Exercise 0.1 *For open $V \subset U \subset \mathbb{R}^n$ we define*

$$\mathcal{F}_n(U) = \begin{cases} \mathbb{C} & U \neq \emptyset \\ \{0\} & U = \emptyset \end{cases},$$

and

$$res_V^U = \begin{cases} id & V \neq \emptyset \\ \{0\} & V = \emptyset \end{cases}.$$

Find all n for which $\mathcal{F}_n$ is a sheaf on $\mathbb{R}^n$.

Exercise 0.2 *Let X be a topological space, let $U \subset X$ be an open subset, and let $\mathcal{F}$ be a sheaf on X. For any open $V \subset U$ we define $\mathcal{F}_U(V) = \mathcal{F}(V)$. We have natural restriction maps $res_W^V : \mathcal{F}_U(V) \to \mathcal{F}_U(W)$ for $W \subset V \subset U$ open. Show that $\mathcal{F}_U$ is a sheaf on the topological space U.*

Exercise 0.3 *Let X be a topological space, and let $X = \cup U_\alpha$ be an open covering. We denote $U_{\alpha_1 \dots \alpha_k} = \cap_{i=1}^k U_{\alpha_i}$. Assume that we are given the following data*

- *A sheaf $\mathcal{F}^\alpha$ on U_α for each α,*
- *An isomorphism $i_{\alpha\beta} : \mathcal{F}^\alpha_{U_{\alpha\beta}} \to \mathcal{F}^\beta_{U_{\alpha\beta}}$ for any α, β,*

satisfying

$$i_{\alpha\gamma} = i_{\beta\gamma} \circ i_{\alpha\beta} : \mathcal{F}^\alpha_{U_{\alpha\beta\gamma}} \to \mathcal{F}^\gamma_{U_{\alpha\beta\gamma}},$$

for any α, β, γ. Show that there exists a sheaf $\mathcal{F}$ on X and an isomorphism $i_\alpha : \mathcal{F}_{U_\alpha} \to \mathcal{F}^\alpha$ for any α, such that

$$i_{\alpha\beta} = i_\beta \circ i_\alpha^{-1}.$$

Show that such $\mathcal{F}$ is unique up to unique isomorphism.

Let Σ be a fan, and let $\mathcal{F}$ be an equivariant invertible sheaf on X_Σ. We remind that for any $\sigma^* \in \Sigma$ the sheaf $\mathcal{F}_{X_\sigma}$ is invertible and equivariant on X_σ, which is equivalent to the fact that $L_\sigma = \mathcal{F}_{X_\sigma}(X_\sigma)$ is an equivariant invertible module over $\mathcal{O}_{X_\sigma}(X_\sigma) = \mathbb{C}[\sigma \cap M]$. In the lecture we saw that such modules are in one-to-one correspondence with the elements of $M/M_{\check{\sigma}}$, where $M_{\check{\sigma}}$ is M intersection with the maximal subspace of σ. In the lecture I gave several exercises proving this correspondence. Solve these exercises.

As we can see an equivariant invertible sheaf $\mathcal{F}$ provides us with a set $m(\mathcal{F}) = (m_\sigma)_{\sigma^* \in \Sigma}$.

Exercise 0.4 *Let τ^* be a face of σ^*. Show that $M_{\check{\sigma}} \subset M_{\check{\tau}}$ and*

$$m_\sigma \equiv m_\tau \mod (M_{\check{\tau}}) \tag{1}$$

Set of all sequences $(m_\sigma \in M/M_{\check{\sigma}})_{\sigma^* \in \Sigma}$ satisfying (**??**) is denoted by $\varprojlim M/M_{\check{\sigma}}$.

Exercise 0.5 *Let* $\underline{m} \in \varprojlim M/M_{\check{\sigma}}$ *be any element. Show that there exists an equivariant invertible sheaf* $\mathcal{F}$ *on* X_Σ *such that* $m(\mathcal{F}) = \underline{m}$. *Show that such sheaf is unique up to unique isomorphism (We remind that equivariant invertible sheaves are considered up to multiplication by elements of* $\mathbb{C}^*$ *).*

Remark 0.6 *An element* $\underline{m} \in \varprojlim M/M_{\check{\sigma}}$ *can be considered as a cone-wise linear function on* Σ *having integral values at integral points (=points of* M*).*

Let $\Delta \subset M_{\mathbb{R}}$ be an integral polytop (i.e. a polytop with vertices in M). In the lecture we constructed the fan Σ_Δ and the toric variety X_Δ. We should mention that the big cones (=cones of biggest dimension) in Σ correspond to the vertices of Δ. Consider the linear function on big cone given by the corresponding vertex.

Exercise 0.7 *Show that all these functions are glued together into a cone-wise linear function on* Σ. *Show that this function has integral values at integral points.*

As a conclusion we see that on X_Δ there is a natural invertible equivariant bundle.