1}(f))$$

d) We define an imbedding $i : \mathcal{S}(\mathbf{K}) \hookrightarrow \mathcal{S}'(\mathbf{K})$ by

$$i(f)(g) :=< f, g^- >$$

e) We define the Fourier transform $\mathcal{F}' : \mathcal{S}'(\mathbf{K}) \to \mathcal{S}'(\mathbf{K})$ by

$$\mathcal{F}' := \mathcal{F}^\vee$$

Exercise 1.17. E:F a) The imbedding i identifies $\mathcal{S}(\mathbf{K})$ with the subspace $M(\mathbf{K}) \subset \mathcal{S}'(\mathbf{K})$.

b) For any $\mu', \mu'' \in M(\mathbf{K})$ we have $\mu' \star \mu'' \in M(\mathbf{K})$.

c) For any $\mu', \mu'' \in M(\mathbf{K})$ we have $\mathcal{F}(\mu' \star \mu'') = \mathcal{F}(\mu')\mathcal{F}(\mu'')$

Exercise 1.18. Show that $i_0(\mathcal{F}(f)) = \mathcal{F}(i_0(f^-)), \forall f \in \mathcal{S}(\mathbf{K})$.

1.3. **The multiplicative group $\mathbf{K}^\star$.** Using the valuation $v : \mathbf{K}^\star \to \mathbb{Z}$ we see that the multiplicative group $\mathbf{K}^\star$ is an extension of $\mathbb{Z}$ be the compact group $\mathcal{O}^\star := v^{-1}(0)$. A choice of a generator of $\mathfrak{m}$ defines a splitting $\mathbf{K}^\star = \mathbb{Z} \times \mathcal{O}^\star$.

Definition 1.19. a) We define a representation $\rho : \mathbf{K}^\star \to Aut(\mathcal{S}(\mathbf{K}))$ by

$$\rho(c)(f)(a) := f(c^{-1}a)\|c\|^{-1/2}$$

b) We denote by $\rho^\vee : \mathbf{K}^\star \to Aut(\mathcal{S}'(\mathbf{K}))$ the dual representation $c \to \rho^\vee(c) := \rho(c)^\vee$

Exercise 1.20. Show that $i_0(\rho(c)(f)) = \rho^\vee(c)(i_0(f))$.

Definition 1.21. D:mult1 a) A *multiplicative character* of $\mathbf{K}$ is a continuous group homomorphism $\chi : \mathbf{K}^\star \to \mathbb{C}^\star$.

b) For any multiplicative character χ we define

$$\mathcal{S}'_\chi(\mathbf{K}) = \{\alpha \in \mathcal{S}' | \rho^\vee(c)(\alpha)\chi(c)\alpha, \forall c \in \mathbf{K}^\star\}$$

Examples 1.22. For any $z \in\to \mathbb{C}^\star$ the map $a \to z^{v(a)}, a \in \mathbf{K}^\star$ is multiplicative character. For any multiplicative character χ and $z \in \mathbb{C}^\star$ we define the multiplicative character χ_z by

$$\chi_z(a) = \chi(a) z^{v(a)}$$

Exercise 1.23. a) Show that $\mathcal{F}'(\mathcal{S}'_\chi(\mathbf{K})) = \mathcal{S}'_{\chi^{-1}}(\mathbf{K})$.

b) Prove that for any multiplicative character χ we have

$$dim(\mathcal{S}'_\chi(\mathbf{K})) = 1$$

Definition 1.24. D:FV1 a) We denote by $\mathcal{S}(V)$ be the space of complex-valued locally constant functions on V with compact support, by $M(V)$ the space of locally constant measures on V with compact support by $M(V)$ the space of locally constant measures on V with compact support [we say that a measure μ on V is locally constant μ is invariant under shifts by $u \in U$ where $U \subset V$ is an open subgroup] and by $\mathcal{S}'(V)$ the space of linear functionals on by $\mathcal{S}(V)$. We have a natural imbedding $M(V) \to \mathcal{S}'(V)$. As we know we can identify the space $M(V)$ with $\mathcal{S}(V) \otimes \|\Lambda^d(V^\vee)\|$.

b) For any $\mu \in M(V)$ we define it's *Fourier transform* as a function $\mathcal{F}_V(f)$ on the dual space $V^\vee$ by

$$\mathcal{F}_V(\mu)(v^\vee) := \int_{v \in V} \psi(< v^\vee, v >)\mu$$

We can consider $\mathcal{F}_V$ as a map from $\mathcal{S}(V) \otimes \|\Lambda^d(V^\vee)\|$ to $\mathcal{S}(V\vee)$ or as map from $\mathcal{S}(V)$ to $\mathcal{S}(V\vee) \otimes \|\Lambda^d(V)\|$ (see Exercise 1.8).

We can consider $\mathcal{F}_{V\vee} \circ \mathcal{F}_V$ as a map from $\mathcal{S}(V)$ to $\mathcal{S}(V) \otimes \|\Lambda^d(V)\| \otimes \|\Lambda^d(V^\vee)\| = \mathcal{S}(V)$

c) Let dv be a Haar measure on V. Then we can identify the space $\mathcal{S}(V)$ with $M(V)$ by $f \to fdv$ and define the scalar product $<,>_V$ on $\mathcal{S}(V)$ by

$$< f', f'' >=:= \int_V f'(v)f''(v)dv$$

d) We define representation of the group $Aut(V)$ of linear automorphisms of V on $\mathcal{S}(V)$ by

$$\rho(g)(f)(v) = f(g^{-1}v)\|det(g)\|^{-d/2}$$

and by $\rho^\vee$ the representation of $Aut(V)$ on $\mathcal{S}'(V)$ by

$$\rho^\vee(g)(\alpha)(f) := \alpha(\rho(g^{-1})(f))$$

e) We denote by $\mathcal{F}'_V \in Aut(\mathcal{S}'(V))$ an operator defined by

$$\mathcal{F}'_V(\alpha)(f) := \alpha(\mathcal{F}_V^{-1}(f))$$

Exercise 1.25. a) For any $f \in \mathcal{S}(V)$ we have $\mathcal{F}_V(f) \in \mathcal{S}(V^\vee)$.

b) Show that $\mathcal{F}_{V\vee} \circ \mathcal{F}_V(f) = f^-$.

c) There exists a Haar measure $dv^\vee$ on $V^\vee$ such that

$$\mathcal{F}_{V^\vee} \circ \mathcal{F}_V(f) = f^-, f \in \mathcal{S}(V)$$

d) For any $f', f'' \in \mathcal{S}(V)$ we have

$$< f', f'' >_V =< \mathcal{F}_V(f'), \mathcal{F}_V(f'') >_{V^\vee}$$

e)

$$\mathcal{F}(\rho(g)f)) = \rho^\vee(g)(\mathcal{F}(f), f \in \mathcal{S}(V)$$

2. Fourier transform over 2-dimensional local fields.

Let $\mathbf{F} = \mathbf{K}((t))$ be the field of formal Laurent series over $\mathbf{K}$. For any $x(t) = \sum_i a_i t^i, a_i \in \mathbf{K}$ in $\mathbf{F}$ we define $[x] := a_0 \in \mathbf{K}$ to be the constant term of $x(t)$.

Given a finite-dimensional $\mathbf{F}$-vector space $\mathbf{V}$ one can try to associate to it a topological vector space $\mathbf{S(V)}$ and to define the Fourier transform $\mathcal{F} : \mathbf{S(V)} \to \mathbf{S(V^\vee)}$.

Remark. *It is more correct to consider* $\mathbf{S(V)}$ *as a pro-vector space*

We consider first the special case when $\mathbf{V} = \mathbf{F}$.

Definition 2.1. D:S2 a) Let $\mathcal{O}_{\mathbf{F}} := \mathbf{K}[[t]]$ be the ring of formal Taylor series over $\mathbf{K}$. and as before da be the Haar measure on $\mathbf{K}$ such that $\int_{a\in\mathcal{O}_K} da = 1$.

b) For any pair $m, n \in \mathbb{N}$ the quotient $N_n^m := t^{-m}\mathcal{O}/t^n\mathcal{O}$ is isomorphic to the $n+m$-dimensional $\mathbf{K}$-vector space of polynomials $x(t) = \sum_{-m}^{i=n-1} a_i t^i, a_i \in \mathbf{K}$.

c) We have an natural imbeddings $i : N_n^m \to N_n^{m+1}$ and also projections $p : N_{n+1}^m \to N_n^m$ when we forget the coefficients at t^n.

d) For any pair $m, n \in \mathbb{N}$ we define a vector space $\mathcal{S}_n^m := \mathcal{S}(N_n^m)$ of locally constant complex-valued functions f on V with compact support. The imbeddings i defines the restrictions

$$i^* : \mathcal{S}_n^{m+1} \to \mathcal{S}_n^m$$

and projections p define the pushforwards $p_* : \mathcal{S}_{n+1}^m \to \mathcal{S}_n^m$

where for any $f \in \mathcal{S}_{n+1}^m, x = \sum_{i=n-1}^{-m} a_i t^i \in N_n^m$

$$p_*(f)(x) := \int_{a\in\mathbf{K}} f(x + at^n)da$$

e) We define now the space $\mathbf{S}(\mathbf{F})$ as the vector space of sequences $\{f_n^m\} \in \mathcal{S}_n^m$

$$i^*(f_n^{m+1}) = p_*(f_{n+1}^m) = f_n^m$$

such that for all $(m, n) \in \mathbb{N}$.

The space $\mathbf{S}(\mathbf{F})$ has a natural topology such that open sets are unions of fibers of the projections $\mathbf{S}(\mathbf{F}) \to S_n^m$.

f) For any pair $m, n \in \mathbb{N}$ we define a pairing $<,>: N_n^m \times N_m^n \to \mathbf{K}$ of $\mathbf{K}$-vector spaces as follows. Given $\bar{x} \in N_m^n, \bar{y} \in N_n^m$consider the product $xy \in \mathbf{F}$ of representatives $x, y \in \mathbf{F}$ of $\bar{x}, \bar{y}$. It is easy to see that the constant term $< \bar{x}, \bar{y} >:= [xy]$ does not depend on a choice of representatives of $\bar{x}, \bar{y}$.

g) Since the pairing $<,>: N_n^m \times N_m^n \to \mathbf{K}$ is non-degenerate we identify the N_m^n with the dual space to N_n^m and denote by $\mathcal{F} : S_n^m \to S_m^n$ the Fourier transform.

Exercise 2.2. a) For any $f \in S_n^m$ we have

$$i^*(\mathcal{F}(f)) = \mathcal{F}(p_*(f)), (p_*(\mathcal{F}(f)) = \mathcal{F}(i^*(f))$$

b) For any sequence $\{f_m^n\} \in \mathbf{S}(\mathbf{F})$ the sequence $\{\phi_n^m\}, \phi_n^m := \mathcal{F}(f_m^n)$ belongs to $\mathbf{S}(\mathbf{F})$

Definition 2.3. D:F2 We define the Fourier transform $\mathcal{F} : \mathbf{S}(\mathbf{F}) \to \mathbf{S}(\mathbf{F})$ by
$\mathcal{F}\{f_n^m\} := \{\phi_n^m\}$ where $\phi_n^m := \mathcal{F}(f_m^n)$

2.1. **The multiplicative group $\mathbf{F}^*$.** The multiplicative group $\mathbf{F}^*$ acts naturally on $\mathbf{F}$ and we can ask whether this action induces a representation of the group $\mathbf{F}^*$ on the space $\mathbf{S}(\mathbf{F})$.

The group $\mathbf{F}^*$ is a direct product of the subgroup of generated by an element t and the subgroup $\mathcal{O}^*$ of series $x((t))$ of the form $x((t)) = \sum_{i\geq 0} a_i t^i, a_i \in \mathbf{K}, a_0 \neq 0$.

The multiplication by t defines an isomorphism of vector spaces $N_n^m \to N_{n+1}^{m-1}$ and therefore of an isomorphism $\hat{t} : S_{n-1}^{m+1} \to S_n^m$ such that $\hat{t}(f)(x) := f(t^{-1}x), x \in N_n^m$. It is clear that the operators $\hat{t}$ commute with the operators i^* and p_*. In other word for any sequence $\mathbf{f} = \{f_n^m\} \in \mathbf{S}(\mathbf{F})$ the sequence

$$\tilde{t}(\mathbf{f}) := \{\phi_n^m\},\ \phi_n^m = \hat{t}(f_{n-1}^{m+1})$$

belongs to $\mathbf{S}(\mathbf{F})$.

Let us now try to define a representation of the group $\mathcal{O}^*$ on $\mathbf{S}(\mathbf{F})$. For any $u = \sum_{i\geq 0} u_i t^i \in \mathcal{O}^*$ the multiplication by u defines an isomorphism of vector spaces $N_n^m \to N_n^m$ and therefore of an isomorphism $\tilde{u} : S_n^m \to S_n^m$ such that $\tilde{u}(f)(x) := f(u^{-1}x), x \in N_n^m$. The operators $\tilde{u}$ commute with operators i^*. On the other hand, the operators p_* are defined as an integration by the Haar measure da which is preserved under the multiplication by u only if $\|u_0\|_p = 1$. So if $\|u_0\|_p \neq 1$ and $\mathbf{f} = \{f_n^m\} \in \mathbf{S}(\mathbf{F})$ the sequence

$$\tilde{u}(\mathbf{f}) := \{\phi_n^m\}, \phi_n^m = \tilde{u}(f_n^m)$$

does not belong to $\mathbf{S}(\mathbf{F})$.

One can correct the operators $\tilde{u}$ and consider operators $\hat{u} := \|u_0\|_p^{-n}\tilde{u} : S_n^m \to S_n^m$.

The operators $\hat{u}$ commute with the operators i^* and p_* and we obtain a representation of the group $\mathcal{O}^*$ on the space $\mathbf{S}(\mathbf{F})$. But the operators $\hat{t}$ and $\hat{u}$ on $\mathbf{S}(\mathbf{F})$ do not commute if $\|u_0\|_p \neq 1$. One can easily check that the operators $\hat{t}$ and $\hat{u}, u \in \mathcal{O}^*$ generate a group $\tilde{\mathbf{F}}^*$ which is a central extension of the group $\mathbf{F}^*$ by a cyclic group. In other words there exists a central extension

$$\{0\} \to \mathbb{Z} \to \tilde{\mathbf{F}}^* \to \mathbf{F}^* \to \{0\}$$

and we obtain a representation of the group $\tilde{\mathbf{F}}^*$ on $\mathbf{S}(\mathbf{F})$ such that the generator $\bar{1} \in \mathbb{Z}$ acts by the multiplication by p.

Lecture 2.

3. Some Analysis

Let $\mathbf{K}$ be a local, non-archimedean field. We want to do analysis on $X(\mathbf{K})$ where X is an algebraic variety defined over $\mathbf{K}$.

Definition 3.1.
(1) An l-space is a topological space which is Hausdorff, locally compact and 0-dimensional (i.e. totally disconnected: any point has a basis of open compact neighborhoods).
(2) An l-group is a Hausdorff topological group such that e (= identity) has a basis of neighborhoods which are open compact groups.

Fact. If G is an algebraic group over $\mathbf{K}$ then $G(\mathbf{K}))$ is an l-group.

Exercise 3.2. Let X be an l-space.
(1) If $Y \subset X$ is locally closed (i.e. the intersection of an open and a closed subset), then Y is an l-space.
(2) If $K \subset X$ is compact and $K \subset \bigcup_\alpha U_\alpha$ is an open covering, then there exists *disjoint* open compact $V_i \subset X$, $i = 1 \ldots k$ such that $V_i \subset U_\alpha$ for some α and $\bigcup V_i \supset K$.
(3) Let G be an l-group which is countable at infinity (i.e. G is a countable union of compact sets). Suppose that G acts on an l-space X with a finite number of orbits. Then G has an open orbit $X_0 \subset X$ so that $X_0 \approx G/H$ for some closed subgroup $H \subset G$.

It is obvious that by applying this lemma to $X \setminus X_i$, we can get a stratification $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X$ such that $X_i \setminus X_{i-1}$ is an orbit.

Example. Let $G = \mathrm{GL}(n, K)$, B = the set of upper triangular matrices. Then we may set $X = G/B$ and consider the action of B on X. When $n = 2$, $X = \mathbb{P}^1$ and there are two orbits: a single point and the complement of that point.

3.1. Distributions.

If X is an l-space, let $S(X)$ be the algebra of locally constant, compactly supported, complex-valued functions on X. $S(X)$ will serve as the "test functions" for our analysis on X. Thus, $S^*(X)$ = the set of functionals on $S(X)$ are called *distributions*. Note that as $S(X)$ has no topology, there is obviously no continuity assumed.

Exercise 3.3. E:exact [Exact Sequence of an Open Subset] a) Let $U \subset X$ be open and $Z = X \setminus Z$. Then

$$0 \to S(U) \to S(X) \to S(Z) \to 0$$

is exact.

b) If A, B are compcat l-spaces then $S(A \times B) = S(A) \otimes S(B)$.

Proof. For the injection at $S(U)$ just extend functions on U by zero to all of X. For the surjection at $S(Z)$ we must explain how to extend functions from a *closed* subset. Since $f \in S(Z)$ is locally constant and compactly supported, we may assume that Z is compact and has a covering by a finite number of open sets U_α with $f|_{U_\alpha} = c_\alpha$ constant. Let V_i be as in Lemma 1 (2). Then we can extend f by defining $f(x) = c_\alpha$ if $x \in V_i \subset U_\alpha$ and zero otherwise. □

Corollary 3.4. *The sequence of distributions*

$$0 \to S^*(Z) \to S^*(X) \to S^*(U) \to 0$$

is exact.

3.2. **Idempotented Algebras.** Unless X is compact, $S(X)$ has no identity element; $\mathbf{1}$ is not compactly supported. However, if $K \subset X$ is open and compact, e_K = characteristic function of K is an idempotent in $S(X)$.

Definition 3.5.
(1) An algebra $\mathcal{H}$ is an *idempotented algebra* if for every finite collection of elements of $\mathcal{H}$, $\{f_i\}$, there exists an idempotent $e \in \mathcal{H}$ such that $ef_i = f_ie = f_i$ for all i.
(2) A module M of an idempotented algebra $\mathcal{H}$ is called *non-degenerate* or *unital* if $\mathcal{H}M = M$.

It is clear that $S(X)$ is an idempotented algebra: let K be an open compact set containing the support of the f_i's; then $e = e_K$ works.

If $\mathcal{H}$ is an idempotented algebra, we will denote by $\mathcal{M}(\mathcal{H})$ the category of non-degenerate $\mathcal{H}$-modules.

4. Smooth Representations of l-Groups: Definitions.

Definition 4.1. Let V be a representation of an l-group G. A vector $v \in V$ is *smooth* if its stabilizer in G is open.

We will denote the set of smooth vectors in V by $V_{\mathrm{sm}} \subset V$.

Exercise 4.2.
(1) V_{sm} is a G-invariant subspace of V.
(2) If V is a topological representation, then V_{sm} is dense in V.

We will study *smooth representations*, that is, representations V such that $V_{\mathrm{sm}} = V$.

4.1. **Smooth Representations of l-Groups: the Hecke Algebra.** We are interested in studying $\mathcal{M}(G)$ = the category of smooth representations of an l-group G.

Definition 4.3. If X is an l-space, define the *support of a distribution* $\mathcal{E} \in S^*(X)$ by $\operatorname{Supp} \mathcal{E}$ = the smallest closed subset S such that $\mathcal{E}|_{X \setminus S} = 0$.

If $\mathcal{E}$ is distribution supported on an open compact set A, then it defines a functional on the set of locally constant functions, $C^\infty(X)$ by

$$< \mathcal{E}, f > \stackrel{\text{def}}{=} < \mathcal{E}, e_A f > .$$

Definition 4.4. a) For any distributions $\alpha, \beta \in S^*(G)_c$ we define a distribution $\alpha \star \beta \in S^*(G)_c$ by

$$\alpha \star \beta(f) := (\alpha \otimes \beta)(m^\star f)_{A \times B}$$

where $m : G \times G \to G$ is the product and $A, B \subset G$ are compacts such that $supp(\alpha) \subset A, supp(\beta) \subset B$.

b) G acts on the space $S^*(G)_c$ by left translation. We define $\mathcal{H}(G) = (S^*(G)_c)_{sm}$.

It is clear that $\mathcal{H}(G) \subset S^*(G)_c$ is a subalgebra.

c) The algebra $\mathcal{H}(G)$ is called the *Hecke Algebra*.

Exercise 4.5.
(1) Multiplication by Haar measure gives an isomorphism $S(G) \to \mathcal{H}(G)$.
(2) Any $h \in \mathcal{H}(G)$ is locally constant with respect to the right action.
(3) Suppose K is a compact open subgroup of G and h is a K-invariant distribution with compact support. Then there exist $g_1, \ldots, g_k \in G$ and $a_1, \ldots, a_k \in \mathbb{C}$ such that

$$h = \sum_{i=1}^{k} a_i (e_K * e_{g_i}).$$

If (π, V) is a smooth representation of G, we can give V the structure of an $\mathcal{H}(G)$-module (in fact, of an $S^*(G)_c$-module) as follows. Choose $\mathcal{E} \in S^*(G)_c$ and fix an open compact $K \subset G$ containing the support of $\mathcal{E}$.

For fixed v, $\pi(g)v$ may be considered as a locally constant function on G with values in V, and thus the restriction of $\pi(g)v$ may be considered as as an element of $C^\infty(K) \otimes V$. Therefore, it makes sense to define

$$\pi(\mathcal{E})v =< \mathcal{E}, \pi(g)v >$$

Proposition 4.6. *P:Heck Let G be an l-group.*

1. *$\mathcal{H}(G)$ is an idempotented algebra.*
2. *If V is a smooth G-module, then the associated $\mathcal{H}(G)$-module is non-degenerate.*
3. *This gives an equivalence of categories*

$$\mathcal{M}(G) \cong \mathcal{M}(\mathcal{H}(G))$$

between smooth representations of G and non-degenerate $\mathcal{H}(G)$-modules.

Proof.

If Γ is a compact subgroup, then normal Haar measure on Γ, $e_\Gamma \in S^*(G)_c$; if Γ is open and compact, then $e_\Gamma \in \mathcal{H}(G)_c$. Moreover, if $g \in G$, the δ distribution at g, $e_g \in S^*(G)_c$ (but not in $\mathcal{H}(G)$; it is not locally constant) . These satisfy the relations $e_\Gamma * e_\Gamma = e_\Gamma$, $e_\Gamma * e_g = e_g * e_\Gamma = e_\Gamma$ if $g \in \Gamma$.

Clearly, if K is a compact open subgroup of G, e_K is an idempotent in $\mathcal{H}(G)$. In fact, e_K is the unit for the algebra $\mathcal{H}_K = e_K \mathcal{H}(G) e_K$. As $\mathcal{H}(G) = \bigcup \mathcal{H}_K$, we see that $\mathcal{H}(G)$ *is an idempotented algebra.*

Lemma 4.7. *Let $\mathcal{H}$ be any idempotented algebra, $A\colon \mathcal{H} \to \mathcal{H}$ an operator commuting with the right action of $\mathcal{H}$. Then there exists unique way to associate to any non-degenerate $\mathcal{H}$-module M a morphism $A_M : M\colon M \to M$ such that for all morphisms $\varphi\colon M \to N$ there is a commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{\varphi} & N \\ {\scriptstyle A_M}\downarrow & & \downarrow{\scriptstyle A_N} \\ M & \xrightarrow{\varphi} & N \end{array}$$

and $A_{\mathcal{H}} = A$.

Proof. I'll show the existence of A_M and leave for you to check the uniqueness. Since M is non-degenerate, each $m \in M$ has the form hm for some $H \in \mathcal{H}$. Thus, we may define $A(hm) = (Ah)m$. □

Now we can finish the proof of the Proposition.

Proof. Given an $\mathcal{H}(G)$-module M, we must show how to define a G-module. First observe that if $\mathcal{E} \in S^*(G)_c$, then $h \mapsto \mathcal{E} * h$ is an operator on $\mathcal{H}(G)$ commuting with convolution on the right. By the lemma, this extends to an operator $\mathcal{E}\colon M \to M$. Specializing $\mathcal{E} = e_g$, this gives M the structure of a G-module. □

4.1.1. *Contragredient Representations.* It is easy to see that the dual $V^\star$ of a smooth representation V may not be smooth.

Definition 4.8. If (π, V) is a smooth representation, the *contragredient representation* $(\tilde{\pi}, \tilde{V})$ is given by $\tilde{V} = V^*_{sm}$ and $\tilde{\pi} = \pi^*(g^{-1})|_{\tilde{V}}$.

Exercise 4.9. (1) For all compact open subgroups K, $\tilde{\pi}(e_K)\tilde{V} = (\pi(e_K)V)^*$.

(2) For any smooth representations V, W of G we have a natural isomorphism

$$\mathrm{Hom}_G(V, \tilde{W}) = \mathrm{Hom}_G(W, \tilde{V})$$

(3) For any smooth representation V we have a natural imbedding $V \hookrightarrow \tilde{\tilde{V}}$

(4) The functor $V \to \tilde{V}$ is exact.

Proof. (1) $\pi(e_K)V = V^K$ so this reduces to $\tilde{V}^K = (V^K)^*$. Which is obvious from the definitiion the of smoothness.

For (2),

$$\begin{aligned} \mathrm{Hom}_G(V, \tilde{W}) &= \mathrm{Hom}_G(V, W^*) \\ &= \mathrm{Hom}_G(V \otimes W, \mathbb{C}) \\ &= \mathrm{Hom}_G(W, V^*) \\ &= \mathrm{Hom}_G(W, \tilde{V}) \end{aligned}$$

where the first and last equalities follow since the image of a smooth module is always smooth.

(3) Just restrict to an appropraite compact open subgroup. □

Theorem 4.10. *$\mathcal{M}(\mathcal{H})$ has enough injectives.*

4.2. **Applications.** Recall that if $\mathcal{M}$ is an abelian category, an object $P \in \mathrm{Ob}(\mathcal{M})$ is *projective* if the functor

$$\begin{aligned} \mathcal{M} &\to \mathrm{Ab} \qquad \text{given by} \\ X &\mapsto \mathrm{Hom}(P, X) \end{aligned}$$

is exact and an object $Q \in \mathrm{Ob}(\mathcal{M})$ is *injective* if the functor

$$\begin{aligned} \mathcal{M} &\to \mathrm{Ab} \qquad \text{given by} \\ X &\mapsto \mathrm{Hom}(X, Q) \end{aligned}$$

is exact.

Theorem 4.11. *T:inj If $\mathcal{H}$ is an idempotented algebra, then the category $\mathcal{M}(\mathcal{H})$ has enough projectives and injectives.*

Proof. To prove that $\mathcal{H}$ has enough projectives we have to show that fro any non-degenerate $\mathcal{H}$-module X we can find a projective non-degenerate $\mathcal{H}$-module P and a surjective map $P \to X$.

Let $e \in \mathcal{H}$ be any idempotent. Consider the $\mathcal{H}$-module $P_e = \mathcal{H}e$. This is projective since $\operatorname{Hom}(P_e, X) = eX$ is clearly exact. Note that the direct sum of any collection of the P_e is also projective.

If $X \in \operatorname{Ob} \mathcal{M}(\mathcal{H})$ and $\xi \in X$, then it follows from non-degeneracy that there exists an idempotent e so that $e\xi = \xi$. Then ξ is in the image of the map $P_e \to X$ given by $he \mapsto h\xi$. Taking the direct sum over all $\xi \in X$ of the associated P_e, we see that X is a quotient of a projective object.

In order to prove that we have enough *injectives*, we use the following result.

Lemma 1. If P is a projective object, then $\tilde{P}$ is an injective object.

Proof. We must show that $X \mapsto \operatorname{Hom}(X, \tilde{P})$ is exact. But $\operatorname{Hom}(X, \tilde{P}) = \operatorname{Hom}(P, \tilde{X})$ by the earlier claim, so this is clear. □

Fix X. As we have enough projectives, there is an epimorphism $P \to \tilde{X}$. Now just consider the composition

$$X \hookrightarrow \tilde{\tilde{X}} \hookrightarrow \tilde{P}$$

where $\tilde{P}$ is injective by the lemma. □

4.2.1. *Admissible Representations.*

Definition 4.12. A smooth representation (π, V) of G is called *admissible* if for every open compact subgroup K, the space V^K is finite dimensional.

Exercise 4.13. An equivalent definition is that V is admissible if $V \to \tilde{\tilde{V}}$ is an ismorphism.

Remark. It is a (hard to prove) fact that every irreducible representation is admissible.

Let (ρ, W) be a representation of G. Consider $\operatorname{End} W$ as a $G \times G$-module under the action $(g_1, g_2)(a) = \rho(g_1) a \rho(g_2)^{-1}$. Let $\operatorname{End} W_{\mathrm{sm}}$ be the smooth part of this module. Then there exists a natural morphism of $G \times G$-modules

$$\alpha \colon W \otimes \tilde{W} \to \operatorname{End} W_{\mathrm{sm}}$$

such that $\alpha(w \otimes \tilde{w})(v) := \tilde{w}(v) w$.

Exercise 4.14. Let (ρ, W) be an irreducible representation of G.

a) Show that the linear map $\alpha\colon W \otimes \tilde{W} \to \operatorname{End} W_{\mathrm{sm}}$ is an imbedding.

b) Then there exists a natural morphism of $G \times G$-modules

$$\varphi\colon S(G) \cong \mathcal{H}(G) \to W \otimes \tilde{W}.$$

Moreover, φ is unique up to scaler and can be normalized so that $\operatorname{tr}\rho(h) =<,> \circ(\varphi(h))$. Here $<,>\colon W \otimes \tilde{W} \to \mathbb{C}$ is the natural pairing.

Lemma 4.15. *L:tensor If (ρ, W) is an admissible irreducible representation of G then the linear map $\alpha\colon W \otimes \tilde{W} \to \operatorname{End} W_{\mathrm{sm}}$ is onto.*

Proof. To prove the surjectivity of α it is sufficient to show that for any open compact subgroup $K \subset G$ the restriction

$$\alpha_K\colon W^K \otimes \tilde{W}^K \to (\operatorname{End} W)^{K \times K}$$

is surjective.

Since α is an imbedding it is sufficient to show that

$$dim((\operatorname{End} W)^{K\times K}) \leq (\dim W^K)^2 \leq \dim W \otimes \tilde{W}$$

. But it is clear that $(\operatorname{End} W)^{K\times K}$ is a subset of $\operatorname{End} W^K$. □

5. Induction and Jaquet Functors.

5.0.2. *Inductive limits.*

Definition 5.1. Let $\mathcal{C}$ be a category

a) We denote by $Pro^{\aleph_0}(\mathcal{C})$ the *pro-completion* of $\mathcal{C}$ which is the full subcategory of the category of functors $\mathcal{C} \to Sets$ which consists of objects isomorphic to ones of the form

$$X \mapsto \varprojlim Hom_{\mathcal{C}}(X_i, X)$$

where $\to X_i \to Xi-1 \to$ is a sequence of objects of $\mathcal{C}$.

b) We denote by $Ind^{\aleph_0}(\mathcal{C})$ the *ind-completion* $\mathcal{C}$ which is the full subcategory of the category of contravariant functors $\mathcal{C} \to Sets$ which consists of objects isomorphic to ones of the form

$$X \mapsto \varinjlim Hom_{\mathcal{C}}(X, X_i)$$

where $\to X_i \to Xi-1 \to$ is a sequence of objects of $\mathcal{C}$.

Remark 5.2. a) Instead of a sequence one has sometimes consider more general filtering sets of indexes.

Exercise 5.3. a) $Ind^{\aleph_0}(\mathcal{C}) = (Pro^{\aleph_0}(\mathcal{C}^0))^0$

b) $Ind^{\aleph_0}(Vect_f) = Vect_{\aleph_0}$ where $Vect_f$ is the category of finite-dimensional $\mathbb{C}$-vector spaces and $Vect_{\aleph_0}$ is the category of $\mathbb{C}$-vector spaces of finite or countable dimension.

c) Let $\hat{V}$ be an object of $Pro^{\aleph_0}(Vect)$ represented by the sequence $\to \mathbb{C}^i \to \mathbb{C}^{i-1} \to$ where $\mathbb{C}^i \to \mathbb{C}^{i-1}$ is the projection to the first $i-1$ coordinates and $V \in Ob(Vect)$ be the vector space of sequences $c_1, \dots, c_i, \dots c_i \in \mathbb{C}$ such that $c_i = 0 fori >> 0$. Construct a natural morphism $p : V \to \hat{V}$, show that p is surjective and describe the $Ker(p)$

Proposition 5.4. *In $\mathcal{C}$ is an abelian category then the categories $Pro^{\aleph_0}(\mathcal{C})$ and $Ind^{\aleph_0}(\mathcal{C})$ are also abelian.*

5.1. **Induction.** Remark. The way to make an advance in representation theory is to find a way to construct representations. Practically our only tool is induction.

If H is a closed subgroup of G, we may restrict G-modules to H. This gives a functor $\mathrm{Res} = \mathrm{Res}_H^G \colon \mathcal{M}(G) \to \mathcal{M}(H)$.

Exercise 5.5. Res has a right adjoint: $\mathrm{Ind} = \mathrm{Ind}_H^G \colon \mathcal{M}(H) \to \mathcal{M}(G)$.

More precisely for any smooth H-module (ρ, V) consider

$$L(V) = \{f \colon G \to V | f(gh) = \rho(h^{-1}) f(g)\}$$

with the action $\pi(g)f(x) = f(xg^{-1})$ and define $\mathrm{Ind}_H^G(\rho, V) := L(V)_{sm}$. Let $ev : L(V)_{sm} \to V$ be the evaluation at e.

Show that for any G-module (π, W) the composition $\phi \to ev \circ \phi$ defines a bijection

$$Hom_{\mathcal{M}(G)}(W, \mathrm{Ind}_H^G(\rho, V)) \to Hom_{\mathcal{M}(H)}(W, V)$$

There is another functor $\mathrm{ind} \colon \mathcal{M}(H) \to \mathcal{M}(G)$ given by

$$\mathrm{ind}(V) = \{f \in L(V) | f \text{ has compact support modulo } H\}.$$

Exercise 5.6. Show that in the case when H is an open subgroup of G then the functor $\mathrm{ind}(V)$ is the left adjoint to RES.

If G is not discrete the functor

$$Fiber : \mathcal{M}(G) \to Vect, (\rho, V) \to V$$

is not representable and therefore the functor RES_e^G of the restriction to $\{e\}$ does not have a left adjoint. But the functor $Fiber$ is pro-representable. Really let K_n be decreasing sequence of open compact subgroups which consitute a basis of neighborhoods of e. Let $R_n \subset \mathcal{G}$ be the representation of G on the space of right K_n-invariant functions and $p_n : R_n \to R_{n-1}$ be the averaging by the right action of the group K_{n-1}.

Exercise 5.7. For any representation ρ, V the morhisms p_n define an inductive system

$$\to Hom_G(R_n, V) \to Hom_G(R_{n+1}, V) \to$$

such that $\varinjlim Hom_G(R_n, V) = V$.

Proposition 5.8. *These functors have the following properties.*

(1) $\mathrm{ind}_H^G \subset \mathrm{Ind}_H^G$
(2) *Both are exact.*
(3) *If $H\backslash G$ is compact,* Ind = ind.
(4) *If $H\backslash G$ is compact, induction maps admissible representations to admissible representations.*

Proof. (1) and (3) are obvious. To prove (2) use arguments analogous to ones used in the proof of Exercise 3.3.

Now we prove (4). Let V be an admissible representation of H and fix $K \subset G$ a compact open subgroup. Let $\{Hg_iK\}$ be a system of coset representatives for $H\backslash G/K$. By our assumption, this is a finite set. It is clear that an element, $f \in L(V)^K$ is determined by its values on the g_i. Moreover, the image of g_i must lie in the subset of V fixed by $H \cap g_iKg_i^{-1}$. Since V is admissible this subspace is finite dimensional. Therefore, there can be only finitely many linearly independent such $f \in L(V)^K$. □

5.2. **Jacquet Functor.** If G is any group, and W is a vector space we denote by $W^G \in Ob(\mathcal{G})$ the trivial representation on the space W. When G is a finite group, it is often usefull to consider the functor $V \to V^G = \mathrm{Hom}_G(\mathbb{C}_G, V)$ of *invariants*. It turns out that when an l-groups is not comact this functor is never exact and therefore almost useless. However, we often use the functor of *coinvariants*, $V \to V_G = V/V(G)$ where $V(G)$ is the subspace spanned by $\pi(g)v - v$. If $H \subset G$ is a subgroup we have a natural imbedding $V(H) \hookrightarrow V(G)$ and therefore a natural projection $J_H(V) \to J_G(V)$. It is quite easy to see that this functor is equivalent to the *Jacquet Functor*

$$J_G \colon \mathcal{M}(G) \to Vect, J_G(V) = \mathbb{C}_G \otimes_G V$$

Exercise 5.9. If $G = \bigcup U_i$ then $J_G(V) = \varinjlim J_{U_i}(V)$ and

$V(G) = \{v \in V | \exists i$ such that $\int_{g \in U_i} \pi(g)v dg = 0\}$ where dg is a Haar measure on U_i.

Proposition 5.10. *P:J The Jacquet functor J has the following properties.*

(1) *J is right exact.*
(2) *If G is compact then J is exact.*
(3) *If G is a union of an increasing family of compact groups U_i, then J_G is exact.*

Proof. (1) It is clear that for any vector space W the natural map

$$Hom_{Vect}(J_G(V), W) \to Hom_{\mathcal{G}}(V, W^G)$$

is a bijection. So the functor J_G has a right adjoint. Therefore it is right exact.

(2) We have an exact sequence

$$0 \to V^G \to V \to V_G \to 0.$$

When G is compact, this implies that $e_G V = V^G = V_G$. But V^G is clearly left exact.

(3) As follows from (2) and the previous Exercise the functor J a direct image of exact functors. But the direct image of exact functors is exact. So (3) follows from (2). □

Remark. Another way to prove (3) is to construct explicitly a functor $\alpha : Vect \to Pro\mathcal{M}(G)$ which is right adjoint to J_G. To do this consider the representations ρ_i of G in the subspace $R_i \subset \mathcal{G}$ of functions which are right U_i-invariant. If $j > i$ then we have a natural imbedding $R_j \hookrightarrow R_i$.

Exercise 5.11. $J_G(V) = \varinjlim Hom_{\mathcal{M}(G)}(R_i, V)$

5.3. **Unipotent groups.**

Definition 5.12. D:un a) For any $n > 0$ we denote by $\underline{U_n} \subset \underline{GL_n}$ the subgroup of upper-triangular unipotent matricies.

b)If $\mathbf{K}$ is a local field we define for any $r \geq 0$ a subgroup $U_n^r \subset U_n := \underline{U_n}(\mathbf{K})$ the subset of matricies $u = (u_{ij}), 1 \leq i, j \leq n$ such that

$$v(u_{ij}) \geq r(i - j)$$

c) An algebraic $\mathbf{K}$-group $\underline{U}$ is *unipotent* if U is isomorphic to a subgroup of $\underline{U_n}$ for some $n > 0$

Remark 5.13. If U is a topological group of the form $\underline{U}(\mathbf{K})$ where $\underline{U}$ is a unipotent $\mathbf{K}$-group we say that U is a *unipotent group*

Exercise 5.14. a) U_n^r is a compact open subgroup of U_n and U_n is equal to the union of its subgroups U_n^r.

b) Any unipotent group can be written as a union of its subgroups compact open subgroups

We see that for any unipotent group U the functor of U-coinvariants is exact.

Lecture 3.

6. Some Representation Theory

6.1. **Jordan-Holder Content.**

Definition 6.1. A representation (ρ, V) is *irreducible* if it is algebraically irreducible, that is if it has no invariant subspaces.

(1) If G is an l-group, let $\operatorname{Irr} G$ be the set of *equivalence classes* of irreducible representations G.
(2) If $M \in \mathcal{M}(G)$, then the *Jordan-Holder content* of M, $\mathrm{JH}(M)$, is the subset of $\operatorname{Irr} G$ consisting of all irreducible subquotients of M.

Lemma 6.2. *L:JH If* $M \neq 0$ *then* $\mathrm{JH}(M) \neq \emptyset$ *then .*

Proof. Choose $v \in M\{0\}$ and denote by $\tilde{M} \subset M$ the subrepresentation generated by v. Since $\tilde{M}$ is generated by v it is clear that for any increasing sequence $N_j, j \in J$ of proper submodules of $\tilde{M}$ the union $\cup_j N_j \neq \tilde{M}$. By the Zorn's lemma there exists a submodule $N \subset \tilde{M}$ which a maximal proper submodule of $\tilde{M}$. Then the quotient $\tilde{M}/N$ is irreducible. □

Exercise 6.3. a) If N is a subquotient of M, then $\mathrm{JH}(N) \subset \mathrm{JH}(M)$.

b) If $M = \sum_\alpha M_\alpha$ then $\mathrm{JH}(M) = \bigcup_\alpha \mathrm{JH}(M_\alpha)$.

Examples 6.4. If G is a compact group then every smooth G-module M is completely reducible, that is $M \cong \bigoplus W_\alpha$ where the W_α are irreducible. Thus, the representation theory is entirely controlled by the irreducibles and in a simple way.

6.2. **Decomposing Categories.**

In this section we discuss categories which satisfy enough conditions so that our definitions make sense (called AB4, or something like that). We will only be interested in categories of modules as we have been discussing.

Let $\mathcal{M}$ be a category. Then $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ means that for any object $M \in \mathcal{M}$, there exist subobjects $M_i \in \mathcal{M}_i$ so that $M = M_1 \oplus M_2$. Of course, if $V \in \operatorname{Irr} \mathcal{M}$ = isomorphism classes of irreducible objects in $\mathcal{M}$, then this implies that either $V \in \mathcal{M}_1$ or $V \in \mathcal{M}_2$. This leads to a decomposition

$$\operatorname{Irr} \mathcal{M} = \operatorname{Irr} \mathcal{M}_1 \coprod \operatorname{Irr} \mathcal{M}_2.$$

(Here $\coprod$ means 'disjoint union'.) Conversly, we will see that such a decomposition on the level of sets completely determines the decomposition on the level of categories.

Let $S \subset \operatorname{Irr} \mathcal{M}$. Denote by $\mathcal{M}(S)$ the full subcategory of $\mathcal{M}$ consisting of objects M with $\mathrm{JH}(M) \subset S$.

Exercise 6.5. If $S, S' \subset \operatorname{Irr} \mathcal{M}$ do not intersect, then the categories $\mathcal{M}(S)$ and $\mathcal{M}(S')$ are orthogonal, i.e. $M \in \mathcal{M}(S)$ and $M' \in \mathcal{M}(S')$ imply $\operatorname{Hom}(M, M') = 0$.

Proof. Suppose $\alpha \in \operatorname{Hom}(M, M')$. Set $N = \alpha(M)$. So, $\mathrm{JH}(N) \subset \mathrm{JH}(M) \subset S$ and also $\mathrm{JH}(N) \subset \mathrm{JH}(M') \subset S'$. But $S \cap S' = \emptyset$ so by the last lemma, $N = 0$. □

If $S \subset \operatorname{Irr} \mathcal{M}$, $M \in \mathcal{M}$, we will denote by $M(S)$ the union of all subobjects of M which lie in $\mathcal{M}(S)$. By the lemma, this is the maximal submodule with Jordan-Holder content lying in S.

Definition 6.6. D:split Let $S \subset \operatorname{Irr} \mathcal{M}$ and $S' = \operatorname{Irr} \mathcal{M} \setminus S$. S is called *splitting* if $\mathcal{M} = \mathcal{M}(S) \times \mathcal{M}(S')$. That is, if $M = M(S) \oplus M(S')$ for each $M \in \mathcal{M}$. In this case we say that S *splits* M.

Exercise 6.7. A decomposition of categories $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ is equivalent to a decomposition of sets $\operatorname{Irr} \mathcal{M} = S \coprod S'$ where S *is a splitting subset.*

Proof. Obvious. □

Examples 6.8. $G = F^*$ (This is "almost" compact.) Let π be a generator for the maximal ideal in the ring of integers $\mathcal{O} \subset F$. Then

$$F^* \cong \pi^{\mathbb{Z}} \times \mathcal{O}^*.$$

Here $\mathcal{O}^*$ is compact and $\mathcal{M}(\mathbb{Z})$ coincides with the category $\mathcal{M}(\mathbb{C}[t, t^{-1}])$ of sheaves on $\mathbb{C}^*$. Thus,

$$\mathcal{M}(G) \cong \prod_{\substack{\text{irred}\\ \text{reps of}\\ \mathcal{O}^*}} \mathcal{M}(\mathbb{Z}) = \prod_{\substack{\text{irred}\\ \text{reps of}\\ \mathcal{O}^*}} \mathcal{M}(\mathbb{C}[t, t^{-1}]).$$

The point here is that the structure of the representations is half discreet and half continuous. Specifically, it is a discrete sum of the category of sheaves on some space. We will see that this is a typical situation.

6.3. **Results on Irreducible Modules.**

Lemma 6.9 (Schur's Lemma)**.** *Suppose G is countable at infinity. Let (ρ, V) be an irreducible representation of G. Then $\operatorname{End}_G V = \mathbb{C}$.*

Proof. Since V is irreducible, $\mathcal{A} = \operatorname{End} V$ is a skew-field. Moreover, $\mathcal{A}$ has countable dimension over $\mathbb{C}$. Indeed, by irreducibility, it is enough to show that the dimension of V is countable. If $\xi \in V$, then V is spanned by the $\rho(g)\xi$ for $g \in G$. But since G is countable at infinity and the function $g \mapsto \rho(g)\xi$ is locally constant (smoothness), we can find a countable spanning set.

Thus we are reduced to proving

Lemma 6.10. *If $\mathcal{A}$ is a skew-field of countable dimension over $\mathbb{C}$, then $\mathcal{A} = \mathbb{C}$.*

Proof. Let $a \in \mathcal{A}$. Suppose $a - \lambda \neq 0$ for any $\lambda \in \mathbb{C}$. Since $\mathcal{A}$ has only countable dimension, the elements $(a-\lambda)^{-1}$ cannot be linearly independent. Thus, there are $c_i \in \mathbb{C}$ so that

$$\sum_{i=1}^{k} c_i (a - \lambda_i)^{-1} = 0.$$

Multiplying through by $\prod (a - \lambda_i)$, we get a non-zero polynomial over $\mathbb{C}$ with a as a root. Factoring this polynomial, we see that there are $\mu_j \in \mathbb{C}$ so that

$$\prod_j (a - \mu_j) = 0.$$

Now one of these factors must be zero because otherwise $\mathcal{A}$ would have zero divisors. Hence $a \in \mathbb{C}$. □

□

Remarks. 1. We will eventually show that the irreducible representations of any reductive p-adic group are admissible. Then we will be able to stop worrying about technical conditions like "countable dimension".

2. One can show that the condition that G is countable at infinity is necessary for the validity of the Schur's Lemma.

The next proposition which we call *the separation lemma* shows that our Hecke algebra resembles a semisimple algebra.

Proposition 6.11. *P:Separation Suppose that G is countable at infinity. Let $h \in \mathcal{H}(G)$, $h \neq 0$. Then there exists an irreducible representation ρ such that $\rho(h) \neq 0$.*

Proof. We will give the proof for unimodular groups.

Choose an open comact subgroup K of G such that h is two-sided K-invariant. Then we can consider th as an element of $\mathcal{H}_K$. We start with the following result

Lemma 6.12. *L:gh Let (ρ, W) be an irreducible representation of G. Then $(\rho|_{\mathcal{H}_K}, W^K)$ is either 0 of an irreducible representation of $\mathcal{H}_K$. Every irreducible representation of $\mathcal{H}_K$ appears this way.*

Proof. Let w_1, $w_2 \in W$ and denote by $\tilde{w}_1$, $\tilde{w}_2$ their images in W^K. There is a $h \in \mathcal{H}$ such that $hw_1 = w_2$. Then, $e_K h e_K \tilde{w}_1 = \tilde{w}_2$. Thus, W^K is irreducible.

Let $V \in \mathcal{M}(\mathcal{H}_K)$ be irreducible. Set $U = \mathcal{H} \otimes_{\mathcal{H}_K} V \in \mathcal{M}(\mathcal{H})$. It is obvious that $V \in \mathrm{JH}(U^K)$. Moreover, taking K-fixed vectors gives an onto map $\mathrm{JH}(U) \to \mathrm{JH}(U^K)$. □

We see that it is sufficient to prove the existence of an irreducible representation ρ of the $\mathcal{H}_K$ such that $\rho(h) \neq 0$.

Assume first that $h \in \mathcal{H}_K$ is not nilpotent. Then the following general result implies the existence of ρ. f from the

Lemma 6.13. *Let $\mathcal{A}$ be an algebra of countable dimension over $\mathbb{C}$ with unit. Let $a \in \mathcal{A}$ be not nilpotent. Then there exists a simple $\mathcal{A}$-module M such that $a|_M \neq 0$.*

Proof. The proof is similar to that of Schur's lemma. First show the existence of $\lambda \in \mathbb{C} \setminus 0$ such that $a - \lambda$ is not invertible in $\mathcal{A}$.

If $a \in \mathbb{C}$, this is trivial. Otherwise, by countable-dimensionality, the elements the $(a-\mu)^{-1}$ are linearly dependent. Thus there exists $c_i \in \mathbb{C}$ so that

$$\sum_{i=1}^{k} c_i (a - \mu_i)^{-1} = 0.$$

Multiplying through by $\prod(a - \mu_i)$, we get a non-zero polynomial over $\mathbb{C}$ with a as a root. Thus, there are $\lambda_j \in \mathbb{C} \setminus 0$ and integers $n_j \geq 0$ so that

$$a^{n_0} \prod_j (a - \lambda_j)^{n_j} = 0$$

As a is not nilpotent, the $(a - \lambda_j)$ are zero divisors, and hence not invertible.

So we may suppose that $a - \lambda$ is not invertible for some $\lambda \in \mathbb{C}^\star$. As in the proof of Lemma 6.2 one can show the existence of an irreducible quotient M of $\mathcal{A}/(a-\lambda)\mathcal{A}$. Then $(a-\lambda)1 = 0$ in M and so $a1 = \lambda 1 \neq 0$. Hence, a acts non-trivially on M. □

To finish the proof of the Proposition it is sufficient to show that for any non-zero $h \in \mathcal{H}_K$ there exists $h^+ \in \mathcal{H}_K$ such that the element $a := hh^+ \in \mathcal{H}_K$ is not nilpotent. In the proof we use the notion of positivity which is define for real-valued functions. One can give a proof which works for the category of K-representations where K is an arbitrary algebraically closed field of characteristic zero.

Consider the map $\mathrm{inv}\colon G \to G$ given by $\mathrm{inv}\colon g \mapsto g^{-1}$. This induces a map $\mathrm{inv}\colon \mathcal{H}(G) \to \mathcal{H}(G)$. Set $h^+ = \mathrm{inv}(h)$, and $u = hh^+$.

Let $h = \varphi\mu_G$ for some Haar measure μ_G and some $\varphi \in S(G)$. Since G is unimodular, $h^+ = \varphi^+\mu_G$ where $\varphi^+(g) = \overline{\varphi(g^{-1})}$. Thus, $u = hh^+ = \psi\mu_G$ for some ψ where

$$\psi(g) = \int_{r\in G} \varphi(r)\overline{\varphi(gr)}dr.$$

Setting $g = 1$, it is obvious that this is not the zero function. What we have shown is that $h \neq 0$ implies $u \neq 0$.

It is enough to find a representation ρ so that $\rho(u) = \rho(h)\rho(h^+) \neq 0$. Note that $u^+ = u$. Thus, from the last paragraph it follows that $u^2 = uu^+ = (hh^+)(hh^+)^+ \neq 0$ and more generally that $u^n \neq 0$. Thus we are reduced to proving the following result.

This ends the proof of Proposition 6.11 □

6.4. **Schur's Lemma Revisited.** To see how Schur's lemma can fail (really the only way that it can fail), let $\mathcal{K}$ be a field properly containing another field k. Consider the discreet group $G = \mathcal{K}^*$ and its representation in the k-vector space $\mathcal{K}$. This representation is obviously irreducible, but Schur's lemma fails: the intertwining operators are $\mathcal{K}$, strictly bigger than k.

We do, however, have the following extensions of Schur's lemma. The key point is that we need some sort of finite-type control to get Schur's lemma.

Lemma 6.14 (Quillen's Lemma)**.** *Let $\mathcal{K}$ be an algebraically closed field of characteristic* 0*, $\mathfrak{g}$ a finite dimensional lie algebra over $\mathcal{K}$ with $U = U(\mathfrak{g})$ its universal enveloping algebra. Then for any irreducible U-module M, $End_U(M) \cong \mathcal{K}$.*

Also, if G is a reductive p-adic group, $\mathcal{H}(G, \mathcal{K})$ the Hecke algebra with coefficients in any algebraically closed field of characteristic 0, then Schur's lemma holds. Again, in this case we have some finite-type control.

Exercise 6.15. Prove the Separation Lemma without the assumption that G is unimodular.

7. Compact Representations.

Our goal is to show that any irreducible representation is admissible and to decompose the category $\mathcal{M}(G)$ into managable pieces, (something like for compact groups.)

Definition 1. Let (π, V) be a smooth representation of G. For any $v \in V, \tilde{v} \in \tilde{V}$ we consider a function $m_{v\tilde{v}}$ on G by $m_{v\tilde{v}}(g) := \tilde{v}(\pi(g)(v))$. We say that a representation π is *compact* if for every $v \in V, \tilde{v} \in \tilde{V}$ the function $m_{v\tilde{v}}$ on G has compact support.

Remarks. 1. Some say *finite* instead of *compact*.

2. We will see that compact representations are completely reducible and so resemble representations of compact groups.

3. It is obvious that if π is compact then so is any subquotient of π.

Proposition 7.1. *A representation (π, V) is compact for any $v \in V$ and any compact open subgroup $K \subset G$ the subset*
$C(v, K) := \{g \in G | \pi(e_K)\pi(g)(v) \neq 0\}$ *is compact.*

Proof. It is easy to see that any representation (π, V) satisfying the condition of the Proposition is compact.

Conversely assume that (π, V) is compact. We want to show that for any $v \in V$ and any compact open subgroup $K \subset G$ the subset $C(v, K)$ is compact. If not then we can find a sequence $g_i \in G, i > 0$ such that $\pi(e_K)\pi(g_i)(v) \neq 0$ and the subset $\{g_i\}$ is not contained in any compact. Choose a linear functional $\lambda \in V^\vee$ such that $\lambda(\pi(e_K)\pi(g_i)(v)) \neq 0$ and define $\tilde{v} \in \tilde{V}$ by

$$\tilde{v}(v) := \int_{k \in K} \lambda(\pi(k)v)$$

It is clear that $m_{v\tilde{v}}(g_i) = \lambda(\pi(e_K)\pi(g_i)(v)) \neq 0$. But this would contrudict the compactness of π. □

Exercise 7.2. E:com a) for any compact admissible representation (π, V) the dual representation $(\tilde{\pi}, \tilde{V})$is compact.

b) Any finitely generated compact representation is admissible.

Remark. If $\xi \in V$, $\tilde{\xi} \in \tilde{V}$ then the function $\varphi_{\tilde{\xi},\xi}(g) =< \tilde{\xi}, \pi(g^{-1})\xi >$ is called a *matrix coefficient*.

Exercise 7.3. Every compact representation has compactly supported matrix coefficients. Conversly, if all matrix coefficients of π are compactly supported, then π is compact.

We assume now that G is a unimodular group so that we may choose a Haar measure μ_G and identify $S(G)$ with $\mathcal{H}(G)$.

7.1. **The Formal Dimension.** Assume that G is unimodular and choose a Haar measure μ_G. It defines an isomorphism $S(G) \cong \mathcal{H}(G)$. If (ρ, W) is an irreducible compact representation then the last proposition gives a map

$$\varphi \colon \mathcal{H}(G) \to W \otimes \tilde{W}.$$

On the other hand, there is a map in the other direction which assigns to two vectors the associated matrix coefficient:

$$m \colon W \otimes \tilde{W} \to S(G) \cong \mathcal{H}(G)$$

given by

$$(\xi, \tilde{\xi}) \mapsto m_{\xi,\tilde{\xi}}(g) =< \rho(g^{-1})\xi, \tilde{\xi} > .$$

It is natural to consider the composition $\varphi \circ m \colon W \otimes \tilde{W} \to W \otimes \tilde{W}$.

Claim 1. Given G, ρ and W as above, there exists a nonzero number $d(\rho)$, called the *formal dimension* of (ρ, W), such that $\varphi \circ m = d(\rho) \cdot$ Identity.

Proof. As $W \otimes \tilde{W}$ is an irreducible representation of $G \times G$, the existence of the formal dimension follows from Schur's lemma. We must show that it is non-zero. Let $e \in W \otimes \tilde{W}$ be so that $h = m(e)$ is non-zero. We will be finished if we show that $\varphi(h) \neq 0$. By the definition of φ, it is enough to show prove that $\rho(h) \neq 0$. We will prove this be showing that for any irreducible representation, (τ, V), *not* equivalent to ρ, $\tau(h) = 0$. Then, by the separation lemma, $\rho(h) \neq 0$.

Lemma 7.4. *L:tau Let (τ, V) be any irreducible representation of G not equivalent to ρ, then $\tau(h) = 0$.*

Proof. Let $v \in V$ and consider the morphism of G-modules $W \otimes \tilde{W} \to V$ given by $\xi \otimes \tilde{\xi} \mapsto \tau(m(\xi \otimes \tilde{\xi}))v$. Here we have thought of m as a map into $\mathcal{H}(G)$ and (τ, V) as an $\mathcal{H}(G)$-module. As a G-module, $W \otimes \tilde{W}$ is completely reducible and is a sum of copies of W. Thus, the same is true of $\Im W \otimes \tilde{W}$. In particular, when V is irreducible and not equivalent to W, $\Im W \otimes \tilde{W} = 0$. In particular, $\tau(h)v = 0$ for all $v \in V$. Thus $\tau(h) = 0$. □

□

REMARKS. 1. $d(\rho)$ depends on the choice of Haar measure.

2. If G is compact and we normalize so the measure of G is 1, then the formal dimension is the reciprocal of the actual dimension of the representation.

3. The formal dimension can be defined more generally for representations whose matrix coefficients are L^2 but not necessarily compactly supported.

4. If W is unitarizable, $\xi \in W$ and $\tilde{\xi} \in \tilde{W}$ then it can be shown that

$$\int_G |m_{\xi,\tilde{\xi}}|^2 d\mu_G = d(\rho).$$

This gives another proof that the formal dimension is non-zero.

7.2. **The decomposition theorem.**

Theorem 7.5. *T:decomp Any irreducible compact representation* (ρ, W) *of* G *splits* $\mathcal{M}(G)$ *[see Definition 6.6].*

Proof. We start with the following result.

Exercise 7.6. Let A be a k-algebra, $e \in A$ be a central idempotent such that the A-module $Ae \subset A$ is is a finete multiple of an irreducible A-module W. Then for any A-module V there exists a decomposition of V as the direct sum $V = V_0 \oplus V_1$ of A-submodules such that V_0 is a direct sum of copies of W and V_1 does not have subquotients isomorphic to W.

To prove the decomposition theorem it is sufficient to show that for any open compact subgroup K of G there exists a decomposition of V^K as the direct sum $V = V_0 \oplus V_1$ of subrepresentations of $\mathcal{H}_K$ such that V_0^K is a direct sum of copies of W^K and V_1^K does not have subquotients isomorphic to W^K. Set

$$e = e_{W,K} = d(\rho)^{-1} m(\varphi(e_K)) \in \mathcal{H}(G)$$

Exercise 7.7. a) $e \in \mathcal{H}_K$ is a central idempotent.

b) The map $W \otimes \tilde{W}^K \to \mathcal{H}_K$ given by $v \times \tilde{v} \to m_{v\tilde{v}}$ defines an isomorphism

$$W \otimes \tilde{W}^K \to \mathcal{H}_K e$$

The decomposition theorem now follows. □

Corollary 7.8. *Any compact representation* $(\pi, V) \in \mathcal{M}(G)$ *is both projective and injective.*

Exercise 7.9. E:decomp a) Any finite set S of irreducible compact representations of G splits $\mathcal{M}(G)$.

b) Any compact representation is direct sum of irreducible ones.

8. The Geometry of reducive groups.

The results I'll explain are true for an arbitrary split reductive group G. I'll illustrate the definitions for the case $G = GL(n)$.

We consider various subgroups. Set $K_0 = G(\mathcal{O})$; this is a maximal compact subgroup. For any $i > 0$ we denote by $K_i \subset K_0$ the *congruence subgroup* which is equal to the kernel of the homomorphism $G(\mathcal{O}) \to G(\mathcal{O}/\pi^i)$. The K_i are open compact normal subgroups of K_0 and shifts of K_i form a basis of the topology on G.

If K is a congruence subgroup; we write $a(g) = e_K * \mathcal{E}_g * e_K \in \mathcal{H}_K(G)$ for any $g \in G$ where $\mathcal{E}_g$ is the δ-function at g.

We denote by T a split torus over $\mathcal{O}$ and by $B = TU$ the Borel subgroup of G. Then $T^c = T \cap K_0$ is the maximal compact subgroup of T. Let $\Lambda := X_\star(T)$ be the group coroots- that is algebraic homomorphisms of $F^\star = \mathbb{G}_m(F)$ to T. Then $X_\star(T) = T/T^c \cong \mathbb{Z}^n$ where $n = rank(G)$. The imbedding $\mathbb{Z} \to F^\star, r \to \pi^r$ defines an imbedding $\Lambda \hookrightarrow T$. A choice of a Borel subgroup $B = TU$ defines a subsemigroup $\Lambda^+ \subset \Lambda$ of positive coroots.

Exercise 8.1. E:bl In the case $G = GL(n)$ we have $K_0 = \mathrm{GL}(n, \mathcal{O})$. If $M_n(\mathcal{O})$ designates the $n \times n$ matrices with entries in $\mathcal{O}$ and π generates the maximal ideal in $\mathcal{O}$, for each $i > 0$ we define $K_i = \{1 + \pi^i M_n(\mathcal{O})\}$. T is the group of diagonal matrices, B is the group of upper-triangular matrices, $\Lambda = \mathbb{Z}^n$ and the subsemigroup $\Lambda^+ \subset \Lambda$ consists of non-decreasing sequences $(e_1, \dots, e_n)|e_1 \leq e_2 \leq \cdots \leq e_n\}$. The imbedding $\Lambda \hookrightarrow T$ is given by

$$\lambda = (l_1, \dots, l_n) \mapsto \begin{pmatrix} \pi^{l_1} & & \\ & \ddots & \\ & & \pi^{l_n} \end{pmatrix}.$$

Under this imbedding.

Our goal is to describe $\mathcal{H}_K(G)$. We start with some simple results.

Exercise 8.2. a) Given a double coset

$$c = KgK \subset G, a(g) = e_K * \mathcal{E}_g * e_K$$

is the unique K-left-and-right-invariant distribution supported on c and with integral 1.

b) As g runs through a system of representatives for the double cosets $K\backslash G/K$, the $a(g)$ form a basis for $\mathcal{H}_K(G)$.

c) $(KgK)(Kg'K) \supset Kgg'K$.

d) $(KgK)(Kg'K) = Kgg'K$ iff $a(gg') = a(g)a(g')$

A description of $K\backslash G/K$ is based on the *Cartan decomposition* $G = K_0\Lambda^+K_0$.

Exercise 8.3. E:Car Prove the Cartan decomposition for $G = GL(n)$

Definition 8.4. D:Par PARABOLIC SUBGROUPS. Let $g \in G$ be a semisimple element. Set

$P_g = \{x \in G | \{\mathrm{Ad}(g^n)x; n \geq 0\}$ is relatively compact in $G\}$

$$U_g = \{x \in G | \lim_{n\to\infty} \mathrm{Ad}(g^n)x = 1\}$$

a) A subgroup $P \subset G$ is *parabolic* if it is equal to P_g for some semisimple element $g \in G$. In this case we say that U_g is the *unipotent radical* of P.

b) A Levi subgroup M of a parabolic P is a subgroup such that $P = M \ltimes U$.

c) A parabolic subgroup is *standard* if it contains the group B of upper triangular matricies.

d) A Levi subgroup M is *standard* if it contains T.

e) Two parabolic subgroups $P = MU, P' = M'U'$ are *associated* if the Levi subgroups M, M' are conjugate.

Exercise 8.5. E:Iw Show that

a) For any semisimple element $g \in G, M_g := P_g \cap P_{g^{-1}}$ is a Levi subgroup M of P_g.

b) If $P = MU_P$ is a parabolic subgroup and K_0 is a maximal compact subgroup of G, then $G = K_0P$. In particular, G/P is compact.

d) For any parabolic subgroup $P = MU_P$ there eixsts unique parabolic subgroup $\bar{P}$ of G such that $P \cap \bar{P} = M$

e) Any semisimple element $g \in G$ is conjugate to an element of the form λc where $\lambda \in \Lambda^+\lambda$ and c commute and c is compact [that is lies in a compact subgroup of G]. In this case $P_{\lambda c} = P_\lambda$.

In particular any parabolic subgroup is conjugate to a standard one.

In the rest of this excersize we assume that $G = GL(n)$.

f) Let $\lambda = \mathrm{diag}(\pi^{m_1}, \dots, \pi^{m_n})$ where the m_i are nondecreasing. Say

$$m_1 = m_2 = \cdots = m_{n_1} > m_{n_1+1} = \cdots = m_{n_1+n_2} >$$
$$> \cdots > m_{n_1+\cdots+n_{k-1}+1} = \cdots = m_{n_1+\cdots+n_k}$$

Then P_λ is the subgroup of upper-triangular block matricies corresponding to the partition $n = \sum_{i=1}^k n_i$, $M_\lambda = Z_G(\lambda)$ is the subgroup of block diagonal matricie and $U_\lambda \subset P_\lambda$ is the subgroup of matricies with block diagonal entries equal to Id_{n_i}.

g) Describe P_λ and U_λ for arbitrary $\lambda \in \Lambda$.

Definition 8.6. D:Weyl a) For any standard Levi subgroup M we define the *Weyl group* $W_M := N_M(T)/T$ and write $W := W_G$.

b) For any two parabolic subgroups P, Q of G we define a partial order on the set $C_{P,Q} := P\backslash G/Q$ by $c' \leq c \leftrightarrow c' \subset \bar{c}$ where $\bar{c}$ is the closure of c in G.

c) For any $w \in W$ we define $X_w := U\tilde{w}B \subset G$ where $\tilde{w} \in N_G(T)$ is a representative of w and define $\bar{X}_w$ as the closure of X_w in G.

Exercise 8.7. Show that

a) The imbedding $M \hookrightarrow G$ induces an imbedding $W_M \hookrightarrow W_G = S_n$.

b) G is a disjoint union of $X_w, w \in W$.

So we can identify $C_{B,B}$ with W and the partial order on $C_{B,B}$ induces a partial order on W.

c) There exists unique element $w_0 \in W$ such that $w \leq w_), \forall w \in W$ and $w_0^2 = e$.

d) For any standard parabolic subgroup P of $G, \tilde{P} := w_0\bar{P}w_0$ is also a standard parabolic subgroup.

In the rest of this excersize we assume that $G = GL(n)$.

e) $dim(\bar{X}_w) = l(w)$ where $l(w)$ is the number of pairs $(i, j) 1 \leq i < j \leq n$ such that $w(i) > w(j)$.

f) If M is a standard Levi subgroup corresoponding to a partition $n = \sum_{i=1}^k n_i$ then $W_M = S_{n_1} \times \cdots \times S_{n_k}$ is a product of symmetric groups. In particualr $W = S_n$.

g) Express the condition $w' \leq w$ in combinatorial terms.

Let $Q = NV$ and $P = MU$ be standard parabolics in G. We define a subset $W_{P,Q}$ of W by

$$W_{P,Q} := \{w \in W | l(w_M w w_N) \geq l(w), \forall w_M \in W_N, w_N \in W_N\}$$

Exercise 8.8. E:rev a) For any $w \in W$ there exists unique $w' \in W_{P,Q}$ such that $W_M w W_N = W_M w' W_N$.

So we can identify the set $W_{P,Q}$ with the set of cosets $W_M \backslash W / W_N$.

b) G is a disjoint union of $Y_w, w \in W_{P,Q}$.

So we can identify the set $P \backslash G/Q$ with the set $W_M \backslash W/W_N$ and the partial order on $C_{P,Q}$ induces a partial order on $W_M \backslash W/W_N$.

c) For any $w', w'' \in W_{P,Q}$ we have $Y_{w'} \subset \bar{Y}_{w''} \leftrightarrow X_{w'} \subset \bar{X}_{w''}$ where $\bar{Y}_w$ is the closure of $Y_w \subset G$.

The left multiplication by w_0 indetifies the sets $W_M \backslash W/W_N$ and $w_0 W_M w_0^{-1} \backslash W/W_N$.

d) Show that this map reverses the order of partially ordered sets $C_{P,Q} = W_M \backslash W/W_N$ and $C_{\tilde{P},Q} = w_0 W_M w_0^{-1} \backslash W/W_N$.

Choose a set of representatives $x_1, \ldots, x_r$ for $K \backslash K_0$. Let $\mathcal{H}_0 = \mathcal{H}_K(K_0) \subset \mathcal{H}_K(G)$ be the finite-dimensional subalgebra spanned by the $a(x_i) = e_K * \mathcal{E}_{x_i} * e_K$ (notation from last lecture). Since K_0 normalizes K, $Kx_i = x_i K$ for all i. Therefore, for any $g \in G$, we have $(Kx_iK)(KgK) = Kx_iKgK = Kx_igK$. Equivalently, $a(x_ig) = a(x_i)a(g)$. In the same way, $a(gx_i) = a(g)a(x_i)$.

Let C be the span of $\{a(\lambda) | \lambda \in \Lambda^+\}$. The next proposition is very important.

Proposition 8.9. *P:com*

(1) $\mathcal{H}_K(G) = \mathcal{H}_0 C \mathcal{H}_0$.
(2) *C is a commutative, finitely generated algebra.*

Remark. This is saying that $\mathcal{H}_K(G)$ is somehow of finite type but it is neither generated over C on the left nor on the right but rather "in the middle".

Proof. (1). By the Cartan decomposition, $G = \bigcup_{\lambda \in \Lambda^+} K_0 \lambda K_0$. Moreover, $K_0 = \bigcup_i Kx_i = \bigcup_i x_i K$. Therefore,

$$G = \bigcup_{\substack{\lambda \in \Lambda^+ \\ i,j}} Kx_i \lambda x_j K.$$

This implies that the $a(x_i \lambda x_j)$ form a basis for $\mathcal{H}_K(G)$. But we showed above that $a(x_i \lambda x_j) = a(x_i)a(\lambda)a(x_j)$ which implies (1). □

For the proof of part (2) of the proposition we must show that $(K\lambda K)(K\mu K) = K\lambda\mu K$. This is not trivial because the elements of Λ^+ do not normalize K. The idea is to decompose K into parts that can be moved right or left.

We will use the following notation. K is a congruence subgroup (i.e. $K = K_i$ for some $i > 0$), U is the standard maximal unipotent (i.e. upper triangular matrices with 1's on diagonal) and $\overline{U}$ is the lower triangular unipotent group. Set $K_+ = K \cap U$ and $K_- = K \cap \overline{U}$. Finally, T = diagonal matrices and $K_T = K \cap T$.

Claim 2. $K = K_+ K_T K_-$

Proof. Just do elementary row and column reductions on the elements of K. These correspond to multiplying by K_+ and K_- on the right and left, respectively. □

Corollary 8.10. $K = K_- K_T K_+$

Proof. $K = K^{-1} = K_-^{-1} K_T^{-1} K_+^{-1} = K_- K_T K_+$ □

Claim 3. If $\lambda \in \Lambda^+$ then $\lambda K_+ \lambda^{-1} \subset K_+$ and $\lambda^{-1} K_- \lambda \subset K_-$.

Proof. Observe that $\mathrm{ad}(\lambda)|_U$ is contracting. In fact, if $\lambda = \mathrm{diag}(\lambda_i)$, then $\lambda(u_{i,j})\lambda^{-1} \mapsto (\lambda_i \lambda_j^{-1} u_{i,j})$. But for $j > i$, $\lambda_i \lambda_j^{-1} \in \mathcal{O}$. This proves the first statement. The proof of second is similar. □

We can now prove (2). Since $\lambda\mu = \mu\lambda$ it is suffiicent to show that $K\lambda K\mu K = K\lambda\mu K$. Now,

$$K\lambda K\mu K = K\lambda K_+ K_T K_- \mu K$$

$$= K(\lambda K_+ \lambda^{-1})\lambda\mu(\mu^{-1} K_- \mu)K \subset K\lambda\mu K$$

by the last claim. The reverse inclusion always holds so (2) is proved.
Remark. This proof only works for congruence subgroups. For the maximal compact, i.e. $K = K_0$ we don't have a decomposition $K = K_+ K_T K_-$ and a different proof is needed. In this case, $a(\lambda)a(\mu) = \sum c^{\nu}_{\lambda,\mu} a(\nu)$ where $c^{\nu}_{\lambda,\mu}$ is a polynomial in $q = |\mathcal{O}/\mathfrak{p}|$; when $q = 1$ this polynomial gives the Clebsch-Gordan coefficients.

8.1. Modules.

We start with the following definition.

We have shown that $\mathcal{H}_K = \mathcal{H}_0 C \mathcal{H}_0$ with C commutative and so, in particular, $a(\lambda^n) = a(\lambda)^n$. We want to use this equality to study $\mathcal{H}_K$-modules. Let (π, V) be a representation of G, π_K the associated representation of $\mathcal{H}_K$ on V^K. We will prove the following result.

Proposition 8.11. *P:j For any $\lambda \in \Lambda^+$ we have*

$$\bigcup_r \operatorname{Ker} a(\lambda^r) \cap V^K = V(U_\lambda) \cap V^K$$

Proof. First a couple of simple results. Recall that for any compact subgroup $\Gamma \subset G$, e_Γ is the *unique* bi-Γ invariant distribution which is supported on Γ with the integral equal to 1; i.e. just Haar measure. Using this uniqueness of e_Γ, it is easy to prove the following results

Exercise 8.12. (1) If $\Gamma = \Gamma_1\Gamma_2$, then $e_\Gamma = e_{\Gamma_1} * e_{\Gamma_2}$.

(2) $e_{g\Gamma g^{-1}} = \mathcal{E}_g * e_\Gamma * \mathcal{E}_{g^{-1}}$.

(3)
$$e_K = e_{K_+} * e_{K_T} * e_{K_-}$$
and
$$\mathcal{E}_\lambda * e_{K_+} * \mathcal{E}_{\lambda^{-1}} = e_{\lambda K_+ \lambda^{-1}}$$
for $\lambda \in \Lambda^+$

(4) $e_K * e_{\nu K_+ \nu^{-1}} = e_K$

(5) $e_{\nu^{-1} K_- \nu} * e_K = e_K$

(6) $e_K * e_{K_T} = e_{K_T} * e_K = e_K$.

(7) $e_{\nu K_T \nu^{-1}} = e_{\nu^{-1} K_T \nu} = e_T$

Lemma 8.13. *L:Ker If $\nu \in \Lambda^+$, then* $\operatorname{Ker} a(\nu)|_{V^K} = \operatorname{Ker} e_{\nu^{-1}K_+\nu}|_{V^K}$.

Proof. Using the preceding formulas, we have

$$a(\nu) = e_K * \mathcal{E}_\nu * e_K =$$
$$= e_{K_+} * \mathcal{E}_\nu * e_{\nu^{-1}K_T\nu} * e_{\nu^{-1}K_-\nu} * e_K = \mathcal{E}_\nu * e_{\nu^{-1}K_+\nu} * e_K$$

But on V^K, e_K acts as the identity. Moreover, $\mathcal{E}_\nu$ is invertible. It follows that $\operatorname{Ker} a(\nu)|_{V^K} = \operatorname{Ker} e_{\nu^{-1}K_+\nu}|_{V^K}$. □

Now we can prove Proposition 8.11.

Let $\lambda = \operatorname{diag}(\pi^{m_1}, \ldots, \pi^{m_n})$ where $m_1 = m_2 = \cdots = m_{n_1} > m_{n_1+1} = \cdots = m_{n_1+n_2} > \ldots$. The n_i's give a partition of n.

Set $K_+^P = K \cap U_\lambda$, $K_-^P = K \cap U_{\lambda^{-1}}$ and $K_M^P = K \cap M_\lambda$ (we will often suppress the P and λ). Exactly as before, we can prove

Exercise 8.14. E:dec

(1) $K = K_+K_MK_-$, $\lambda K_+^P \lambda^{-1} \subset K_+^P$

(2) $\lambda^{-1} K_-^P \lambda \subset K_-^P$

(3) $(\operatorname{Ad}\lambda^n)|_{K_+} \to 1$ as $n \to \infty$, and $(\operatorname{Ad}\lambda^{-n})|_{K_-} \to 1$ as $n \to \infty$

(4) $\bigcup_n \operatorname{Ad}(\lambda^{-n}) K_+^P = U_\lambda$

(5) $\bigcup_n \operatorname{Ker} a(\lambda^n) \cap V^K = V(U_\lambda) \cap V^K$.

As follows form Lemma 8.13 applied to the case $\nu = \lambda^n$ we have $a(\lambda^r) = \mathcal{E}_\nu * e_{\nu^{-1}K_+\lambda^r} * e_K$ and therefore

$$\bigcup_r \operatorname{Ker} a(\lambda^r) \cap V^K = V(U_\lambda) \cap V^K$$

□

Definition 8.15. D:G a) Let $G^0 = \{g \in G | \det g \in \mathcal{O}^*\}$ and $Z(G)$ be the center of G.

Remark 8.16. a) $Z(G)G^0 \subset G$ is an open subgroup of finite index.

b) All unipotent radicals of parabolic subgroups of G belong to G^0.

c) All definitions and results of this section are applicable if we replace G by G^0.

Exercise 8.17. Let $\nu_i, i \in I$ be the set indecomposable elelemtns in the semigroup $\Lambda^+(G^0)$. Then $\Lambda^+(G^0)$ is a free abelian semigroup generated by $\nu_i, i \in I$.

8.2. **The Jacquet Functors.** Let $P = MU$ be a standard parabolic subgroup of G.

If (π, V) is a representation of G, the Jacquet module,

$J_U(V) = V/V(U_P)$ is an M-module. This gives the *Jacquet functor* $r_{P,G} \colon \mathcal{M}(G) \to \mathcal{M}(M)$.

Remark 8.18. As follows from Proposition 5.10 (3) the functor $r_{P,G}$ is exact.

Proposition 8.19. *P:adj $r_{P,G}$ has the right adjoint functor, called Jacquet's induction functor, $i_{G,P} \colon \mathcal{M}(M) \to \mathcal{M}(G)$, defined as follows: if L is a representation of M, extend it trivially to P; then $i_{G,P}(L) = \operatorname{ind}_P^G(L)$.*

Remarks. The Iwasawa decomposition [Exercise 8.5 e)] implies that $\operatorname{ind}_P^G = \operatorname{Ind}_P^G$.

Proof. Let $L \in \mathcal{M}(M)$, $V \in \mathcal{M}(G)$. We must show the existence of a functorial isomorphism $\operatorname{Hom}_G(V, i_{G,P}(L)) \tilde{\to} \operatorname{Hom}_M(r_{P,G}V, L)$. To construct sucha morhism observe that

$$\operatorname{Hom}_G(V, i_{G,P}(L)) = \operatorname{Hom}_G(V, \operatorname{Ind}_P^G(L))$$

by Frobenious Reciprocity

$$= \operatorname{Hom}_P(V, L)$$

as U acts trivially

$$
\begin{aligned}
&= \mathrm{Hom}_P(V/V(U), L)\\
&= \mathrm{Hom}_P(r_{P,G}V, L)\\
&= \mathrm{Hom}_M(r_{P,G}V, L)
\end{aligned}
$$

Exercise 8.20. The morphism $\mathrm{Hom}_G(V, i_{G,P}(L)) \tilde{\to} \mathrm{Hom}_M(r_{P,G}V, L)$ we constructed is an isomorphis.

□

Proposition 8.21. *P:ri Let M be a Levi subgroup of G. Then, the functors $r_{P,G}$ and $i_{G,P}$ satisfy the following properties.*

(1) *$r_{P,G}$ is the right adjoint to $i_{G,P}$.*
(2) *If N is a Levi subgroup of M, then $r_{N,M} \circ r_{P,G} = r_{N,G}$ and $i_{G,P} \circ i_{M,N} = i_{G,N}$.*
(3) *$i_{G,P}$ maps admissible to admissible.*
(4) *For any representation (π, V) of G and a congruence subgroup K of G such that V is generated by V^K as a G-module the space $r_{P,G}(V)$ is generated by $r_{P,G}(V)^{K_M}$ as an M-module.*
(5) *$i_{G,P}$ and $r_{P,G}$ are exact and $r_{P,G}$ maps finitely generated to finitely generated.*

Remark 8.22. R:ri You can replace G by G^0 in Proposition 8.21.

Proof. We have already proved (1) and (2) is a simple verification. (3) follows from the compactness of G/P and (5) from Proposition 5.10.

Now we prove (4). Suppose V is generated by V^K as a G-module. Since K is a normal subgroup of K_0 the subspace $V^K \subset V$ is K_0-invariant. Since $G = PK_0$ we see that implies that V is generated by V^K as a P-module. Since U acts trivially on $V/V(U) = r_{G,M}(V)$ we see that $r_{P,G}(V)$ is generated as an M-module by the image of V^K in $r_{P,G}(V)$. But it is clear that the image of V^K in $r_{P,G}(V)$ is contained in $r_{P,G}(V)^{K_M}$.

□

8.3. **Quasi-cuspidality.**

Definition 8.23. A representation (π, V) of G^0 is called *quasi-cuspidal* if $J_U(V) = 0$ for the unipotent radicals of any proper parabolic subgroup of G.

Theorem 8.24. *T:qua A representation (π, V) of G^0 is quasi-cuspidal if and only if it is compact.*

Proof. We first show that any quasi-cuspidal representation is compact.

Assume that (π, V) is a quasi-cuspidal representation, $v \in V$. We may assume that K is a congruence subgroup and, by choosing K small enough, that v is K-invariant. Then, $\pi(e_K)\pi(g)v = \pi(e_K)\pi(g)\pi(e_K)v$.

Let $x_i, i \in I$ be a system of representatives for $K\backslash K_0$ as before. We have seen that and $g \in G$ can be written in the form $g = x_i K\lambda K x_j, \lambda \in \Lambda^+, i, j \in I$. Thus, it is enough to prove that form any $v \in V^K$ the function

$$\lambda \mapsto \pi(x_i)\pi(a(\lambda))\pi(x_j)v$$

on G^0 has compact support Λ^+. In other words we have to show that for any $v \in V^K$ the function $\lambda \mapsto \pi(a(\lambda))v$ has compact support on Λ^+.

Let $\nu_i, i \in I$ be the basis of $\Lambda^+ := \Lambda^+(G^0)$. To prove that (π, V) is compact we have to show the existence of $m \in \mathbb{Z}$ such that for any sequence $\bar{m} = \{m_i\} \in \mathbb{Z}^+, i \in I$ such that $m_i > m$ for some $i \in I$ we have $\pi(a(\lambda_{\bar{m})})v = 0$ where $\lambda_{\bar{m}} := \sum_{i\in I} m_i\nu_i$.

Since V is quasi-cuspidal, for each $\mu \in \Lambda^+$ we have

$$V^K \cap \bigcup_n \operatorname{Ker} a(\mu^n) = V(U_\mu) \cap V^K = V^K.$$

Hence, for any $i \in I$ we can find $r(i) \in \mathbb{Z}_+$ such that $\pi(a(\nu_k^{r(k)}))v = 0$. Since the operators $\pi(a(\nu_i)$ commute, we can find r so large that $\pi(a(\lambda_{\bar{m}})v = 0$ if any $m_i \geq r$ for some $i \in I$.

Conversely, suppose that (π, V) is compact. By reversing the reasoning, we see that $\lambda \mapsto \pi(a(\lambda))v$ has compact-modulo-center support in Λ^+. But, for any non-zero $\lambda \in \Lambda^+$ the sequence λ^n eventually leaves all finite subsets of Λ^+. That is, $a(\lambda^n)$ eventually acts trivially, and so $\operatorname{Ker} a(\lambda^n) = V^K$. However, we know that

$$V^K \cap \bigcup_n \operatorname{Ker} a(\lambda^n) = V(U_\lambda) \cap V^K.$$

Therefore, $V(U_\lambda) \cap V^K = V^K$ for all compact subgroups K. As our representations are smooth, this implies that $V/V(U_\lambda) = 0$ whenever λ is a non-central element of Λ^+. Therefore, (π, V) is quasi-cuspidal. □

Definition 8.25. a) A representation (π, V) of G is called *compact modulo center* if for any open compact subgroup $K \subset G$ and $\xi \in V$,

$$g \mapsto \pi(e_K)\pi(g)\xi$$

has compact support modulo center.

b) A representation of G is quasi-cuspidal if it's restriction to G^0 is quasi-cuspidal.

Exercise 8.26. E:qua a) A representation of G is quasi-cuspidal iff it is compact modulo center.

b) Any cuspidal representation of G can be imbedded in an injective cuspidal representation.

Corollary 8.27. *C:qua For any parabolic subgroup $P = LM \subsetneq G, \rho \in \mathcal{M}(L)$ and a quasi-cuspidal representation $\pi \in \mathcal{M}(G^0)$ we have*

$$Hom_{G^0}(i_{G,P}(\rho), \pi) = \{0\}$$

Proof. As follows from Exercise 7.9 we can assume that π is irreducible and therefore is admissible. As follows from Exercise 13.4 the dual representation $\tilde{\pi}$ is also compact. Since the quotient G/P is compact can identify the smooth dual $\widetilde{i_{G,P}(\rho)}$ of $i_{G,P}(\rho)$ with $i_{G,P}(\tilde{\rho})$. So it is sufficient to show that

$$Hom_{G^0}(\tilde{\pi}, \widetilde{i_{G,P}(\rho)}) = \{0\}$$

But this follows immediately form the Remark 8.22. □

Let $\mathcal{P}$ be the set of standard parabolics of $P \neq G$ of G. For any representation (π, V) of G and and $P \in \mathcal{P}$ we define $\pi_P := r_{P,G}(\pi)$ and denote by $j_P : V \to i_{P,G}(\pi)(\pi_P)$ the morphism corresponding to the identity morphism $Id : \pi_P \to \pi_P$ under the adjunction. Let

$$j := \oplus_{P \in \mathcal{P}} j_P V \to \oplus_{P \in \mathcal{P}} i_{P,G}(\pi)(\pi_P)$$

Exercise 8.28. E:d The representation of G on $Ker(j)$ is quasi-cuspidal.

9. Irreducible Implies Admissible.

Definition 9.1. D:G a) For any representation (π, V) of G^0 and $\bar{g} \in G/Z(G)G^0$ we denote by $(\pi^{\bar{g}}, V)$ a representation of of G given by $\pi^{\bar{g}}(g_0) := \pi(gg_0g^{-1})$ where $g \in G$ is a representative of the class $\bar{g} \in G/Z(G)G^0$. It is clear that the equivalence class of $\pi^{\bar{g}}$ does not depend on a choice of a representative g of $\bar{g}$.

b) For any irreducible representation σ of G^0 we denote by $S(\sigma)$ the set of equivalence classes of irreducible representations of G^0 of the form $\sigma^{\bar{g}}, \bar{g} \in G/Z(G)G^0$.

c) As all unipotent radicals of parabolic subgroups of G belong to G^0, we say that a representation (π, V) of G^0 is *quasi-cuspidal* if $J_{U_P}(V) = 0$ for all proper parabolic subgroups P of G.

Lemma 9.2. *L:du For any proper parabolic subgroups $P = MU$ of $G, \rho \in \mathcal{M}(M)$ and any quasi-cuspidal representation (π, V) of G we have* $\mathrm{Hom}_G(\pi, i_{G,P}(\rho)) = \{0\}$.

Proof. It is sufficient to show that $\mathrm{Hom}_{G^0}(\pi, i_{G^0,P\cap G^0}(\rho)) = \{0\}$ for and any quasi-cuspidal representation (π, V) of G^0 . As follows from Exercise 7.9 we can assume that (π, V) is compact. Since $G^0/P \cap G^0$ is compact we have $i_{G^0,P\cap G^0}(\rho) = i_{G^0,P\cap G^0}(\tilde{\rho})$. So it is sufficient to show that $\mathrm{Hom}_{G^0}(i_{G^0,P\cap G^0}(\tilde{\rho}), \tilde{\pi})) = \{0\}$. Since by Exercise 13.4 the representation $\tilde{\pi}$ is compact it is quasi-caspidal and the last equality follows from Proposition 8.21 (1).

□

Exercise 9.3. For any irreducible representation (π, V) of G there exists an irreducible representation σ of G^0 such that the restriction (π_{G^0}, V) is equivalent to $\oplus_{\pi\in S(\sigma)}\pi$.

Definition 2. A representation is called *cuspidal* if it is both quasi-cuspidal and finitely generated.

Lemma 9.4. *Any irreducible cuspidal representation of G is admissible.*

Proof. Let (ρ, W) be the representation. First we show that $W|_{G^0}$ is finitely generated. Let $Z = Z(G)$. Because $[G : ZG^0]$ is finite, $W|_{ZG^0}$ is finitely generated. But by irreducibility, Z acts as a scaler. Hence W is finitely generated as a G^0-module, as claimed.

By Theorem 8.24, $W|_{G^0}$ is compact. Proposition 1 of lecture 3 says that finitely generated compact representations are admissible. Thus, $W|_{G^0}$ is admissible. As G^0 contains all compact subgroups, the lemma follows. □

This lemma is the first step toward our goal of proving

Theorem 9.5. *T:adm Any irreducible representation of G is admissible.*

9.1. **Proof of Theorem 9.5 .**

Lemma 9.6. *L:in Let (τ, W) be an irreducible representation of G, then there is a parabolic $P = MU$ and an irreducible cuspidal representation (ρ, L) of M, so that there is an embedding $W \hookrightarrow i_{G,P}(L)$.*

Proof. Let M be a Levi subgroup, minimal subject to the condition $L' = r_{P,G} \neq 0$. We claim that L' is cuspidal. These follow from the last proposition. As follows from the part (2) of Proposition 8.21 for any proper Levi subgroup N of M we have

$$r_{N,M}(L') = r_{N,M} \circ r_{P,G}(W) = r_{N,G}(V) = 0$$

. This proves L' quasi-cuspidal. Also, W is irreducible and so certainly finitely generated. Thus, by Proposition 8.21 (4), L' is finitely generated.

Let L be an irreducible quotient of L'. L is an irreducible cuspidal representation of M, as required. Moreover, there is a nonzero map $r_{P,G}(W) = L' \to L$. By the adjunction property (i.e. adjointness), we get a non-zero map $W \to i_{G,P}(L)$. As W is irreducible, this must be an embedding. □

Remark. This use of the adjunction property is typical. Namely, we show that something is non-zero. It never gives more detailed information than that.

Theorem 9.5 now follows immediately. Really let (π, V) be an irreducible representation of G. Let W and L be as in the lemma. L is irreducible cuspidal and therefore admissible (corollary 1). By part (3) of proposition 3, $i_{G,P}(L)$ is also admissible. But $W \hookrightarrow i_{G,P}(L)$ so W is admissible.□

Theorem 9.7. *T:J [Jacquet's lemma]. Let (π, V) be an admissible representation of G and $(\rho, W) = r_{G,P}(\pi, V)$. Then for any parabolic $P = MU$ of G and a congruence subgroup $K \neq K_0$ the map $q : V \to V_U$ induces a surjection $q_K : V^K \to V_U^{K_M}, K_M =: K \cap M$.*

Proof. It is obvious that $q(V_K) \subset V_U^{K_M}$. To prove the opposite inclusion observe that as follows from the Iwasava decomposition we can assume that $P = MU$ is a standard parabolic subgroup P_λ of $G, \lambda \in \Lambda^+$. Let $P^- = MU^-$ be the opposite parabolic. By Exercise 8.14 we have $K = K_+ K_M K_-$ where $K_+ = K \cap U, K_- = K \cap U^-$.

We have to show that for any $w \in V_U^{K_M}$ there exists $v \in V^K$ such that $q(v) = w$. Choose any $v' \in V$ such that $q(v') = w$ and consider the stabilizer K'_- of v' in U^-. It is clear that for sufficiently big $r \in \mathbb{Z}_+$ we have $\lambda^{-r} K'_- \lambda^r \supset K_-$ and therefore $K_- \subset St_G(\lambda^{-r} v')$. Let

$$v'' := \int_{k \in K_M K_+} \pi(k)\pi(\lambda^{-r})v'$$

Since $K_+ = K \cap U, K_- = K \cap U^-$ we see that $v'' \in V^K$. On the other hand $q(\pi(\lambda^{-r})v') = \rho(\lambda^{-r})w$. Since $\lambda \in Z_M$ and $w \in V_U^{K_M}$ we have $q(v'')) = \rho(\lambda^{-r})w$. Let $v = \pi(a_{\lambda^r})v'' \in V^K$. The same arguments as in the proof of Proposition 8.11 show that $q(v) = w$.

□

Corollary 9.8. *For any admissible representation π of G and a parabolic subgroup $P = MU \subset G$ the representation $r_{P,G}(\pi)$ on M is admissible.*

9.2. **Uniform Admissiblity.** The theorem that we just proved says the following: given an open compact subgroup K and an irreducible representation V of G, then V^K is finite dimensional. However, as far as we know, $\dim V^K$ may be arbitrarily large for a given V.

Theorem 9.9. *T:un [Uniform Admissibility] Given an open compact subgroup $K \subset G$, then there is an effectively computable constant, $c = c(G, K)$, so that whenever V is an irreducible representation of G, $\dim V^K \leq c$.*

Reformulation. *All irreducible representations of the algebra $\mathcal{H}_K(G)$ have dimension bounded by $c(G, K)$.*

As it has been for some time, our main tool is the decomposition $\mathcal{H}_K(G) = \mathcal{H}_0 C \mathcal{H}_0$. However, we first need some linear algebra. Consider the following question: Give $N \cong \mathbb{C}^m$ and C a commutative subalgebra of $\operatorname{End} N$, what is the bound for $\dim C$?

Conjecture. *If C is generated by l elements, then* $\dim C \leq m + l$.

Bernstein does not know how to prove this. However, we do have

Proposition 1. If $N \cong \mathbb{C}^n$, $C \subset \operatorname{End} N$ is commutative and generated by l elements, then

$$\dim C \leq m^{2-1/2^{l-1}}.$$

Proof. Omitted. □

We now prove the theorem.

Proof. Let (ρ, V) be an irreducible representation of $\mathcal{H}_K(G)$. Let $k = \dim V$. We want to find $c = c(G, K)$ so that $k \leq c$. We know that $k \leq \infty$. Moreover, it is a general algebraic result (Burnside's Theorem) that $\rho \colon \mathcal{H}_K(G) \to \operatorname{End} V$ is surjective.

We may write $\mathcal{H}_K = \mathcal{H}_0 C \mathcal{H}_0$ with C commutative and finitly generated (say l generators). Let $d = \dim \mathcal{H}_0$. Clearly, $k^2 = \dim \operatorname{End} V = \dim \rho(\mathcal{H}_K) \leq d^2 \dim \rho(C)$. But by the proposition, $\dim \rho(C) \leq k^{2-1/2^{l-1}}$. Thus,

$$k^2 \leq d^2 k^{2-1/2^{l-1}}.$$

Therefore, if we set $c(G, K) = d^{2^l}$, we have $k \leq c$. □

Consider $G^0 \subset G$ as before, and consider $K \subset G^0$ compact. We know that given any irreducible cuspidal representation (ρ, W) of G^0, and $\xi \in W$, then $\rho(e_K)\rho(g)\rho(e_K)\xi$ has compact support in G^0. We will now show how the uniform admissibility theorem can strengthen this result.

Proposition 2. Given $K \subset G^0 \subset G$ as above, there exists an open compact subset $\Omega \subset \Omega(G, K) \subset G^0$ such that $\operatorname{Supp} \rho(e_K)\rho(g)\rho(e_K)\xi \subset \Omega$ for all (ρ, W) irreducible cuspidal and $\xi \in W$.

Proof. It follows from the proof of Harish-Chanda's theorem (and in fact, theorem 1 of the last lecture) that compact representations of G^0 are exactly those for which $\lambda \mapsto \rho(a(\lambda))\xi$ has finite (and hence compact) support in Λ^{+0}. This is in turn equivalent to the statement that for any $\nu \in \Lambda^{+0}\backslash\{1\}$, $\xi \in W^K$ there is a constant $k_{\nu,\xi}$ so that $\rho(\nu^k)\xi = 0$ whenever $k \geq k_{\nu,\xi}$. It is easy to see that our proposition amounts to the statement that these constants can be chosen independent of ξ and W. But this is obvious because we know that there is a constant $c = c(G, K)$ so that $\dim W^K \leq c$. □

Corollary 1. Given $K \subset G^0$, there are only finitly many equivalence classes of irreducible cuspidal representations of $\mathcal{H}_K(G^0)$.

Proof. Since the support of the matrix coefficients of the irreducible cuspidal representations must lie in $\Omega(G, K)$, the corollary follows from the following general lemma.

Lemma 2. The matrix coefficients of any set of pairwise non-isomorphic matrix coefficiets are linearly independent functions.

□

Remark. By working through the proofs in this section, this bound can be made precise.

10. Integration

10.1. **Smooth measures.**

Definition 10.1. D:sm a) The category $\mathcal{A}_F$ of analytic F-manifold is defined exactly as the category of analytic $\mathbb{C}$-manifold. By the definition any point $x \in X$ has an admits a neighborhood $U \ni x$ an an bianalitic bijection $\phi : U \to \mathcal{O}_F^d$. We call such a pair (U, ϕ) a coordinate system at x.

b) For any smooth algebraic F-variety $\underline{X}$ the set $X := \underline{X}(F)$ has a natural structure of an analytic F-manifold.

c) A morhism $f : X \to Y$ in $\mathcal{A}_F$ is *smooth* if for any point $x \in$ the differential $df_x : T_X(x) \to T_Y(y)$ is surjective.

d) A measure μ on an analytic F-manifold X is *smooth* for any point $x \in X$ and a coordinate system $\phi : U \to \mathcal{O}_F^d$ the measure $\phi_\star(\mu)$ on $\mathcal{O}_F^d$ has a form fdx where dx is a Haar measure on $\mathcal{O}_F^d$ and f a locally constant function on $\mathcal{O}_F^d$.

Exercise 10.2. E:int Let $f : X \to Y$ be a smooth morphism and μ a smooth measure on X with compact support. Then the measure $f_\star(\mu)$ on Y is smooth.

Let $G' \subset G$ be the set of regular simisimple elements. For any $x \in G'$ we denote by $\Omega_x \in G$ the conjugacy class of x. Let $B \subset G$ be the subgroup of upper-triangular matricies, $Z := G/B$ and $p : \Omega_x \to Z$ the natural projection.

Lemma 10.3. *L:pr The morphism p is smooth.*

Proof. Let $f^x : G \to Z$ be given by $f^x(g) = p(gxg^{-1})$. It is sufficient to show that the morphism f^x is smooth.

Since $f^x(gg') = f^{g'xg'^{-1}}(g)$ it is sufficient to show that the differential of f^x at e is surjective. By the definition $T_G(e) = \mathcal{G}$ and $T_Z(p(e)) = \mathcal{G}/\mathcal{B}$ where $\mathcal{G}$ is the Lie algebra of G and $\mathcal{B}$ is the Lie algebra of B.

Let $<,>: \mathcal{G} \times \mathcal{G} \to F$ be the bilinear form $< a, b >:= Tr(ab)$. For any subspace L of $\mathcal{G}$ we denote by $L^\perp \subset \mathcal{G}$ the orthogonal complement to L in respect to $<,>$.

Since the map $df_e^x : \mathcal{G} \to \mathcal{B}$ is given by

$$a \to q(Ad(x^{-1})(a) - a)$$

where $q : \mathcal{G} \to \mathcal{G}/\mathcal{B}$ is the natural projection the sujectivity of df_e^x would follow from the sujectivity of the map $\mathcal{G} \oplus \mathcal{B} \to \mathcal{G}$ given by $(a, b) \to (Ad(x^{-1}) - Id)(a) + b$.

Since the form $<,>$ is non-degenerate and $Ag(G)$-invariant it is sufficient to show that

$$Im^\perp(Ad(x) - Id) \cap \mathcal{B}^\perp = \{0\}$$

It is clear that $Im^\perp(Ad(x^{-1} - Id)) = Ker(Ad(x^{-1} - Id)$ does not contain non-zero nilpotent elements. On the other hand the space $\mathcal{B}^\perp = \mathcal{U}$ consists of upper-triangular nilpotent matricies. So $Im^\perp(Ad(x) - Id) \cap \mathcal{B}^\perp = \{0\}$. □

Corollary 10.4. *C:pr The morphism* $m : \Omega_x \times B \to G$ *given by* $(y, b) \to yb$ *is smooth.*

10.2. **Left and right invariant measures.** Let H be an l-group. Choose a right-invariant Haar measure μ_r on H.

For any $h \in H$ we denote by $\hat{h}^r, \hat{h}^l : H \to H$ the right and left shifts by h.

Exercise 10.5. E:mes a) For any automorphism α of H there exists $\Delta(\alpha) \in \mathbb{R}_+$ such that $\alpha_\star(\mu_r) = \Delta(\alpha)(h)\mu_r$.

In other words for any open compact subset C of H we have

$$\mu_r(\alpha(C)) = \Delta^{-1}(\alpha)\mu_r(C)$$

For any $h \in H$ we define $\Delta(h) := \Delta(\alpha_h), \alpha_h(x) := hxh^{-1}$

b) The image μ_l of μ_r under the map $h \to h^{-1}$ is a left-invariant Haar measure on H.

c) $\mu_l = \Delta\mu_r$.

d) Assume that $H = M \ltimes U$ where M and U are unimodular and choose Haar measures dl on M and du on U. Let $m : U \times M \to H$ be the product map. Then $m_\star(dudl)$ is a right-invariant Haar measure on H and $\Delta(mu) = \Delta(\alpha(m))$ where $\alpha(m)$ is an automorphism of U given by $u \to mum^{-1}$.

e) Compute the function Δ in the case when H is a standard parapolic subgroup $P_{\bar{m}}$ of $GL(n, F)$.

f) Let G be a unimodular l-group, $H, K \subset G$ closed subgroups such that K is compact and the product map $m : K \times H \to G$ is surjective. Then $m_\star(dk\mu_r)$ is a Haar Measure on G.

10.3. **Integration over the quotient spaces.** Let X be a locally compact space and H be a locally compact group acting continuously and freely on X in such way that the quotient space X/H is an l-space and the projection $p : X \to X/H$ is locally trivial. We will construct a bijection between the space of $\mathcal{M}(X, H)$ smooth measures on μ on X such that

$$\hat{h}_\star(\mu) = \Delta(h)\mu, \hat{h}(x) = xh$$

and the space $\mathcal{M}(X/H)$ of smooth measures on X/G. Let μ_l be a left-invariant measure on H. For any point $x \in X$ the map $r_x : h \to xh$ identifies the group H with the fiber $p^{-1}(\bar{x}), \bar{x} := p(x)$. It is clear that the measure $\mu_{\bar{x}} := r_{x\star}(\mu_l)$ on $p^{-1}(\bar{x})$ depends only on $\bar{x} \in G/H$.

We denote by $I : \mathcal{S}(X) \to \mathcal{S}(X/H)$ the linear map given by

$$I(f)(\bar{x}) = \int_{p^{-1}(\bar{x})} f\mu_{\bar{x}}$$

We denote by $I^\vee$ the dual map from measures on X/H to measures on X.

Exercise 10.6. E:meas Show that

a) $I^\vee \mathcal{M}(X/H) \subset \mathcal{M}(X,H)$.

An explicit construction of the inverse of $I^\vee : \mathcal{M}(X/H) \to \mathcal{M}(X,H)$ requires some choices.

b) There exists a closed subset S of X such that the multiplication $m : S\times H \to X$ is a homeomorphism. The natural projection $S\times H \to S$ induces a bijection $p_S : S \to X/H$. We identify functions on X with ones on $S \times H$.

Choose an open compact subset C of H such that $\int_C \mu_r = 1$. For any $\mu \in \mathcal{M}(X,H)$ f on X we define a functional $\bar{\mu}$ on the space of locally constant functions ϕ on G/H with compact support by

$$\bar{\mu}(\phi) := \int_{S\times H} p_S^\star(\phi) \times ch_C m_\star^{-1}(\mu)$$

where ch_C is the characteristic function of the set C.

c) The functional $\bar{\mu}$ does not depend on the choice of sets S and C and the map $\mu \to \bar{\mu}$ is the inverse of $I^\vee$.

Remark 10.7. If ν is a smooth H-invariant measure on X which is not zero anywhere on X we can identify the space $\mathcal{M}(X,H)$ with the space of of locally constant functions f on X such that

$$f(xh) = \Delta f(x)$$

Definition 10.8. Let G be a unimodular l-group, $H \subset G$ a closed subgroup and (ρ, V) a representation of H. We denote by $\tilde{i}_H^G(V)$ the space of locally constant functions $f : G \to V$ such that

$$f(gh) = \rho(h^{-1})f(g)\Delta^{1/2}(h)$$

and $g \in G$ acts on $\tilde{i}_H^G(V)$ by the left shift by g^{-1}.

It is clear that for any $f \in \tilde{i}_H^G(V), \tilde{f} \in \tilde{i}_H^G(\tilde{V})$ the function $< \tilde{f}(g), f(g) > dg$ belongs to $\mathcal{M}(X,H)$.

Exercise 10.9. E:un Let $W = \tilde{i}_H^G(V), \tilde{W} = \tilde{i}_H^G(\tilde{V})$. We define a pairing $[,] : W \times \tilde{W} \to \mathbb{C}$ by

$$[f, \tilde{f}] := \int \overline{\tilde{f}(g)(f(g))}$$

Show that

a) The pairing $[,]$ is G-invariant.

b) For any unitary representation (ρ, V) of H the representation $\tilde{i}_H^G(V)$ of G has a natural unitary structure.

c) Generalize the definition of $\tilde{i}_H^G(V)$ to the case when G is not unimodular.

Assume now that $G = GL_n(F), K_0 = GL_n(\mathcal{O})$ and $P \subset G$ is a parabolic subgroup. If $m \in M$ is regular semisimple element then the centralizer $Z(m)$ of m in G lies in M. We fix a Haar measure dz on Z. Since both group G, M and $Z(m)$ are unimodular a choice of Haar measures dg, dm, dz on G, M and Z defines G-invariant measure dx on $G/Z(m)$ and an M-invariant measure dy on $M/Z(m)$. For any function $f \in \mathcal{S}(G)$ we define a function f_P om M by

$$f_P(m) := \int_{k \in K_0, u \in U} f(k^{-1}muk)dmdudk$$

Exercise 10.10. E:orb For any regular element $m \in M$ and $f \in \mathcal{S}(G)$ we have

$$\int_{x \in G/Z(m)} f(xmx^{-1})\|det(Ad(m) - Id)_{\mathcal{U}}\| = \int_{y \in M/Z(m)} f_M(ymy^{-1}$$

where $\mathcal{U}$ is the Lie algebra of U.

11. Characters of representations

We start with the following result of Harish-Chandra. Let (π, V) be an admissible finitely generated representation of G. Consider a function $T : G' \to End(V)$ given by $T(x) = \int_{k \in K_0} \pi(kxk^{-1})dk$ where dk is the Haar measure on K_0. It is clear that operators T_x commute with the action of K_0.

Theorem 11.1. *T:H a) For any $x \in G'$ we have $T_x \in End(V)_{sm}$*

b) The function T on G' is locally constant.

Proof. We prove the part a). The proof of the part b) is completely analogous.

Choose a congruence subgroup K such that V is generated by V^K. Since (π, V) is an admissible $dimV^K < \infty$. Let $B_0 = B \cap K$. It is clear that $\lambda b \lambda^{-1} \in B_0$ for all $\lambda \in \Lambda^+, b \in B_0$. Let α_x be the $K_0 \times B_0$-invariant measure on $\Omega_x \times B$ of volume 1 which is supported on $A_x \times B_0, A_x = \{kxk^{-1}\}, k \in K_0 \subset \Omega_x$. As follows from Corollary 10.4 the morphism $m : \Omega_x \times B \to G$ given by the multiplication is smooth. Therefore the measure $m_\star(\alpha_x)$ has a form $f_x(g)dg$ where dg is a Haar measure on G

and $f_x(g)$ is a locally constant function of $G' \times G$ such that for any $x \in G'$ the function $f_x(g)$ has compact support.

Let K' be the congruence subgroup of G such that the function f_x is right K'-invariant. It is sufficient to show that for any $v \in V^K$ and $g \in G$ we have $T_x\pi(g)v \in V^{K'}$. Since the subspaces $V^K, V^{K'}$ are K_0-invariant it follows from the Cartan decomposition that it is sufficient to show that for any $v \in V^K, \lambda \in \Lambda^+$ we have $T_x\pi(\lambda^{-1})v \in V^{K'}$. In particular it is sufficient to show that for all $\lambda \in \Lambda^+$ we have

$$T_x\pi(\lambda^{-1})v = \pi(f_x)\pi(\lambda^{-1})v$$

For any $\lambda \in \Lambda^+, b \in B_0, v \in V^K$ we have

$$\pi(b\lambda^{-1})v = \pi(\lambda^{-1})\pi(\lambda b\lambda^{-1})v = \pi(\lambda^{-1})v$$

since $\lambda b\lambda^{-1} \in B_0 \subset K$ for all $\lambda \in \Lambda^+, b \in B_0$. So

$$T_x\pi(\lambda^{-1})v = \int_{k\in K_0, b\in B_0} \pi(kxk^{-1}b)v$$

By the definition the measure $m_\star(\alpha_x)$

$$\int_{K_0\times B_0} F(kxk^{-1})bdkdb = \int_G f_x(g)F(g)dg$$

we have for any $F \in L^1_{loc}(G)$. Therefore

$$T_x\pi(\lambda^{-1})v = \pi(f_x)\pi(\lambda^{-1})v$$

□

If (π, V) is an admissible representation of G define the *character* χ_π of π as a linear functionl on $\mathcal{H}(G)$ given by

$$\chi_\pi(h) := Tr(\pi(h))$$

Remark 11.2. Characters χ_π are functionals on the space of smooth measures on G with compact support. So they are "generalized functions" on G.

Exercise 11.3. C:H Let $G' \subset G$ be the set of regular semisimple elements. Then for any irreducible representation (π, V) of G the restriction of χ_π on G' is given by a locally constant function. [That is there exists a locally constant function t_π on G' such that for any $h \in \mathcal{H}(G)$ supported on G' we have $\chi_\pi(h) = \int_G t_\pi h$].

Remark 11.4. R:H One can show that the function t_π belongs to $L^1_{loc}(GL(n, F))$ and for any $h \in \mathcal{H}(G)$ we have $\chi_\pi(h) = \int_G t_\pi h$. For other groups the analogous result is known only if F is a field of characteristic zero.

Lemma 11.5. *L:Del Let (π, V) be a cuspidal irreducible representation of G and $g \in G$ be a semisimple elelment such that $P_g = M_g U_g \neq G$. Then $\chi_\pi(a_r(g)) = 0$ for $r >> 0$ where $a_r(g) = e_{K_r} * \mathcal{E}_g * e_{K_r} \in \mathcal{H}_{K_r}(G)$.*

Proof. The same arguments as in the proof of Proposition 8.9 show that there exists r_0 such that for any $r > r_0$ and any $k > 0$ we have $a_r(g^k) = a_r^k(g)$. Since the representation (π, V) is cuspidal it follows from Proposition 8.11 that there exist $k(r) \in \mathbb{Z}$ such that $a_r^{k(r)}(g) = 0$. But then $a_r^{k(r)}(g)$. Therefore $Tr(\pi(a_r(g))) = 0$. □

Exercise 11.6. E:del Let $G_c \subset G_n$ be the set of elements such that the image $\bar{g}$ of g in $PGL(n, F)$ generates a compact subgroup. Show that

a) $G_c = \{g \in G_n | U_g = \{e\}\}$.

b) If (π, V) is a cuspidal irreducible representation of G then

$$supp(\chi_\pi) \subset G_c$$

c) For any regular semisimple $g \in G$ we have $\chi_\pi(g) = \chi_{J_{U_g}(\pi)}(g)$.

11.1. **Characters of square-integrable representations.** For compact groups one can write characters as averages of matrix coefficients. For locally compact groups such a construction exists for *square-integrable* representations. Let G be a locally compact group and Z be the center of G.

Definition 11.7. D:sq a) For a unitary character $\theta : Z \to \mathbb{C}^\star$ we denote by $L^2_\theta(G)$ the space of complex-valued functions f on G such that $f(zg) = \theta(z)f(g), g \in G, z \in Z$ and $|f| \in L^2(G/Z)$. The group G acts naturally on $L^2_\theta(G)$ by left shifts.

b) An irreducible representation (π, V) is of a locally compact group G is square-integrable if there exists an imbedding of (π, V) into $L^2_\theta(G)$ for some unitary character θ.

Exercise 11.8. a) An irreducible representation (π, V) of an l-group is square-integrable iff for any pair $v \in V, \tilde{v} \in \tilde{V}$ we have

$$|m_{v,\tilde{v}}| \in L^2(G/Z)$$

b) If (π, V) is an irreducible representation of an l-group such that $|m_{v,\tilde{v}}| \in L^2(G/Z)$ for some non-zero $v \in V, \tilde{v} \in \tilde{V}$ then (π, V) is square-integrable.

c) For any irreducible square-integrable representation (π, V) there exists unique (up to a scalar) Hermitian G-invariant form $<, >$ on V

d) If (π, V) is an irreducible square-integrable representation and $d\bar{g}$ a Haar measure on G/Z then there exists a constant $d_\pi > 0$ such that for any $v, v', v'' \in W$ we have

$$\int_{G/Z} < \pi(g)v, v' >< v'', \pi(g)v > d\bar{g} = d(\pi) < v'', v' >$$

[the Schur orthogonality relations].

Proposition 11.9. *P:sq For any irreducible square-integrable representation (π, V) any $f \in \mathcal{H}(G)$ and $v \in V$ the integral $\int_{G/Z}\{\int_G < \pi(g^{-1}hg)v, v > f(h)\}d\bar{g}$ is absolutely convergent and*

$$(\star)\chi_\pi(f) < v, v >= d(\pi) \int_{G/Z} \{ \int_G < \pi(g^{-1}hg)v, v > f(h)\}d\bar{g}$$

Proof. Let v_i be an orthonormal basis of $V, q_{ij} :=< \pi(f)(v_i), v_j >$. Since the representation (π, V) is unitary we have

$$< \pi(g^{-1})\pi(f)v, v >=< \pi(f)v, \pi(g)v >$$

Therefore the right side of $(\star)$ equals to

$$\sum_i < \pi(f)v, v_i >< v_i, \pi(g)v >$$

which in turn equals to

$$\sum_{ij} < \pi(g)v, v_j > q_{ji} < v_i, \pi(g)v >$$

Since the (π, V) is admissible $q_{ij} = 0$ for almost all pairs i, j and both series there is only a finite number of non-zero terms. So we have

$$\int_{G/Z} \{ \int_G < \pi(g^{-1}hg)v, v > f(h)\}d\bar{g} = \sum_{ij} q_{ji} \int_{G/Z} < \pi(g)v, v_j >< v_i, \pi(g)v >$$

Applying the Schur orthogonality relations we see that the integral in the right side of $\star$ is absolutely convergent and is equal to

$$1/d(\pi) \sum_{ij} q_{ij} < v_i, v_j >= 1/d(\pi) \sum_i q_{ii} = 1/d(\pi)Tr(\pi(f))$$

Since

$$< \pi(g^{-1})\pi(f)\pi(g) = \int_G \pi(g^{-1}hg)v, v > f(h)$$

the Proposition follows. ∎

11.2. **Characters of induced representations.** Let G be an unimodular l-group, $P \subset G$ a closed cocompact subgroup. As follows from Proposition 8.21 (4) for any admissible representation (ρ, V) of P the induced representations $\tilde{i}_P^G(\rho)$ is also admissible. We show now how to express the character $\tilde{i}_H^G(\rho)$ in terms of χ_ρ. We fix a Haar measure dg on G and a right-invariant Haar measure $d_r p$ on P.

Let (ρ, V) be an admissible representation of P and

$$(\pi, W) = (\tilde{i}_P^G(\rho), \tilde{i}_P^G(V))$$

For any function $a \in \mathcal{S}(G)$ and a pair $(x, g) \in G$ we define $A^a_{x,g} \in End(V)$ by

$$A^a_{x,g} := \int_{p \in P} a(xpg^{-1})\rho(p)\Delta^{1/2}(p) d_r p$$

Exercise 11.10. E:ind a) For any $a \in \mathcal{S}(G)$ we have

$$A^a_{xp',gp''} = \Delta^{1/2}(p'p'')\rho^{-1}(p')A^a_{x,g}\rho(p'')$$

As follows from a) for any $f \in W, x \in G$ we have

$$A^a_{x,gp} f(gp) = \Delta^{-1}(p) A^a_{x,g} f(g), g \in G, p \in P$$

and therefore (see Exercise 10.6) we can consider $A^a_{x,g} f(g)$ as a V-valued measure on G/P. Analogously we can consider $A^a_{x,x}$ as as a $End(V)_{sm}$-valued measure on G/P.

b) For any $h = adg \in \mathcal{H}(G)$ we have

$$(\pi(h)f)(x) = \int_{G/P} A^a_{x,g} f(g)$$

c) For any $h = adg \in \mathcal{H}(G)$ we have

$$Tr(\pi(h)) = \int_{G/P} Tr(A^a_{x,x})$$

Assume now that P is a semidirect product $P = M \ltimes U$ where M and U are unimodular and there exists a compact subgroup K_0 of G such that $K_0 H = G$. We choose a Haar measures dm on m and du on U. We define $dg := m_\star(dkdmdu)$ where dk is the Haar measure on K_0 of volume equal to 1 and $m : K_0 \times U \times M \to G$ is the product. By Exercise 10.5 f) dg is a Haar measure on G. As before for any representation ρ of M we define $\tilde{i}_{G,P}(\rho) := \tilde{i}_P^G(\rho \circ q)$ where $q : P \to M = P/U$ is the canonical projection.

d) For any $h = fdg \in \mathcal{H}(G)$ we define $h_M \in \mathcal{S}(M)$ by $h_M = f_M dm$ where

$$f_M(m) := \int_{k \in K_0, u \in U} f(k^{-1}muk)dmdudk$$

we have

$$\chi_\pi(h) = \chi_\rho(h_M)$$

Assume now that $G = GL_n(F), K_0 = GL_n(\mathcal{O})$ and $P \subset G$ is a parabolic subgroup. If $m \in M$ is regular semisimple element then the centralizer $Z(m)$ of m in G lies in M. We fix a Haar measure dz on Z. It defines a G-invariant measure dx on $G/Z(m)$ and an M-invariant measure dy on $M/Z(m)$.

Exercise 11.11. E:orb For any regular element $m \in M$ and $f \in \mathcal{S}(G)$ we have

$$\int_{x \in G/Z(m)} f(xmx^{-1})\|det(Ad(m) - Id)_{\mathcal{U}}\| = \int_{y \in M/Z(m)} f_M(ymy^{-1})$$

where $\mathcal{U}$ is the Lie algebra of U.

Corollary 11.12. *If $P = MU, P' = MU'$ are two parabolics with the same Levi subgroup then for any admissible representation π of M the characters of representations $\tilde{i}_{G,P}$ and $\tilde{i}_{G,P'}$ coinside on the set $G' \subset G$ of regular elements.*

12. The Geometric Lemma

12.1. **The case $G = GL(2, F)$.** Let $G = SL(2, F), B = TU \subset G$ be the upper-triangular subgroup, where T is the group of diagonal matricies and U is the group of unipotent upper-triangular matricies. How to describe $r_{B,G} \circ i_{G,B}(\mathbb{C}, \chi)$ where $\chi : T \to \mathbb{C}^\star$ is a character?

By the defintion, $i_{G,B}(\mathbb{C}, \chi)$ is the space of smooth functions f on G such that

$$f(gtu) = \chi(t^{-1})f(g), g \in G, t \in T, u \in U$$

For any $f \in i_{G,B}(\mathbb{C}, \chi)$ we denote by f_B the restriction of f on $B \subset G$.

Let $V_0 = \{f \in i_{G,B}(\mathbb{C}, \chi) | f_B \equiv 0\}$. Then V_0 is a B-invariant subspace of $i_{G,B}(\mathbb{C}, \chi)$.As follows from Exercise 3.3 we can identify V_0 with the space of functions f on $G - B$ such that

$$f(gtu) = \chi(t^{-1})f(g), g \in G - B, t \in T, u \in U$$

for which there exists a compact subset C of $G - B$ such that

supp$(f) \subset CB$. It is clear that we can identify the quotient $V_1 = i_{G,B}(\mathbb{C}, \chi)/V_0$ with the space of functions h on B such that

$$f(xtu) = \chi(t^{-1})f(x), x \in B, t \in T, u \in U$$

In other words $V_1 = \mathbb{C}$ and the group B acts by the character χ.

Since $G - B = UwB, w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the restriction to Uw identifies the space V_0 with the space of functions f with compact support on U. We identify F with U by $u : x \to u(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $F^\star$ with T by $d : (t) \to d(t, t^{-1}) := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$

The group $B = TU$ acts on V_0 by the representation $\rho_w(\chi)$ where

$$(\rho_w(\chi)(d(t)u(x))f)(y) = \chi(d(t^{-1}))f(t^2(y - x))$$

Let $\bar{V_0} := r_{B,G}(V_0)$. Since the functor $r_{B,G}$ is exact $\bar{V_0}$ is a T-invariant subspace of $r_{B,G} \circ i_{G,B}\chi$ and we can identify the quotient $r_{B,G} \circ i_{G,B}\chi/\bar{V_0}$ with $r_{B,G}(V_1)$. It is clear that $\bar{V_1} = (\mathbb{C}, \chi)$

Exercise 12.1. Show that

a) The integration over U defines an equivalence $\bar{V_0} \to (\mathbb{C}, \tilde{\chi})$ where $\tilde{\chi}(t) = \chi^{-1}(t) \ |t\ |^{-2}$.

b) The representation $L(\chi) := \tilde{r}_{B,G} \circ \tilde{i}_{G,B}(\mathbb{C}, \chi)$ of T has a T-invariant subspace isomorphic to $(\mathbb{C}, \chi^{-1})$ and the quotient is isomorphic to $(\mathbb{C}, \chi)$.

c) If $\chi \neq \chi^{-1}$ then the representation $L(\chi)$ of T is isomorphic to $\chi \oplus \chi^{-1}$.

d) If $\chi = \chi^{-1}$ but $\chi \neq Id$ then $L(\chi) = \chi \oplus \chi$.

e) If $\chi = Id$ then $L(\chi) \neq Id \oplus Id$.

f) The representation $\tilde{i}_{G,B}(\mathbb{C}, \chi)$ of G is completely reducible iff $\chi = \chi^{-1}$ but $\chi \neq Id$.

g) The representation $\tilde{i}_{G,B}(\mathbb{C}, \chi)$ of G is reducible but not completely reducible iff $\chi(t) = \ |t\ |^{\pm 1}$.

12.2. **The case of the Borel subgroup.** Let $B = TU \subset G$ be the Borel subgroup. How to describe $\tilde{r}_{B,G} \circ \tilde{i}_{G,B}(\mathbb{C}, \chi)$ where $\chi : T \to \mathbb{C}^\star$ is a character?

By the defintion, $\tilde{i}_{G,B}(\mathbb{C}, \chi)$ is the space V of smooth functions f on G such that

$$f(gtu) = \chi(t^{-1})f(g)\Delta^{1/2}(t), g \in G, t \in T, u \in U$$

For any $w \in W$ we choose a representative $\tilde{w} \in N_G(T)$ and we define $X_w := UwB \subset G$. It is clear that X_w does not depend on a choice of

$\tilde{w}$. Let $\bar{X}_w$ be the closure of X_w in G. We write $w' \leq w$ for $w', w \in W$ if $X_{w'} \subset \bar{X}_w$. For any $w \in W$ we define

$$V_w = \{f \in V | f_{X_w} \equiv 0\}$$

and define V_w^+ as the intersection of $V_{w'}$ for $w' < w$.

Exercise 12.2. Show that

a)) $V_w^+ \supset V_w$.

b) $\tilde{r}_{B,G}(V_w^+/V_w) = (\mathbb{C}, \chi^w)$.

12.3. **The general case.** Let $Q = NV$ and $P = MU$ be parabolics in G containing T. We use the notations from Definition 8.6

For any $w \in W$ we consider subgroups

$$N'_w = M \cap w^{-1}Nw \subset Q'_w = M \cap w^{-1}Qw \subset M$$

and

$$M'_w = N \cap wMw^{-1} \subset P'_w = N \cap wPw^{-1} \subset N$$

Exercise 12.3. E:Basic a) For any $w \in W_{P,Q}$ subgroups

$$P'_w = M'_w U'_w \subset N, Q'_w = N'_w V'_w \subset M$$

are parabolic subgroups of N and M respectively, and $\operatorname{Ad} w : N'_w \to M'_w$ is an isomorphism.

b) There is a finite filtration of the functor

$$F := \tilde{r}_{P,G} \circ \tilde{i}_{G,Q}$$

from $\mathcal{M}(N) \to \mathcal{M}(M)$ by subfunctions with quotients

$$F_w = \tilde{i}_{M,Q'_w} \circ \tilde{w} \circ \tilde{r}_{P'_w,N}, w \in W_{P,Q}$$

where $\tilde{w}_i \colon \mathcal{M}(M'_w) \to \mathcal{M}(N'_w)$ is the equivalence of categories associated with the isomorphism $\operatorname{Ad} w_i \colon N'_w \to M'_w$.

Let $\bar{P} = M\bar{U} \subset G$ be the parabolic subgroup opposite to P [so $\bar{P}$ contains the opposite Borel subgroup $\bar{B}$]. Let $Y := P\bar{P} \subset G$ and $V' \subset \tilde{i}_{G\bar{P}}(\rho)$ be the subspace of functions f such that $f_{G-Y} \equiv 0$.

c) Construct a functorial morphism $\beta_M(\rho) : \rho \hookrightarrow \tilde{r}_{PG}\tilde{i}_{G\bar{P}}(\rho)$.

13. The second adjointness

If M is a Levi of G we denote by $P = MU \subset G$ the parabolic subgroup containing the Borel subgroup B and by $\bar{P} = M\bar{U} \subset G$ the parabolic subgroup containing the Borel opposite subgroup $\bar{B}$. For simplicity we write $i_{GM}, r_{MG}, \bar{i}_{GM}$ instead of $\tilde{i}_{GP}, \tilde{r}_{PG}$ and $\tilde{i}_{G\bar{P}}$. In this section we prove the folowing result.

Theorem 13.1. *T:second The functor $\bar{i}_{GM}$ is left adjoint to r_{MG}.*

The proof will be by induction on the semi-simple rank of G. To prove the theorem we have to construct a functorial isomorphism

$$\alpha_{\rho,\pi} \operatorname{Hom}_G(\bar{i}_{GM}(\rho), \pi) \simeq \operatorname{Hom}_M(\rho, r_{MG}(\pi))$$

The proof consists of several steps:
Step 1: Construction of a morphism. For any $\phi : \operatorname{Hom}_G(\bar{i}_{GM}(\rho), \pi)$ we can define

$$\alpha_{\rho,\pi}(\phi) := r_{M,G}(\phi) \circ \beta_M(\rho)$$

We need to show that this morphism is an isomorphism.

We shall say that π is *good* if $\alpha_{\rho,\pi}$ is an isomorphism for all ρ. If in an exact sequence $\{0\} \to \pi_1 \to \pi \to \pi_2 \to \{0\}$ in $\mathcal{M}(G)$ representations π_1, π_2 are *good* it follows from the exactness of functors r and i that π is also *good*.

Any quasi-cuspidal representation π of G is *good* since $\operatorname{Hom}_G(\bar{i}_{GM}(\rho), \pi) = \{0\}$ [see Corollary 8.27] and $r_{MG}(\pi)) = \{0\}$.

As follows from Exercise 8.28 there exists a morphism

$$j : V \to \oplus_{P \in \mathcal{P}} i_{G,P}(\pi_P)$$

such that the representation of G on $Ker(j)$ is quasi-cuspidal and therefore is *good*. Moreover as follows from Theorem 9.9 it is sufficient to show that $Im(j)$ is *good*.

Step 2. We claim that it is enough to prove that for every π there exists an imbedding $\pi \hookrightarrow \pi'$ where π' is *good*. We first show that the existence of such a morphism for all $\pi \in \mathcal{M}(G)$ implies that $\alpha_{\rho,\pi}(\phi)$ is an imbedding for all $\pi \in \mathcal{M}(G), \rho \in \mathcal{M}(M)$.

Let $\bar{\pi} \subset \pi'$ be the image of t. Consider the diagram

$$\begin{array}{ccccc} 0 \to \operatorname{Hom}_G(\bar{i}_{GM}(\rho), \bar{\pi}) & \longrightarrow & \operatorname{Hom}_G(\bar{i}_{GM}(\rho), \pi') & \longrightarrow & \operatorname{Hom}_G(\bar{i}_{GM}(\rho), \pi'/\bar{\pi}) \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to \operatorname{Hom}_M(\rho, r_{MG}(\bar{\pi})) & \longrightarrow & \operatorname{Hom}_M(\rho, r_{MG}(\pi')) & \longrightarrow & \operatorname{Hom}_M(\rho, r_{MG}(\pi'/\bar{\pi})) \end{array} \quad (1)$$

in which both rows are exact and the middle vertical arrow is an isomorphism. It shows immediately that the left vertical arrow $\alpha_{\rho,\pi}(\bar{\phi})$ is an embedding.

Consider now the diagram

$$\begin{array}{ccccc} 0 \to \mathrm{Hom}_G(\bar{i}_{GM}(\rho), \bar{\pi}) & \longrightarrow & \mathrm{Hom}_G(\bar{i}_{GM}(\rho), \pi') & \longrightarrow & \mathrm{Hom}_G(\bar{i}_{GM}(\rho), \pi'/\bar{\pi}) \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to \mathrm{Hom}_M(\rho, r_{MG}(\bar{\pi})) & \longrightarrow & \mathrm{Hom}_M(\rho, r_{MG}(\pi')) & \longrightarrow & \mathrm{Hom}_M(\rho, r_{MG}(\pi'/\bar{\pi})) \end{array} \quad (2)$$

once more. Since$\alpha_{\rho,\pi}(\phi)$ is an imbedding for all $\pi \in \mathcal{M}(G), \rho \in \mathcal{M}(M)$ the map $\alpha_{\rho,\pi'/\bar{\pi}}(\phi)$ is also an imbedding. This easily implies that the left vertical arrow $\alpha_{\rho,\pi}(\bar{\phi})$ is an isomorphism.

Step 3. We see that it is sufficient to show that for any proper parabolic subgroups $Q = NV$ of G and a representation σ of M the representation $i_{G,P}(\sigma)$ is good. The same arguments as in the end of the Step 1 show that we may assume that σ is quasi-cuspidal. As follows from Exercise 8.26 we can also assume that σ is injective.

Step 4. As follows from Proposition 8.21 we can consider $\alpha_{\rho,i_{G,N}(\sigma)}$ as a morphism from $\mathrm{Hom}_N(r_{N,G} \circ \bar{i}_{GM}(\rho), \sigma)$ to $\mathrm{Hom}_M(\rho, r_{MG} \circ i_{G,N}(\sigma))$.

By Exercise 12.3 the representation $r_{MG} \circ i_{G,N}(\sigma)$ has a filtration by subrepresentations

$$F_w(r_{MG} \circ i_{G,N}(\sigma)) = \tilde{i}_{M,Q'_w} \circ \tilde{w} \circ \tilde{r}_{P'_w,N}(\sigma), w \in W_{P,Q}$$

Since σ is injective all the factors $F_w(r_{MG} \circ i_{G,N}(\sigma))$ are injective and therefore $r_{MG} \circ i_{G,N}(\sigma)$ is isomorphic to the direct sum

$$\oplus_{w \in W_{P,Q}} \tilde{i}_{M,Q'_w} \circ \tilde{w} \circ \tilde{r}_{P'_w,N}(\sigma)$$

So $\mathrm{Hom}_M(\rho, (r_{MG} \circ i_{G,N}(\sigma))$ has a filtration by subspaces

$$\mathrm{Hom}_M(\rho, F_w(r_{MG} \circ i_{G,N}(\sigma))), w \in W_{P,Q}$$

On the other hand as follows from Exercise 12.3 the representation $r_{N,G} \circ \bar{i}_{GM}(\rho)$ has a filtration by subrepresentations $F_v(r_{N,G} \circ \bar{i}_{GM}(\rho))$ with subquotients $\bar{F}_v(r_{N,G} \circ \bar{i}_{GM}(\rho))$ isomorphic to

$$\tilde{i}_{N,P'_v} \circ \tilde{v} \circ \tilde{r}_{Q'_v,M}(\rho), v \in W_{Q,\tilde{P}}$$

Since σ is injective the space $\mathrm{Hom}_N(r_{N,G} \circ \bar{i}_{GM}(\rho), \sigma)$ has a filtration by subspaces

$$\mathrm{Hom}_N(F_v(r_{N,G} \circ \bar{i}_{GM}(\rho), \sigma)), v \in \bar{W}_{Q,\tilde{P}}$$

where $\bar{W}_{Q,\tilde{P}}$ is a partially ordered set obtained from the the partially ordered set $W_{Q,\tilde{P}}$ by the reversal of the ordering.

Step 5. As follows from Exercise 8.8 $W_{P,Q} = W_M \backslash W / W_N, W_{Q,\tilde{P}} = W_N \backslash W / W_{w_0 M w_0^{-1}}$ and the bijection $w \to r(w) := w^{-1} w_0$ from $W_{P,Q}$ to

$\bar{W}_{Q,\tilde{P}}$ preserves the partial order. So the spaces $\mathrm{Hom}_N(r_{N,G}\circ\bar{i}_{GM}(\rho),\sigma)$ and $\mathrm{Hom}_M(\rho, r_{MG}\circ i_{G,N}(\sigma))$ both have filtartions

$$F_w(\mathrm{Hom}_N(r_{N,G}\circ\bar{i}_{GM}(\rho),\sigma)) \subset \mathrm{Hom}_N(r_{N,G}\circ\bar{i}_{GM}(\rho),\sigma)$$

and

$$F_w(\mathrm{Hom}_M(\rho, r_{MG}\circ i_{G,N}(\sigma))) \subset \mathrm{Hom}_M(\rho, r_{MG}\circ i_{G,N}(\sigma))$$

parametrized by the partial ordered set $W_{P,Q}$. Moreover the subquotiets of the first filtration are isomorphic to $\mathrm{Hom}_N(\tilde{i}_{N,P'_{r(w)}}\circ\tilde{v}\circ\tilde{r}_{Q'_{r(w)},M}(\rho)$ and the subquotiets of the second filtration are isomorphic to $\mathrm{Hom}_M(\rho,\tilde{i}_{M,Q'_w}\circ\tilde{w}\circ\tilde{r}_{P'_w,N}(\sigma)$. Since σ is cuspidal we can consider only $w\in W_{P,Q}$ such that $N\subset wPw^{-1}$. In this case

$$\bar{F}_w(\mathrm{Hom}_M(\rho, r_{MG}\circ i_{G,N}(\sigma))) = \mathrm{Hom}_M(\rho, i_{M,w^{-1}Qw\cap M}(\tilde{w}\circ\sigma)$$

and

$$\bar{F}_w(\mathrm{Hom}_N(r_{N,G}\circ\bar{i}_{GM}(\rho),\sigma)) = \mathrm{Hom}_N(r_{Q,Q\cap wMw^{-1}},\sigma)$$

Exercise 13.2. a) The map $\alpha_{\rho,i_{G,N}(\sigma)}$ is compatible with the filtrations.

b) The induced morphism

$$\mathrm{Hom}_N(\tilde{w}\circ r_{w^{-1}Qw\cap M,M}(\rho),\sigma) \to \mathrm{Hom}_M(\rho, i_{w^{-1}Qw\cap M,M}(\tilde{w}^{-1}\circ(\sigma))$$

is given by the adjunction.

Theorem 13.1 is proven.

Corollary 13.3. *C:ad a) The functor $i_{G,P}$ maps projective objects into projective.*

b) The functor $r_{P,G}$ commutes with infinite direct products.

Proof. a) Any functor which has a right adjoint maps projective objects into projective.

b) Any functor which has a left adjoint commutes with infinite direct products. □

Exercise 13.4. E:com Let $M^0 := M\cap G^0$. Then the functor $r_{P^0,G^0}:\mathcal{M}(G^0)\to\mathcal{M}(M^0)$ of U-coinvariants commutes with infinite direct products.

14. Decomposition of the category $\mathcal{M}(G)$

14.1. **Uniform admissibility.** By the definition for any admissible compact representation of G^0 a compact $C\subset G^0$ such that any $v\in V^K$ we have $\pi(e_K)\pi(g_0)v=0$ for all $g_0\notin C$. Now we can prove a stronger result.

Lemma 14.1. *L:uad For any congruence subgroup K of G there exists a compact $C \subset G^0$ such that for any compact representation (π, V) of G^0 and any $v \in V^K$ we have $\pi(e_K)\pi(g_0)v = 0$ for all $g_0 \notin C$.*

Proof. Fix a congruence subgroup K and choose an increasing sequence of open K-biinvariant subsets C_n of G^0 such that $\cup_n C_n = G^0$. We have to show that existence of n such that for any irreducible cuspidal representation (π, V) of G and any $v \in V^K$ we have $\pi(e_K)\pi(g_0)v = 0$ for all $g_0 \notin C_n$.

Assume that this is false.

Then there exists a sequence of cuspidal representations (π_n, V_n), vectors $v_n \in V_n^K$ and elements $g_n \notin Z(G)C_n$such that $\pi_n(e_K)\pi(g_n)v_n \neq 0$ for all $n > 0$. Let $(\pi, V) = \prod_n (\pi_n, V_n)$. As follows from Corollary 13.3 for any proper parabolic subgroup P of G we have $r_{P,G}(V) = \{0\}$. Therefore by Theorem 8.24 the representation (π, V) is compcat modulo center.

On the other hand it is clear that for any $n > 0$ we have $\pi(e_K)\pi(g_n)v \neq 0$ where $v = \prod_n v_n \in V$. This contrudiction proves the Lemma. □

Exercise 14.2. For any congruence subgroup K of G there exists a compact $C \subset G^0$ such that for any compcat representation (π, V) of G and any $v \in V^K$ we have $\pi(e_K)\pi(g)v = 0$ for all $g \notin C$.

Corollary 14.3. *C:uad For any congruence subgroup K of G there exists only a finite number of compact irreducible representations (π, V) of G^0 such that $V^K \neq \{0\}$.*

Proof. Suppose that there eixists a sequence an infinite sequence of nonequivalent compcat representations (π_n, V_n) of G^0 such that $V_n^K \neq \{0\}$. Choose $v_n \in V_n^K \neq, \tilde{v}_n \in \tilde{V}_n^K \neq$ such that $< \tilde{v}_n, v_n > \neq 0$ and consider functions $f_n := < \tilde{v}_n, \pi_n(g)v_n >$ on G^0. Since representations (π_n, V_n) are nonequivalent functions f_n are linearly independent. On the other hand there exists a compact $C \subset G^0$ such that $\pi(e_K)\pi_n(g)v_n = 0$ for all $g \notin C, n > 0$. We obtain a contradiction since all the functions f_n are right K-invariant. □

Exercise 14.4. E:uad a) For any pair K, K' of congruence subgroups of G there exists a constant $d(K, K')$ such that for any cuspidal irreducible representation (π, V) of G such that $V^K \neq \{0\}$ we have $dim(V^{K'}) \leq d(K, K')$.

b) One can omit the condition of the cuspidality in a)

14.2. **The decomposition of the cuspidal part of $\mathcal{M}(G)$.**

Theorem 14.5. *T:dc The set $Irr_{comp} \subset Irr(G^0)$ of compact irreducible representations of G^0 splits $\mathcal{M}(G^0)$ [see Definition 6.6].*

Proof. For any congruence subgroup $K \subset G^0$ we denote by $S(K)$ the set of compact irreducible representations which have a non-zero K-invariant vector. As follows from Corollary 14.3 this set is finite. Therefore by Theorem 7.5 it splits the category $\mathcal{M}(G^0)$ and for any representation V of G we have a decomposition $V = V(K) \oplus V(K^{\perp})$ where V_K is a direct sum of representation from $S(K)$ and $V_K^{\perp}$ does not have subquotients from $S(K)$.

Let $K_1 \supset K_2 \supset \dots K_r \supset \dots$ be a sequence congruence subgroups such that $\cap_r K_r = \{e\}$. For any representation V of G we define

$$V_c := \cup_r V(K_r), V_c^{\perp} := \cap_r V(K_r)$$

It is clear that all irreducible subquotients of V_c are compact and that $V_c^{\perp}$ does not have irreducible compact subquotients. So it only remains to show that $V = V_c \oplus V_c^{\perp}$. Fix $v \in V$. Then $v \in V^{K_r}$ for some $r > 0$ and we have

$$v = v_{c,K} \oplus v_{c,K}^{\perp}$$

. We will write v' for $v_{c,K}^{\perp}$. It is obvious that $v_{c,K} \in V_c$. We will be done if we show that $v' \in V_c^{\perp}$.

Let V' be the module generated by v'. We must show that JH(V') does not contain any cuspidal components D. Since v is K-invariant, any projection of v (and so, a fortiori, v') onto a representation without K-fixed vectors must be zero. Thus, JH(V') does not contain any D without K-fixed vectors. On the other hand, v' is, by definition, the part of v which has zero projection onto the cuspidal components with K-fixed vectors. Thus, JH(V') does not contain any D *with* K-fixed vectors. This proves $v' \in V_c^{\perp}$ as needed. □

Let Irr_{cusp} be the set of cuspidal irreducible representations of G.

Exercise 14.6. E:j a) A representation π of G belongs to $\mathcal{M}(Irr_{cusp}(G))$ iff the restriction of π to G^0 belongs to $\mathcal{M}_c(G^0)$.

b) The set Irr_{cusp} of cuspidal irreducible representations of G splits $\mathcal{M}(G)$.

For any $(\pi, V) \in \mathcal{M}(G), P \in \mathcal{P}$ we define $\pi_P := r_{P.G}(\pi)$. As follows from Proposition 8.21 the identity map $Id : r_{P.G}(\pi) \to r_{P.G}(\pi)$ induces a morphism $j_P : \pi \to i_{G,P}(\pi_P)$ and therefore a morphism

$$j : \pi \to \oplus_{P \in \mathcal{P}} i_{G,P}(\pi_P)$$

c) $V_c = Kerj$ and $V_c^{\perp} = Im(j)$.

Definition 14.7. For any representations (π, V) of G we denote by V_{cusp} the projection of V to $\mathcal{M}(Irr_{cusp}(G))$.

14.3. **The decomposition of $\mathcal{M}(G)$.**

Definition 14.8. a) We denote by $\mathcal{L}$ be the set of conjugacy classes of Levi subgroups of G.

b) For any $l \in \mathcal{L}$ we denote by $\mathcal{P}_l$ the set of standard parabolic subgroups $P = MU$ such that U belongs to l.

c) For any $l \in \mathcal{L}$ we denote by $Irr_l(G)$ the set of irreducible representations of G which appear as subquotients of representations $iG, P(\sigma)$ for some $P \in \mathcal{P}_l$ for some cuspidal $\sigma \in \mathcal{M}(M)$ and define $\mathcal{M}_l(G) := \mathcal{M}(Irr_l(G))$.

d) For any representations (π, V) of G and $P = MU \in \mathcal{P}$ we define $\pi_P^0 = (r_{G,P}(V))_{cusp}$, denote by $j_P^0 : V \to i_{G,P}(\pi_P^0$ induced by the canonical mprphism j_P [see Exercise 14.6] and denote by V_P^0 the image of j_P^0.
d)For any $P = MU \in \mathcal{P}$ we define a functor $r_{P,G}^0 : \mathcal{M}(G) \to \mathcal{M}(M)_{cusp}$ by $r_{P,G}^0 := r_{P,G_{cusp}}$ and denote by $j_P^0 : V \to i_{G,P}(\pi_P^0)$ the induced by the canonical mprphism.

e) For any representations (π, V) of G we define a functorial morphism $j_l^0 := \oplus j_P^0 : V \to \oplus_{P \in \mathcal{P}_l}$ and denote it's image by V_l.

Example 14.9. If $G = GL(n)$ then two the set $\mathcal{L}$ is the set of unordered partitions of n.

Theorem 14.10. *T:d a) For any representations (π, V) of G we have $V_l \in \mathcal{M}_l(G)$ for any $l \in \mathcal{L}$.*

b) The map $j^0 := \oplus_{l \in \mathcal{L}} j_l : V \to \oplus_{l \in \mathcal{L}} V_l$ is an isomorphism.

Corollary 14.11. *C:d The category $\mathcal{M}$ is the direct product of it subcategories $\mathcal{M}_l(G), l \in \mathcal{L}$.*

Proof. We start with the following useful results.

Lemma 14.12. *A representation (π, V) of G belongs to $\mathcal{M}_l(G)$ iff for any parabolic subgroup $Q = MU \notin \mathcal{P}_l$ we have $r_{Q,G}(V)_{cusp} = \{0\}$.*

Proof of the Lemma. If follows immediately from Exercise 12.3 and the exactness of the functor $r_{Q,G}$ that for any $(\pi, V) \in Ob(\mathcal{M}_l(G))$ we have $r_{Q,G}(V)_{cusp} = \{0\}$ for all $Q \notin \mathcal{P}_l$.

Conversely let (π, V) be a representation of G such that $r_{Q,G}(V)_{cusp} = \{0\}$ for all $Q \notin \mathcal{P}_l$. Consider the morphism

$$j_l := \oplus j_P : V \to \oplus_{P \in \mathcal{P}_l} i_{G,P}(\pi_P), P \in \mathcal{P}_l, \pi_P = r_{G,P}(V)$$

as in Exercise 14.6. Since $r_{Q,G}(V)_{cusp} = \{0\}$ for all $Q \subsetneq P \in \mathcal{P}_l$ we see that the representations π_P are all cuspidal. So $\oplus_{P\in\mathcal{P}_l} i_{G,P}(\pi_P) \in Ob(\mathcal{M}_l(G))$.

Since the functors $r_{Q,G}$ and $V \to V_{cusp}$ are exact we see that for any $P \in \mathcal{P}$ we have $r_{Q,G}(V')_{cusp} = \{0\}, V' := Ker(j_l)$. But then $V' = \{0\}$.□

To show a) it is sufficient to show that $i_{G,P} \circ r_{P,G_{cusp}} \in)b(\mathcal{M}_l(G)$ for any $P \in \mathcal{P}_l$.But this follows immediately from Exercise 12.3.

The injectivity of j^0 follows from Exercise 14.6.

Exercise 14.13. For any $V \in (\mathcal{M}_l(G)$ we have $V_l = V$ and $V_{l'} = \{0\}$ if $l \neq l'; l, l' \in \mathcal{L}$.

Lemma 14.14. *Let* $(\star)\{0\} \to V' \to V \to V'' \to \{0\}$ *be an exact sequence such that* $V' \in Ob(\mathcal{M}_{l'}(G), V'' \in Ob(\mathcal{M}_{l''}(G), l', l'' \in \mathcal{L}$. *If* $l' \neq l''$ *then* $(\star)$ *splits.*

Proof of the Lemma. As we have seen $V''_{l'} = \{0\}$ and $V'_{l'} = V'$. Since the functor $V \to V_{l''}$ is exact the imbedding $V' \hookrightarrow V$ induces an isomorphism $a : V' \to V_{l'}$ and we can define a spliting $a^{-1} \circ j_{l'} : V \to V'$.□

To finish the proof of b) we have to show the surjectivity of j^0. Let $W = Im(j^0)$. By the definition W is a subrepresentation in $\oplus_{l\in\mathcal{L}} V_l$ such that projections of W on any summand V_l is surjective. It follows now from the previos Lemma that $W = \oplus_{l\in\mathcal{L}} V_l$. □

The following result is quite difficult and uses heavily the Harish-Chandra's analysis of representations induces from the tempered ones.

Theorem 14.15. *For any* $l \in \mathcal{L}$ *and any* (π, V) *in* $\mathcal{M}_l(G)$ *we have* $r_{P,G}(V) \neq \{0\}$ *for any* $P \in l$.

Lemma 14.16. *L:sub Let* $K \neq K_0$ *be a congruence subgroup and* (π, V) *be a representation such that* V *is generated by* V^K. *Then* $W^K \neq \{0\}$ *for any irreducible subquotient* W *of* V.

Proof. We first consider the case when the representation (ρ, W) is cuspidal. As follows from Theorem 14.10 we can write (π, V) as a direct sum $(\pi', V') \oplus (\pi'', V'')$ where $V(\pi', V') \in \mathcal{M}_\rho$ and V'' does not have subquotients of the form $\rho \otimes \theta \circ det$ where $\theta : F^\star \to \mathbb{C}^\star$ is an unramified character. Since V is generated by V^K the representation V' is generated by $V^{K'}$. So we can assume that $V \in \mathcal{M}_\rho$.

As follows from Theorem 8.24 π_{G^0} of (π, V) on the subgroup $G^0 \subset G$ is a finite direct sum $\oplus_{i\in I} V_i$ where $(V_i, \rho_i), i \in I$ are non-isomorphic irreducible representations of G^0 and the adjoint action of G on G^0 induces a transitive action of G on I. As follows from the Theorem 7.5 the restriction π_{G^0} of (π, V) on G^0 is a multiple of π_{G^0} of (π, V). Since $V^K \neq \{0\}$ we see that $W^K \neq \{0\}$.

Assume now that the representation (ρ, W) is not cuspidal. As follows from Lemma 9.6 there exists a proper parabolic subgroup $P = MU$ of G such that $r_{P,G}(\rho)$ is a non-zero cuspidal representation. Since the functor $r_{P,G}$ is exact the representation $r_{P,G}(\rho)$ is a subquotient of $r_{P,G}(\pi)$. As follows from Proposition 8.21 the space $r_{P,G}(V)$ is generated by the subspace $r_{P,G}(V)^{K_M}$ as an M-module. Since the representation $r_{P,G}(\rho)$ is cuspidal we conclude that $r_{P,G}(W)^{K_M} \neq 0$. It follows now from Theorem 9.7 that $W^{K_M} \neq 0$. □

Exercise 14.17. Let V be the space of measures on $\mathbb{P}^1(F), V_0 \subset V$ the subspace of measures μ such that $\int_{\mathbb{P}^1(F)} \mu = 0$. The action of $GL(2, F)$ on $\mathbb{P}^1(F)$ induces representations of $GL(2, F)$ on on V and V_0. Show that the space V_{sm} is genrated by V^{K_0} but $V_0^{K_0} = \{0\}$.