Exercises and problems:

The Numbering follows Knuth's notation: (1) very easy, (2) require some thinking and work, (3) More difficult but doable, represent a result published in the literature, (4-5) open.

0. (4) Show that for a d-polytope P, $f_k(P) \geq \min(f_0(P), f_{d-1}(P))$.

1. (2) Prove that a face of a face of a polytope is a face of the polytope.

2. (2) Prove the Local Maximum = Global maximum principle. Namely, let P be a d-polytope, ϕ be a linear functional. (***) If for a vertex v of P $\phi(v) > \phi(u)$ for every neighboring vertex u of v then $\phi(v) > \phi(x)$ for every $x \in P$.

3. (1+) Find the (dual) h-vector of the d-dimensional cube.

4. (3) Let G be the graph of a simple d-polytopes P. A real function ϕ defined on the vertices of G is an *abstract objective function* if (***) is satisfied for P and for every face of P.. Find a way to identify if a real function on G is an abstract objective function just from knowing the graph G.

5. (3) (continuation of 4.) Find a way to determine if a subgraph H of G is the graph of a face of G.

6. (4) Let G be the dual graph for a triangulation K of a $(d-1)$-dimensional sphere. Find a way to determine if a sungraph H of G correspond to the link of a k-face of K.

7. (2+) Let $x(t) = (t, t^2, t^3, \ldots, t^d)$ be the moment curve in $\mathbf{R}^d$. Let $C = conv(x(t_1), x(t_2), \ldots, x(t_d))$, where $t_1 < t_2 < \ldots < t_n$. Here $n > d$. Determine the faces of $C(d, n)$.

8. (2+) Let P be a polytope and let $h(P) = (h_0, h_1, \ldots, h_d)$ be its IH h-vector. Let $g(P) = (h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{[d/2]} - h_{[d/2]-1})$. Show that $g(P * Q) = g(P) * g(Q)$ and $h(P \oplus Q) = h(P)h(Q)$.