Calibrated Forecasts, Regret Matching, Dynamics and Equilibria

Sergiu Hart

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Papers

Papers

- **Dean P. Foster and Rakesh V. Vohra**
  "Asymptotic Calibration"
  - first version: 1991
  - *Biometrika* 1998

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- **Dean P. Foster and Rakesh V. Vohra**
  "Asymptotic Calibration"
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- **Sergiu Hart and Andreu Mas-Colell**
  "A Simple Adaptive Procedure Leading to Correlated Equilibrium"
  - first version: 1996
  - *Econometrica* 2000

  www.ma.huji.ac.il/hart/publ.html#adapt

- **Sergiu Hart and Andreu Mas-Colell**
  Simple Adaptive Strategies: From Regret-Matching to Uncoupled Dynamics
  - World Scientific Publishing 2013

  www.ma.huji.ac.il/hart/publ.html#sas

- **Sergiu Hart and Andreu Mas-Colell**
  Simple Adaptive Strategies: From Regret-Matching to Uncoupled Dynamics
  - World Scientific Publishing 2013

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- **Wojciech Olszewski**
  "Calibration and Expert Testing"
  - in *Handbook of Game Theory IV* 2015

- **Dean Foster and Sergiu Hart**
  "Smooth Calibration, Leaky Forecasts, Finite Recall, and Nash Dynamics"
  - *Games and Economic Behavior* 2018

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- **Dean Foster and Sergiu Hart**
  "Smooth Calibration, Leaky Forecasts, Finite Recall, and Nash Dynamics"
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- **Dean Foster and Sergiu Hart**
  "Forecast-Hedging and Calibration"
  - 2019

  www.ma.huji.ac.il/hart/publ.html#calib-int

The Test of Time

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What is the Test of Time?

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- 25 years of work

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- **25 years of work** ?

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- **25 years of work** ?
- **25 minutes of presentation**
  covering 25 years of work !

The True Test of Time

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I. Calibrated Forecasts

Calibrated Forecasts

Calibrated Forecasts

- Forecaster says: "The chance of rain tomorrow is p"

Calibrated Forecasts

- Forecaster says: "The chance of rain tomorrow is p"
- Forecaster is **CALIBRATED** if for every p: the proportion of rainy days among those days when the forecast was p equals p (or is close to p in the long run)

Calibrated Forecasts

Calibrated Forecasts

CALIBRATION ***can be guaranteed***

(no matter what the weather will be)

Calibrated Forecasts

CALIBRATION *can be guaranteed*

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- Foster and Vohra 1991 [1998]

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NON-Bayesian, NO statistical assumptions !

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- Forecaster uses ***mixed*** forecasting
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  with probability $\mathbf{1/2}$, forecast $= \mathbf{60\%}$)

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  with probability $1/2$, forecast $= 60\%$)

- Foster and Vohra 1991 [1998]
- Hart 1995: proof using Minimax Theorem

Calibration Proof: Minimax

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- **FINITE** δ**-GRID**, **FINITE HORIZON**
  $\Rightarrow$ **FINITE** 2-person 0-sum game

Calibration Proof: Minimax

- FINITE δ-GRID, FINITE HORIZON
  $\Rightarrow$ FINITE 2-person 0-sum game

- IF the strategy of the rainmaker IS KNOWN
  THEN the forecaster can get δ-calibrated forecasts

Calibration Proof: Minimax

- FINITE δ-GRID, FINITE HORIZON
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- MINIMAX THEOREM $\Rightarrow$ the forecaster can GUARANTEE δ-calibrated forecasts (without knowing the rainmaker's strategy)

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II. Regret Matching

Regret Matching

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Setup: n-person game played repeatedly

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REGRET MATCHING =

Switch next period to a different action with a probability that is proportional to the regret for that action

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Setup: n-person game played repeatedly

<u>REGRET MATCHING</u> =
Switch next period to a different action with a probability that is proportional to the regret for that action

<u>REGRET</u> = **increase in payoff had such a change always been made in the past**

Regret

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- $V(k)$ = average payoff if action k had been played instead of **the current action** j every time in the past that j was played
- $R(k) = [V(k) - U]_+$ = **regret** for action k

Regret Matching

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Next period play:

- **Switch** to action k with a probability that is proportional to the regret $R(k)$ (for $k \neq j$)

Regret Matching

Next period play:

- **Switch** to action k with a probability that is proportional to the regret $R(k)$ (for $k \neq j$)
- Play **the same** action j of last period with the remaining probability

Regret Matching Result

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If all players play Regret Matching then the joint distribution of play converges to the set of CORRELATED EQUILIBRIA of the game

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Hart and Mas-Colell 1996 [2000]

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- **Regret Matching** $\Rightarrow$ **all regrets** $\to 0$
  - **Blackwell Approachability**
    for **payoff vector** $=$ **regrets**
    $\Rightarrow$ play: eigenvector of **regret** matrix

Proof of Regret Matching Result

- **Correlated Equilibrium** $\Leftrightarrow$ **all regrets** $= 0$
- **Regret Matching** $\Longrightarrow$ **all regrets** $\longrightarrow 0$
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    for **payoff vector** $=$ **regrets**
    $\Longrightarrow$ play: eigenvector of **regret** matrix
  - $\Longrightarrow$ play: **regrets** (transition probabilities)

Proof of Regret Matching Result

- **Correlated Equilibrium** $\Leftrightarrow$ **all regrets** $= 0$
- **Regret Matching** $\Rightarrow$ **all regrets** $\rightarrow 0$
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    for **payoff vector** $=$ **regrets**
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    $\equiv$ **Regret Matching**

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- Correlated Equilibrium $\Leftrightarrow$ all regrets $= 0$
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simple procedure* . . . *complex proof

III. Dynamics and Equilibria

Dynamics and Equilibria

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There are no general, natural dynamics leading to Nash equilibrium

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- ***"general"***

Dynamics and Equilibria

There are no general, natural dynamics leading to Nash equilibrium

- ***"general"*** : **in all games**

Dynamics and Equilibria

There are no general, natural dynamics leading to Nash equilibrium

- *"general"* : **in all games**
  rather than: in specific classes of games

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- ***"leading to Nash equilibrium"***

Dynamics and Equilibria

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- ***"leading to Nash equilibrium"*** **:**
  at a Nash equilibrium (or close to it) from some time on

Dynamics and Equilibria

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- ***"natural"***

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- ***"natural"***:

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There are no general, natural dynamics leading to Nash equilibrium

- ***"natural"***:
  - **adaptive** (reacting, improving, ...)

Dynamics and Equilibria

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- ***"natural"***:
  - **adaptive** (reacting, improving, ...)
  - **simple and efficient**

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- ***"natural"***:
  - **adaptive** (reacting, improving, ...)
  - **simple and efficient**:
    - **computation** (performed at each step)

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    - **time** (how long to reach equilibrium)

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Uncoupled Dynamics

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Hart and Mas-Colell 2003

Uncoupled Dynamics

UNCOUPLED DYNAMICS :

Each player knows *only* his own payoff (utility) function

(does *not* know the payoff functions of the other players)

(privacy-preserving, decentralized, distributed ...)

Hart and Mas-Colell 2003

Dynamics and Equilibria

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    - information: *uncoupledness* ✓

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    - **time** to reach equilibrium ?
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How Long to Equilibrium?

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- An uncoupled dynamic

  $\approx$

  A distributed computational procedure
  (Conitzer and Sandholm 2004)

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- $\Rightarrow$ **COMMUNICATION COMPLEXITY**

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- An uncoupled dynamic

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- $\Rightarrow$ **COMMUNICATION COMPLEXITY**

Hart and Mansour 2010
Babichenko and Rubinstein 2017

IV. Calibration and Dynamics

No Calibration

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- CALIBRATION ***cannot*** be guaranteed when:

No Calibration

- CALIBRATION ***cannot*** be guaranteed when:
  - Forecast is known before the rain/no-rain decision is made ("LEAKY FORECASTS")

No Calibration

- **CALIBRATION** ***cannot*** be guaranteed when:
  - Forecast is known before the rain/no-rain decision is made ("**LEAKY FORECASTS**")
  - Forecaster uses a ***deterministic*** forecasting procedure

Continuous Calibration

Continuous Calibration

- **CONTINUOUS CALIBRATION**: combine together the days when the forecast was ***close to*** p

Continuous Calibration

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(*smooth out* the calibration score)

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  (*smooth out* the calibration score)

> There exists a ***deterministic*** procedure that is **CONTINUOUSLY CALIBRATED**.

Continuous Calibration

- **Continuous Calibration**: combine together the days when the forecast was ***close to*** p (*smooth out* the calibration score)

There exists a ***deterministic*** procedure that is **continuously calibrated**.

Foster and Kakade 2004, 2006
Foster and Hart 2018, 2019

Calibration and Game Dynamics

General n-person game

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- Players ***forecast*** the play in the next period

General n-person game

- Players ***forecast*** the play in the next period
- Players choose their actions in ***response*** to the forecasts

Calibration and Game Dynamics

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  - ***calibrated forecasts***
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Calibration and Game Dynamics

General n-person game

- Players ***forecast*** the play in the next period
  - ***calibrated forecasts***
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  - ***best response***

$\Rightarrow$ **Long-run play ?**

Calibrated Learning

Calibrated Learning

- Each player makes a ***calibrated forecast*** on the next period play

Calibrated Learning

- Each player makes a ***calibrated forecast*** on the next period play
- Each player ***best replies*** to the forecast

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$\Rightarrow$ **JOINT DISTRIBUTION OF PLAY**
($\equiv$ time-averge of play)
is a **CORRELATED ε-EQUILIBRIUM**
in the long run

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Foster and Vohra 1997

Continuous Calibrated Learning

Continuous Calibrated Learning

- All players make a ***deterministic continuously calibrated forecast*** on the next period play

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Foster and Kakade 2004, 2006
Foster and Hart 2018, 2019

Calibration, Dynamics, Equilibria

MINIMAX universe

Calibration, Dynamics, Equilibria

MINIMAX universe

FIXEDPOINT universe

Calibration, Dynamics, Equilibria

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- ***stochastic***
  forecast-hedging

Calibration, Dynamics, Equilibria

MINIMAX universe

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- ***deterministic***
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MINIMAX universe

- ***stochastic*** forecast-hedging
- ***P***-procedures

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MINIMAX universe	FIXEDPOINT universe
stochastic forecast-hedging	*deterministic* forecast-hedging
P-procedures	*PPAD*-procedures

Calibration, Dynamics, Equilibria

MINIMAX universe

- ***stochastic*** forecast-hedging
- ***P***-procedures
- ***classic*** calibration

FIXEDPOINT universe

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Calibration, Dynamics, Equilibria

MINIMAX universe	FIXEDPOINT universe
• ***stochastic*** forecast-hedging	• ***deterministic*** forecast-hedging
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Calibration, Dynamics, Equilibria

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time-average	

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correlated equilibria	***Nash*** equilibria
time-average	***period-by-period***

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MINIMAX universe	**FIXEDPOINT** universe
stochastic forecast-hedging	***deterministic*** forecast-hedging
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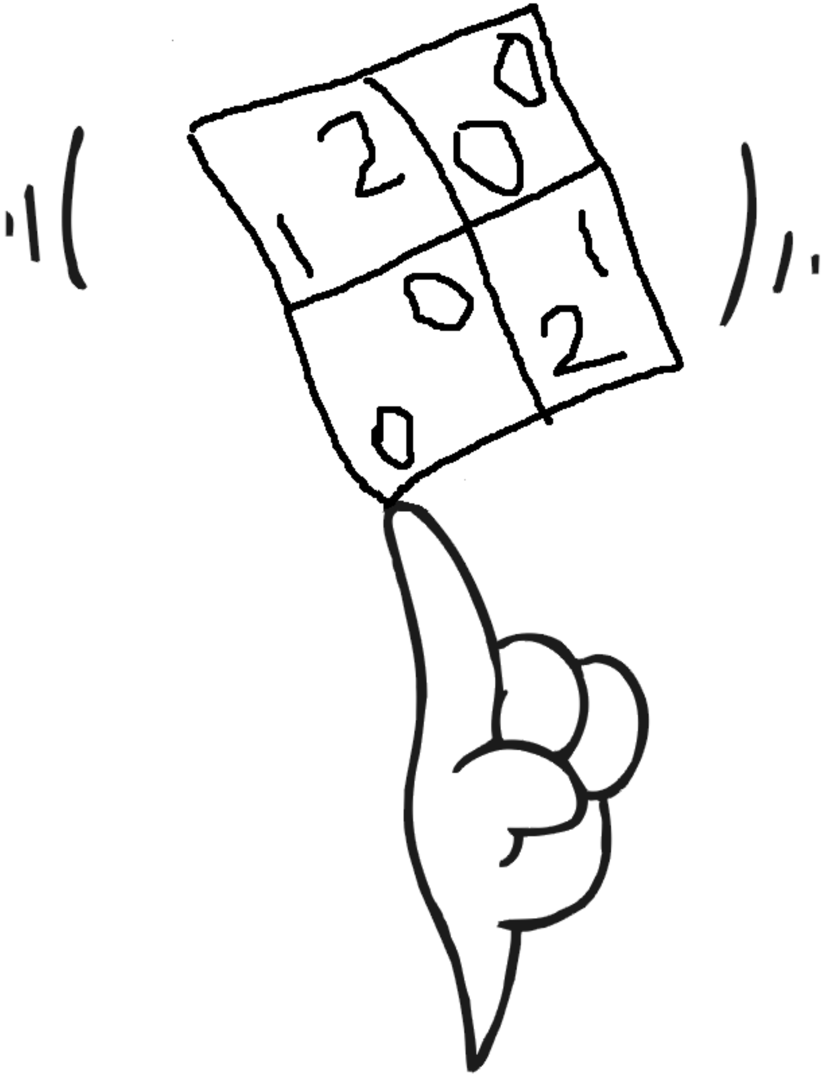
classic calibration	***continuous*** calibration
correlated equilibria	***Nash*** equilibria
time-average	***period-by-period***
from some time on	

Calibration, Dynamics, Equilibria

MINIMAX universe	**FIXEDPOINT** universe
stochastic forecast-hedging	***deterministic*** forecast-hedging
P-procedures	***PPAD***-procedures
classic calibration	***continuous*** calibration
correlated equilibria	***Nash*** equilibria
time-average	***period-by-period***
from some time on	***most of the time***

Game Dynamics and Equilibrium

Game Dynamics and Equilibrium



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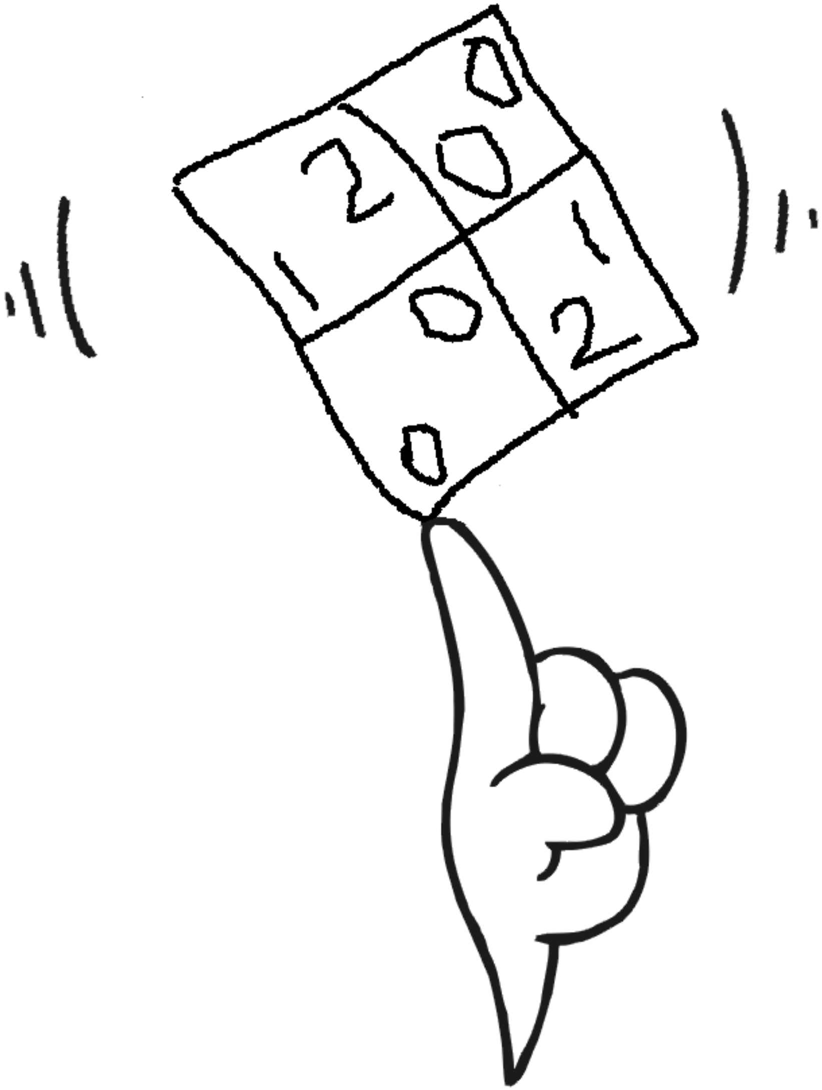

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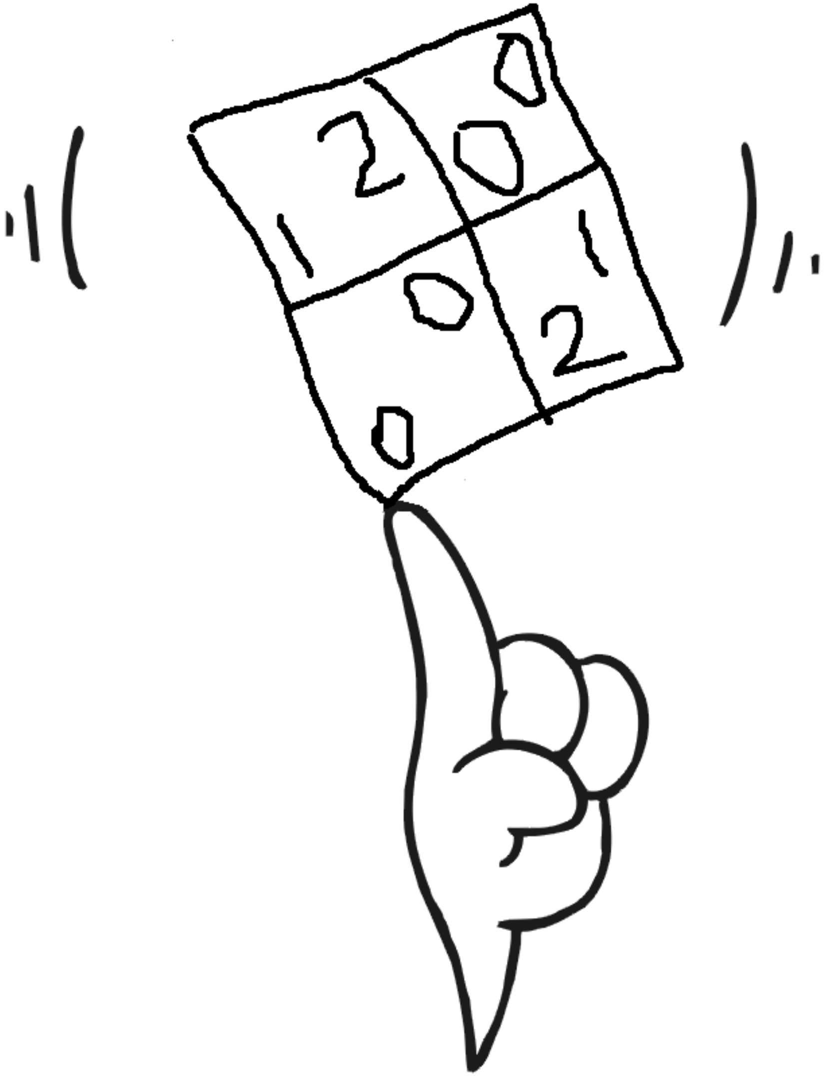

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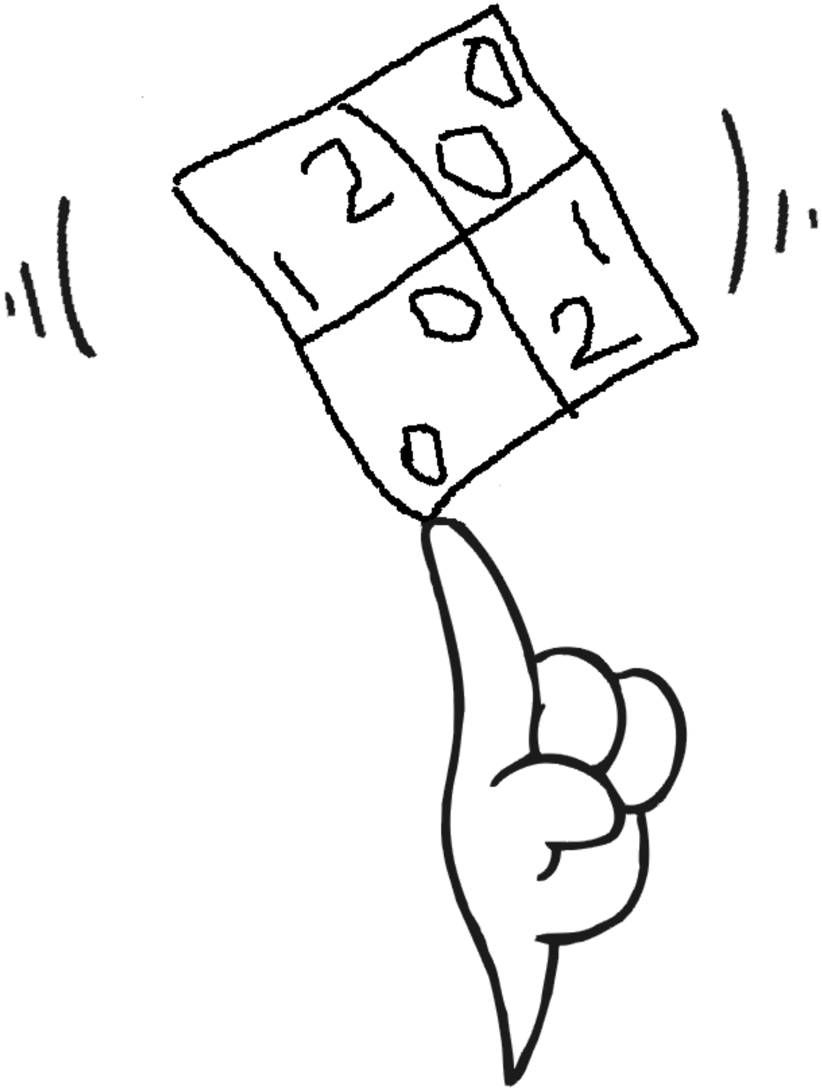

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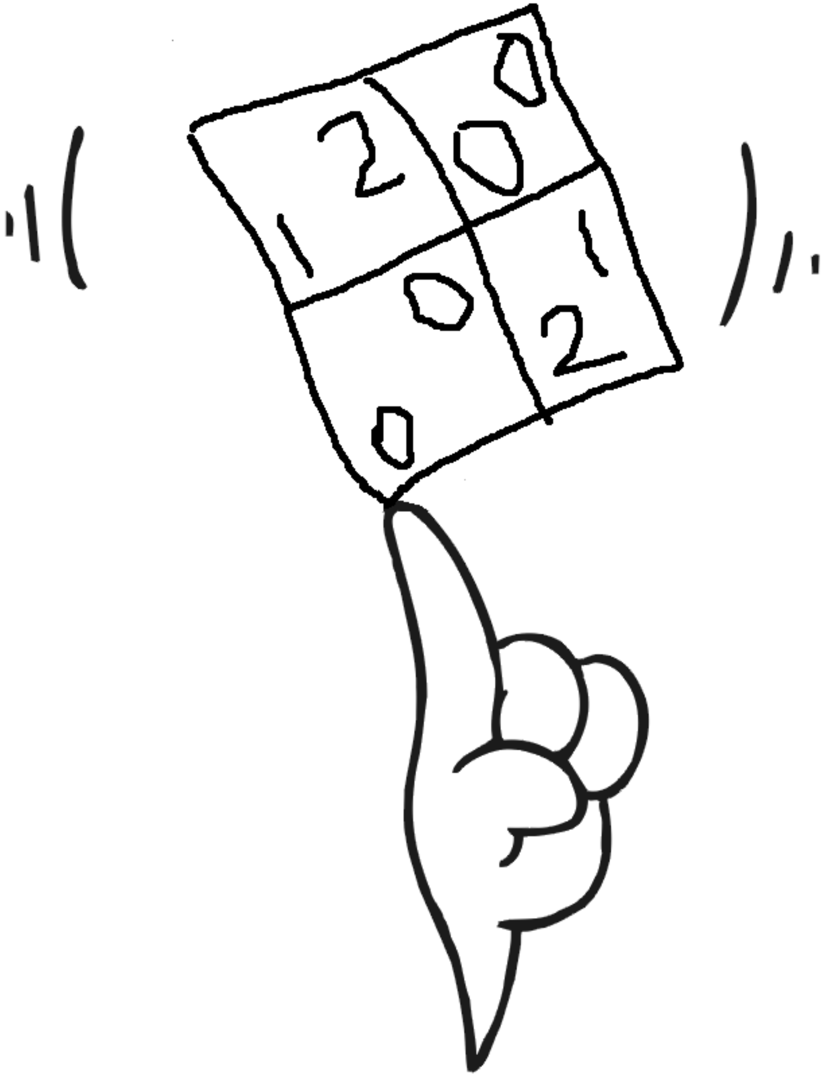

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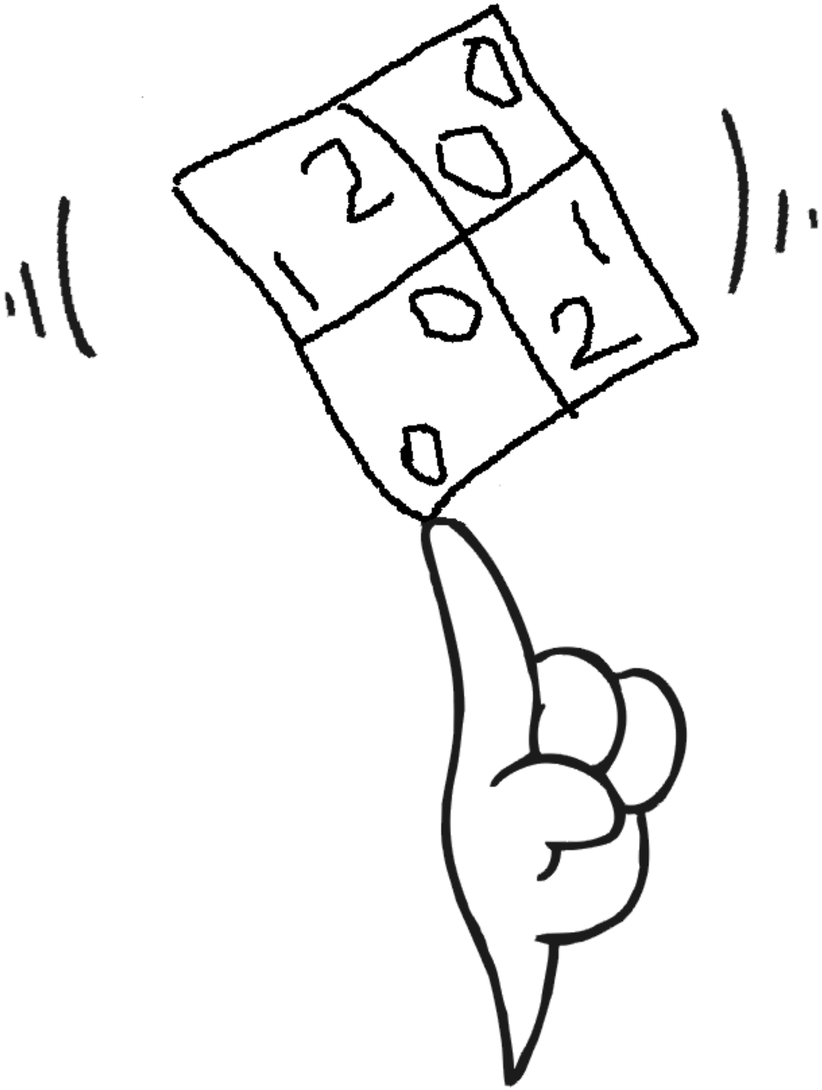

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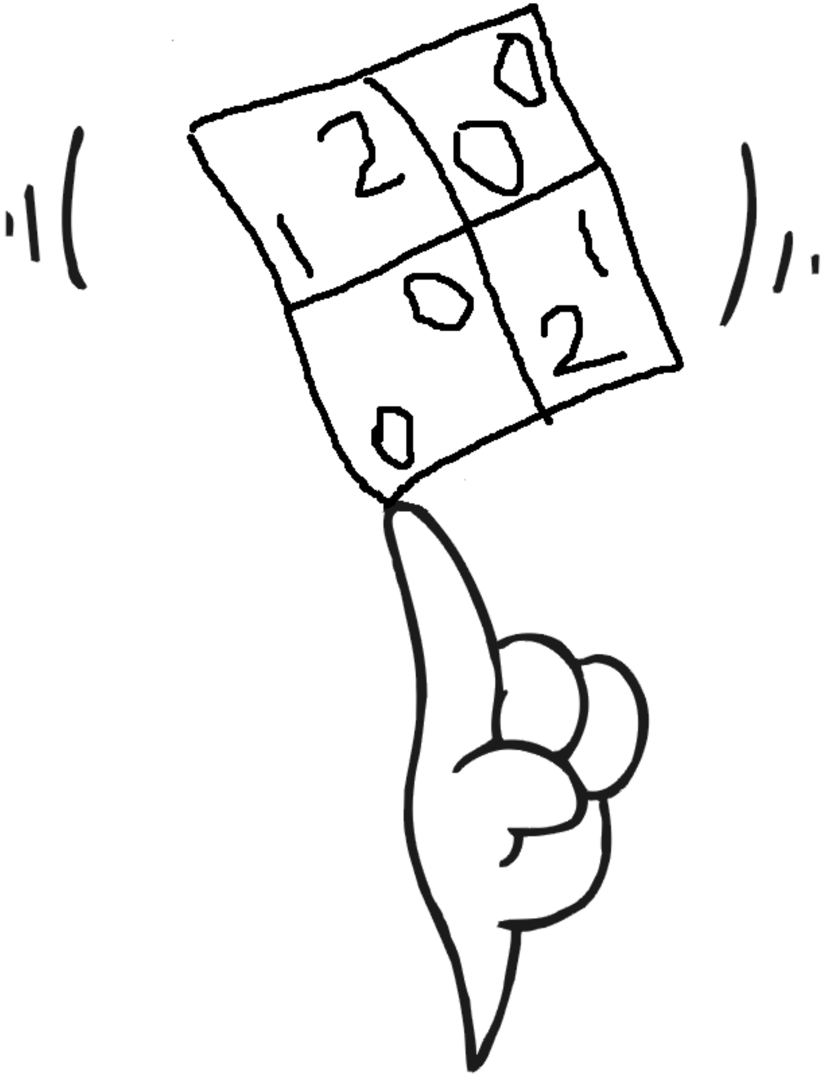

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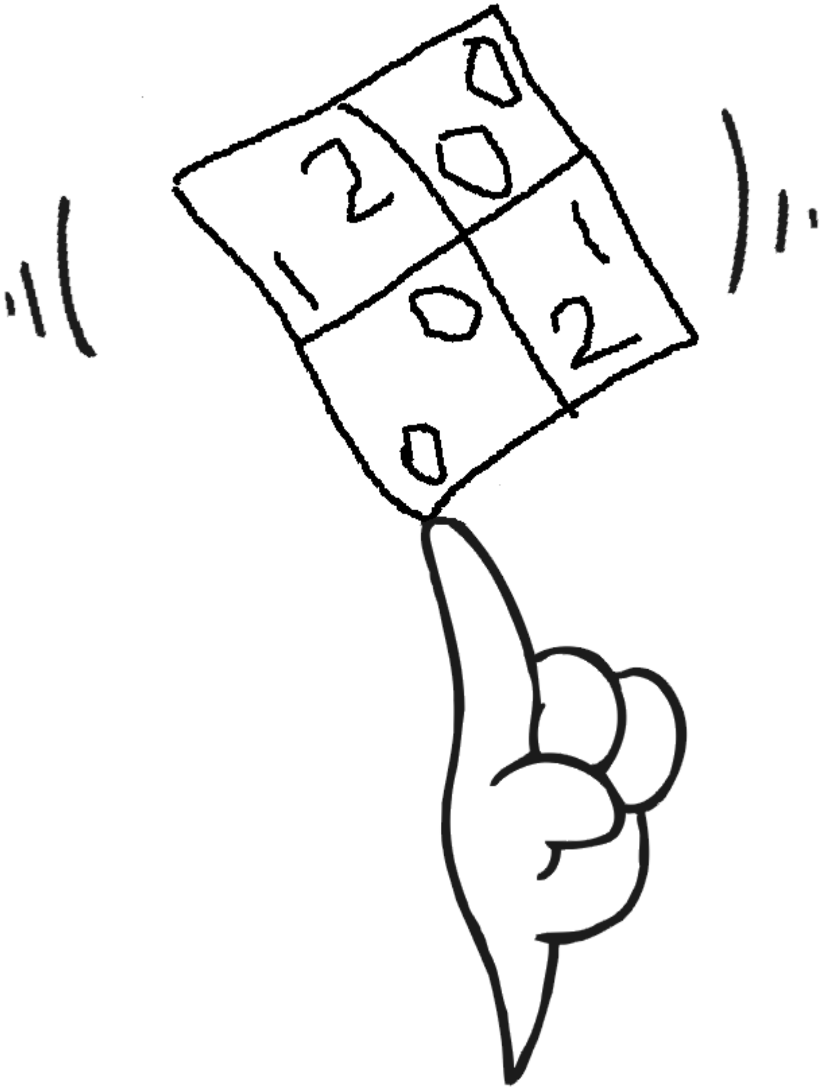

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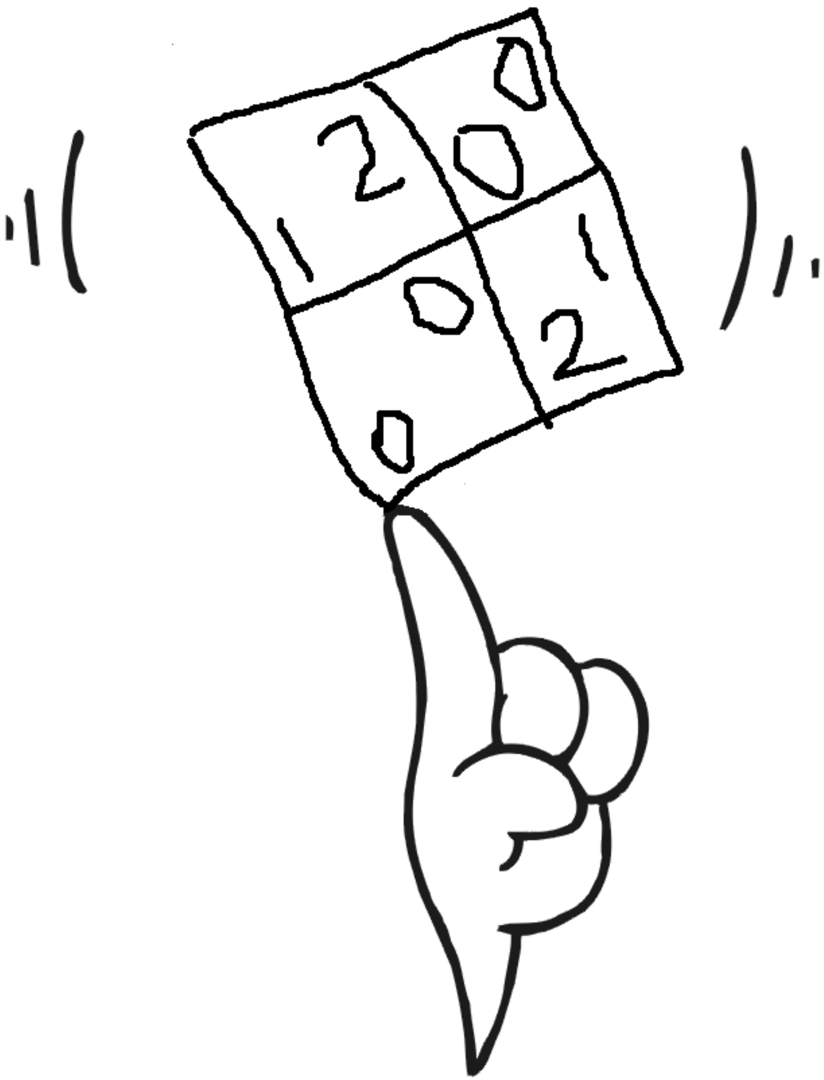

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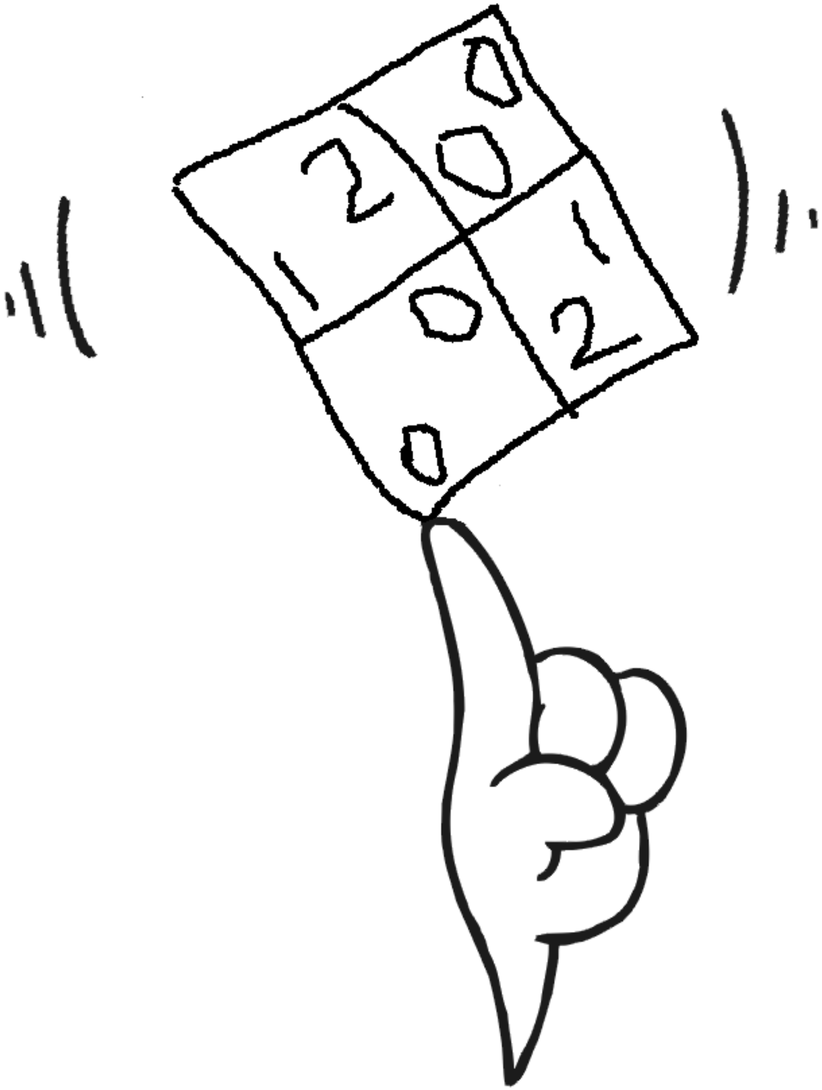

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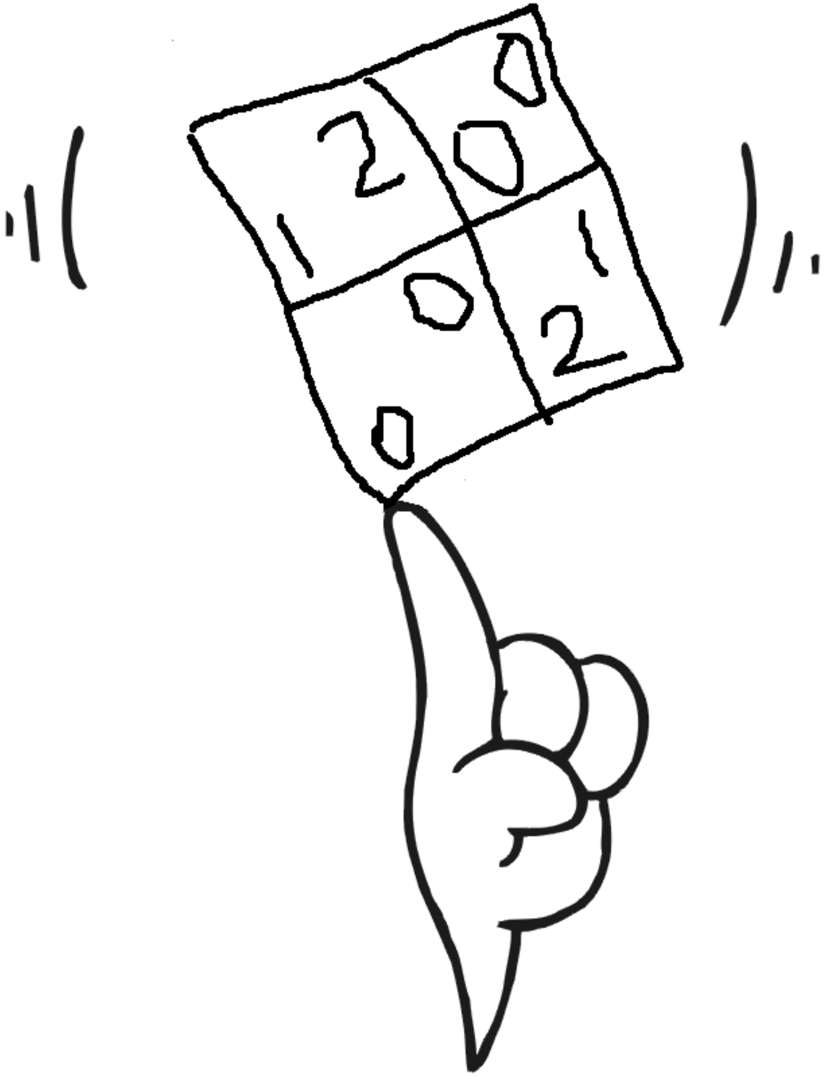

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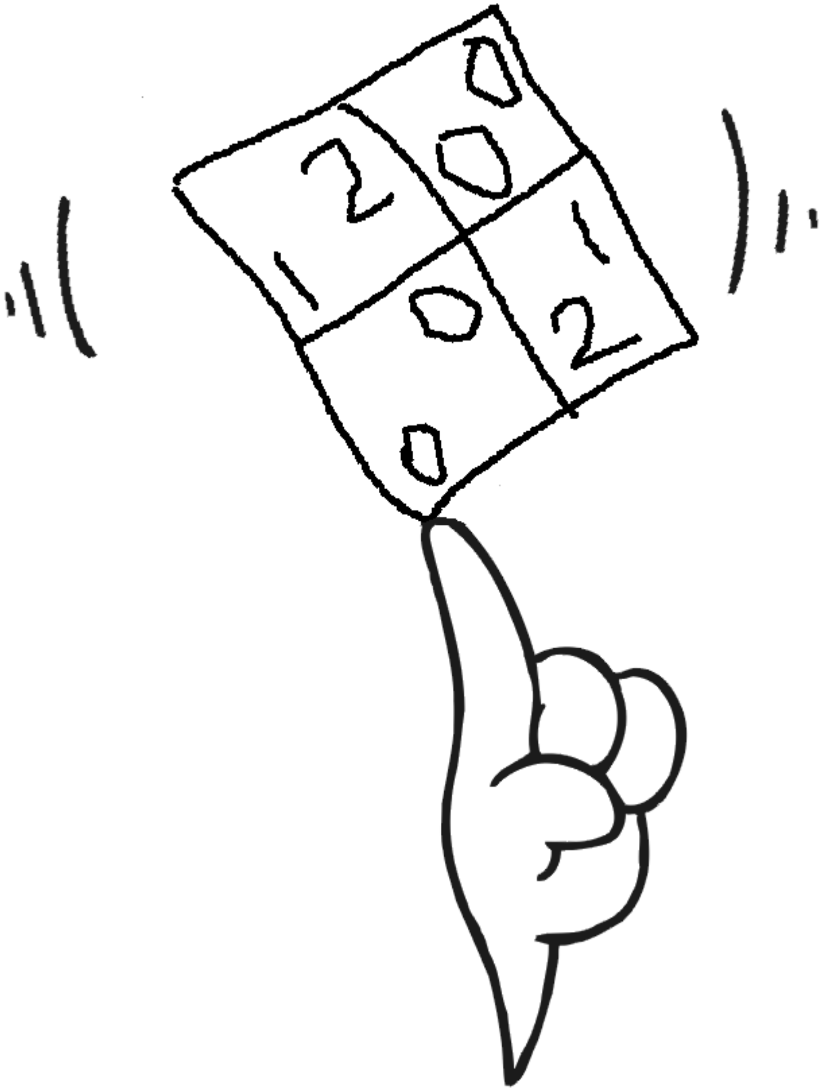
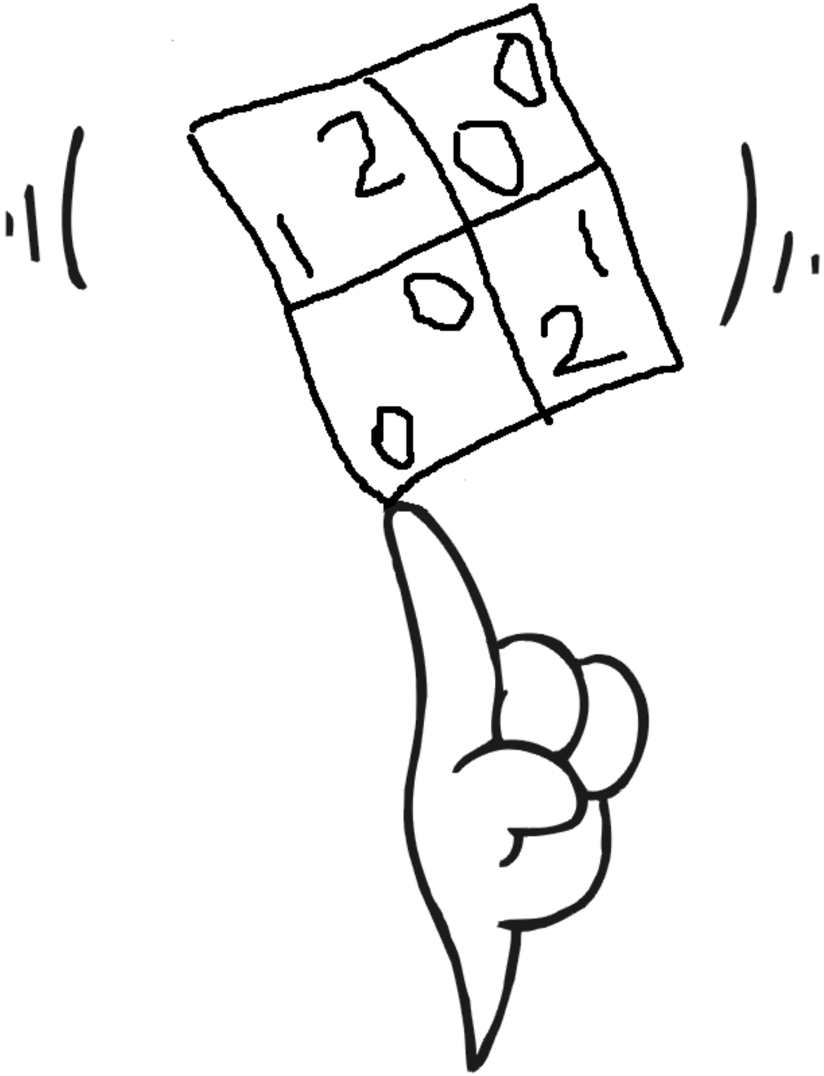


Game Dynamics and Equilibrium



Game Dynamics and Equilibrium