1. Introduction

A collection of finite subsets X_i of groups G_i is a family of *approximate subgroups* if $1 \in X_i = X_i^{-1}$, and the products set X_iX_i is contained in a bounded number of translates of X_i. Freiman, around 1960, classified the approximate subgroups of $\mathbb{Z}$. After considerable activity in the last few years, Breuillard, Green and Tao (BGT, [3]) showed that all approximate subgroups arise from nilpotent groups, and gave a similar classification of approximate subgroups of nilpotent groups. These notes were written for a class given in Jerusalem in Spring 2012, covering their main results. We work entirely in a model-theoretic setting, deducing the finite consequences at the very end.

The proof has three parts. The first is a basic connection between approximate subgroups and locally compact groups. It appeared in [8] and will not be repeated here in detail, though we will describe it and give other references. In class we described a more general connection between approximate equivalence relations and locally compact spaces, that may be incorporated here later.

The second part centers around the Gleason-Yamabe structure theory of locally compact groups. These results were merely cited in [8]; in order to go further, [3] had to enter into their methods and incorporate them in the pseudo-finite setting. A locally compact group is said to be NSS (no small subgroups) if some neighborhood U of the identity contains no nontrivial subgroups. Gleason showed that NSS groups are Lie groups. We will avoid using Lie theory, but the *proof* of Gleason's theorem, and especially the construction of tangent spaces, will be essential. Gleason's proof includes a notion of distance from the identity, with the essential property that the commutator of two elements at very small distance from 1 becomes yet much closer to 1. Equally important will be a related theorem of Yamabe, that connected locally compact groups admit many homomorphisms to NSS groups. We will look at this in the presentation of [13], who in turn largely follow [7]. This route goes through in the pseudo-finite case with almost no change.

§3 contains the proof of nilpotence, and the generalized Freiman theorem. The ultraproduct X of the X_i retains a memory of the finiteness of the X_i, hence must have a non-identity element at minimal distance to 1 for the Gleason distance. It follows that this element commutes with any element in a neighborhood of 1, i.e. is essentially central. This, along with an induction on dimension of the associated NSS group, leads to the proof of nilpotence.

We place ourselves in a setting appropriate both for ultrapowers of locally compact groups (the original setting of Hirschfeld), and for ultraproducts of finite approximate groups (yielding the BGT results.)

This text was written for a seminar in Jerusalem on [3]; thanks to the audience, and also to Immanuel Halupczok and Itay Kaplan for various corrections.

1.1. **Groups with power maps.**

We work with a saturated structure, including a sort $\mathbb{R}^*$ and $\mathbb{N}^* \subset \mathbb{Z}^* \subset \mathbb{R}^*$ that can be taken to be ultrapower of $(\mathbb{N}, \mathbb{Z}, \mathbb{R})$. Thus $\mathbb{N}^*$ is *definably well-ordered*, i.e. every nonempty definable set has a least element. *st* denotes the *standard part map*; it is defined on the convex hull $\widetilde{\mathbb{R}}$ of $\mathbb{R}$ in $\mathbb{R}^*$, and returns the nearest standard real.

For $f \in \mathbb{Z}^*$, let $O(f) = \{n \in \mathbb{Z}^* : (\exists e \in \mathbb{N})(|n| \leq e|f|)\}$, and $o(f) = \{n \in \mathbb{Z}^* : (\forall e \in \mathbb{N})(|n| \leq |f|/e)\}$. Thus for $a \in G$, $a^{\mathbb{Z}^*}$ is a definable subgroup of G, $a^{O(f)}$ is an Ind-definable subgroup, while $a^{\leq f} := \{a^n : |n| \leq f\}$ is a definable subset, generating $a^{O(f)}$. We write $x \approx y$ if $O(x) = O(y)$, i.e. $x/m \leq y \leq mx$ for some $m \in \mathbb{N}$.

We deal here with groups G *with a power map*; i.e. the map $(x, n) \mapsto x^n$, $G \times \mathbb{Z}^* \to G$, is assumed to be given as a binary function of x, n.

For $a \in G$, and U a definable subset of G, $s_U(a)$ is defined to be the smallest $n \in \mathbb{N}^*$ (if any) such that $a^{n+1} \notin U$. Such an n exists, by the definable well-ordering property of $\mathbb{Z}^*$, unless $a^n \in U$ for all $n \in \mathbb{N}^*$.

In fact we will make some use of powers of definable sets. For $Q \subset G$ definable, and $n \in \mathbb{Z}^*$, we have $Q^{\cdot n}$, definable uniformly in n; Again we let $s_U(Q)$ be the least $n \in \mathbb{N}^*$ (if any) such that $Q^{n+1} \not\subseteq U$

In addition, our data includes a definably generated group $\widetilde{G}$, and a normal, $\bigwedge$-definable subgroup Γ of $\widetilde{G}$, with $\widetilde{G}/\Gamma$ bounded in the model theoretic sense: passing to elementary extensions does not increase it. This gives rise to a locally compact structure on $\widetilde{G}/\Gamma$, so that we have a continuous homomrphism $\pi : \widetilde{G} \to \widetilde{G}/\Gamma$ in the sense of continuous logic (see below).

We now explain below how to obtain $\widetilde{G}, \Gamma$ in the two settings. The quotient map $\widetilde{G} \to \widetilde{G}/\Gamma$ will be denoted by π.

1.2. **Ultrapowers of approximate groups.** Let $(G, X, \mathbb{Z}^*)$ be an ultrapower of $(G_i, X_i, \mathbb{Z})$ with X_i a finite subset of G_i, and with ordinary $\mathbb{Z}$-powers. Any definable function into $\mathbb{R}^*$ may be internally summed.

Thus we assume that G is equipped with a summation operator on definable functions into $\mathbb{R}^*$, taking $\mathbb{R}^*$-valued function with support $\subset X^{\cdot m}$ (for some m) to their "sum" in $\mathbb{R}^*$. To enable us to treat the locally compact and pseudo-finite cases together, we denote the summation operator as an integral.

We will use $\int$ directly in the Gleason-Yamabe lemmas. We can also coarsen it to obtain a real-valued (rather than $\mathbb{R}^*$)-valued measure. Let $\widetilde{G}$ be the subgroup of G generated by X, and normalize $\int$ so that $\int(1_X)$ is finite. For a definable $Y \subset \widetilde{G}$, let $\mu(Y) = st(\int(1_Y))$. When (X_i) is a family of approximate subgroups. turns X into a *near-subgroup*, meaning that $0 < \mu(X), \mu(X^{\cdot 3}) < \infty$.

We need this special case of Theorem 3.1 of [8]. It follows a sequence of similar results in model theory, called "stabilizer theorems". We say that a subset X of a group is symmetric if $X = X^{-1}$, and that two symmetric subsets X, Y are *commensurable* if each is contained in the union of finitely many cosets of the other.

Theorem 1.3. *Let (G, μ, X) be a near-subgroup. Then there exist a normal, $\bigwedge$-definable subgroup Γ of G, $\Gamma \subset X^{\cdot 4}$. Any definable D with $\Gamma \subset D \subset \widetilde{G}$ is commensurable with $X^{\cdot 2}$.* [1]

Note, by compactness:

Corollary 1.4. *Let (G, μ, X) be a near-subgroup. Then there exist a μ-wide definable set Y with $Y^8 \subset X^4$.*

Proof. Say $\Gamma = \cap S_n$, with S_n definable; then $\cap S_n^8 \subset S \subset X^4$, so for some n, $S_n^8 \subset X^4$. $\square$

The corollary is in fact easily seen to be equivalent to Theorem 1.3: inductively define Y_n with $Y_{n+1}^8 \subset Y_n^4$, and let $\Gamma = \cap_n Y_n^4$.

For finite approximate subgroups, it was given an independent, direct proof by Sanders [10], following a line in combinatorics starting from Balog-Szemeredi. See [3] , Theorem 5.3 for a self-contained proof in about one page, for ultraproducts of (general) finite approximate subgroups. We will use Theorem 1.3 directly, referring the reader to either [8] or [3] for the proof. (The combinatorial proof appears on the fact of it to be second-order, so does not give definability of Y in a specified language.)

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[1](Moreover there exists a definable Z, $\mu(Z) = 0$ with $\Gamma \subset X^{\cdot 2} \cup Z$.)

We also quote from [8]: $H = \widetilde{G}/\Gamma$ admits a natural locally compact topology. It is characterized by: (*) If $F \subseteq F' \subseteq H$ with F compact and F' open, then there exists a definable D with $\pi^{-1}(F) \subset D \subset \pi^{-1}(F')$. Conversely, the image of a definable subset of $\widetilde{G}$ is compact.

Using Theorem 2.13, we can (easily) improve our $(\widetilde{G}, \Gamma)$ as follows:

Theorem 1.5. *There exist an Ind-definable subgroup $\widetilde{G}'$ of $\widetilde{G}$, and a $\bigwedge$-definable subgroup $\Gamma' \supset \Gamma$, normal in $\widetilde{G}'$, such that $\widetilde{G}'/\Gamma'$ is a connected NSS group with no normal compact subgroups. (*) above holds; any definable D with $\Gamma' \subset D \subset \widetilde{G}'$ is commensurable with $X^{-1}X$.*

Our main goal is to show, in this situation, that $\widetilde{G}'/\Gamma'$ is *nilpotent*.

1.6. **Locally compact groups.** If G_0 is a locally compact group, let $G = G_0^*$ be an ultrapower; let $\widetilde{G} = \cup U^*$, as U runs over compact subsets of G; let $\Gamma = \cap U^*$, as U runs over neighborhoods of 1 in G. This is the setting of [7], [13]. The integral operator on definable functions, in this case is the ultrapower of the Haar integral.

We may remark that the locally compact group obtained above from an ultraproduct of approximate subgroups comes automatically equipped with a left-invariant real-valued integral on Borel functions (ultimately induced from counting); so we do not require Haar's theorem for the results of §3.

The one non-elementary fact we will use is the Peter-Weyl theorem for *compact* groups, or rather the following consequence: for any neighborhood U of the identity, there exists a normal subgroup $N \subset U$ such that G/N is NSS. It seems that the only known proof of this result relies on spectral theory for (self-adjoint, compact) operators on L^2.

2. Gleason-Yamabe structure theory (following Hirschfeld, Goldbring-Van den Dries)

The setting is *either* an ultrapower of a locally compact group, if the goal is the Gleason - Yamabe structure theory of locally compact groups; *or* an ultraproduct of finite approximate subgroups, when the goal is the structure of these. We follow [13] (§5, 5.1-5.8) with very minor variations; in particular we will work in the ultraproduct all along, while [13] work briefly in the ultrafactors (5.1-5.2). The main technical improvement in [13] over [7], is that their treatment essentially covers Yamabe's theorem; we will make this explicit. I had to write the notes to make sure that [13] goes through for the pseudofinite setting, but the reader may prefer to read the well-written notes by Goldbring and Van den Dries and make the necessary transpositions him or herself.

For $\alpha \in \mathbb{N}^*$, let

$$G(\alpha) = \{a \in \Gamma : a^i \in \Gamma \text{ for all } i \in o(\alpha)\}$$

$$\mathfrak{g}(\alpha) = \{a \in \Gamma : a^i \in \Gamma \text{ for all } i \in O(\alpha)\}$$

Note that $G(\alpha), \mathfrak{g}(\alpha)$ depend only on the archimedean class $\alpha/\backsimeq$.

Our goal is Theorem 2.11: $G(\alpha)$ and $\mathfrak{g}(\alpha)$ are normal subgroups of $\widetilde{G}$; $L_\alpha := G(\alpha)/\mathfrak{g}(\alpha)$ is commutative, and in fact Γ acts trivially on L_α by conjugation; and if $\beta = [\alpha/m]$, $m \in \mathbb{N}^*$, then $\underline{m} : x \mapsto x^m$ induces a well-defined, injective map $L_\alpha \to L_\beta$.

In case G is an ultrapower, we will see that $\underline{m}$ is also surjective; the L_α are all isomorphic, and we define the Lie algebra of G to be isomorphic to them. In case of an ultraproduct of finite approximate subgroups, the same picture holds for sufficiently small infinite α. However we will see that there exists a maximal $\lambda/\backsimeq$ with $G(\lambda) \neq 1$. Then $\mathfrak{g}(\lambda) = 1$, and $L(\lambda)$ is *discrete*. The map $\underline{m}$ will yield a discrete subgroup of the Lie algebra of $\widetilde{G}/\Gamma$. This discrete structure will lead to nilpotence in the next section.

We begin with the technical part.

2.1. **An invariant integral.** For definiteness we fix a definable U_1 (with $\Gamma \subset U_1 \subset \widetilde{G}$) and normalize the integral to give $U_1^{\cdot 6}$ measure 1:

$$\int f = \frac{1}{|U^{\cdot 6}|} \sum_{x \in U^{\cdot 6}} f(x)$$

2.2. **Urysohnn's lemma.** We need a definable Urysohn function; i.e. a definable $\tau : \widetilde{G} \to [0,1]^* \subset \mathbb{R}^*$, inducing a continuous $\bar{\tau} : \widetilde{G}/\Gamma \to \mathbb{R}$ with $\bar{\tau}(1) = 1$, and supported on a prescribed definable set containing Γ. The simplest way is to begin with $\bar{\tau}$ provided by Urysohn's lemma, then to produce τ. (Since $\bar{\tau}$ has compact support, τ will vanish outside a certain definable set.) We could construct τ explicitly, but prefer to obtain it from Urysohn via the basic lemma of nonstandard analysis, formulated below in the setting of ultraproducts M of finite structures. We take the definable relations on M to be all ultraproducts of relations on the factors; we could of course describe a countable sublanguage that also works.

Recall that a map from a first-order structure M into a compact space Y is *continous* if for any compact K and open U with $K \subset U \subset Y$, there exists a definable D with $f^{-1}(K) \subset D \subset f^{-1}(U)$.

Lemma 2.3. *Let $M = \lim_{i \to u} M_i$ be an ultraproduct of finite structures. Let $\bar{f} : M \to Y$ be a continuous map into a separable compact space. Then there exists a definable $f : M \to Y^*$ such that $st(f(x)) = \bar{f}(x)$.*

Proof. Fix a metric on Y. For any $\epsilon > 0$, find an ϵ-dense finite subset $F_\epsilon \subset Y$, and a definable map $f_\epsilon : M \to F_\epsilon$ with $d(f, f_\epsilon) < 2\epsilon$. So $f_\epsilon = \lim_{i \to u} f_{\epsilon,i}$, with $f_{\epsilon,i} : M_i \to F_\epsilon$. Write $F_i(n,x) = f_{n^{-1},i}(x)$, and let $F = \lim_{i \to u} F_i$. Then $d(F(n,x), F(n',x)) < 2(n^{-1} + (n')^{-1})$. For standard n we have $F(n,x) = f_{n^{-1}}(x)$. So $\mathrm{st}F(b,x)$ does not depend on $b \in \mathbb{N}^* \smallsetminus \mathbb{N}$, and is infinitely close to $\bar{f}$. Fix some $b \in \mathbb{N}^* \smallsetminus \mathbb{N}$ and let $f(x) = F(b,x)$. □

We obtain, for any definable $U_1 \supset \Gamma$, a definable $\tau : U_1^{\cdot 2} \to [0,1]^*$, with $st \circ \tau = \bar{\tau}$. We have $st\tau(1) = 1$; replacing $\tau(x)$ by $\max(1, \tau(x)/\tau(1))$, we may assume $\tau(1) = 1$. We can extend τ to $\widetilde{G}$ by zero outside $U_1^{\cdot 2}$; we still have $st \circ \tau = \bar{\tau}$. Note in particular that $\tau(x) > 1/2$ on some definable U_2 containing Γ; so

(1) $$\mathrm{st}(\int \tau) > 0$$

Note that for $b \in \Gamma$, $st(\tau(bx) - \tau(x)) = (\bar{\tau}(bx) - \bar{\tau}(x)) = 0$; and similarly on the right for $\tau(xb)$. Thus

(2) $$b \in \Gamma \Rightarrow |\tau(xb) - \tau(x)| \in o(1)$$

2.4. **Gleason-Yamabe lemmas.** For a definable function f into $\mathbb{R}^*$, $||f||$ denotes the supremum of $|f|$.

We will consider definable functions $\phi : \widetilde{G} \to \mathbb{R}^*$ with definable support. For such a ϕ and for $a \in \widetilde{G}$, let $D_a\phi(x) = \phi(a^{-1}x) - \phi(x)$. Also let $|\phi| = \sup\{|\phi(x)| : x \in \widetilde{G}$. Since the support is definable, the supremum is attained in the pseudo-finite case.

We write $x \le o(1)$ to mean: $|x| < 1/n, n = 1, 2, \ldots$.

Fix a definable set $Q \subseteq \Gamma$. The set Q will typically be a finite set $\{1, a, b, a^{-1}, b^{-1}\}$. Only for Yamabe's theorem 2.13 will we need an infinite definable set.

Lemma 2.5. *Let $Q \subset \Gamma$ be definable, symmetric, $e = s_U(Q)$. There exists a definable $\phi : G^* \to \mathbb{R}^*$ such that for $a \in Q, b \in \Gamma$ we have:*

(i) $\phi = 0$ *on* $G^* \setminus U^{\cdot 3}; 0 \leq \phi \leq 1$
(ii) $\phi(1) > 0$, *moreover* $\phi(1)$ *is not infinitesimal.*
(iii) $|D_a \phi| \leq 1/e$
(iv) $|D_b \phi| \leq o(1)$
(v) $|D_b D_a \phi| \leq o(1)1/e$

Proof. Let $\mathbf{u}(x) = \frac{1}{e} d_Q(x, 1)$, i.e. for $k \leq e$, $\mathbf{u}(x) = \frac{k}{e}$ if $x \in D^k \setminus D^{k-1}$; and $\mathbf{u}(x) = 1$ if $x \notin D^e$ (in particular if $x \notin U$.) Clearly:

$$0 \leq \mathbf{u}(x) \leq 1, \mathbf{u}(1) = 0$$

$\mathbf{u} = 1$ outside U_1, and $|D_a u| \leq 1/e$ for $a \in Q$. Let $u = 1 - \mathbf{u}$. We have:

$$u = 0 \text{ outside } U_1,\ 0 \leq u \leq 1, u(1) = 1 \tag{3}$$

$$|u(ya) - u(y)|, |u(ay) - u(y)| \leq 1/e \text{ for } a \in Q \tag{4}$$

Let

$$\theta(x) = \sup_{y \in U_1} u(y)\tau(y^{-1}x) = sup_{y \in \widetilde{G}} u(y)\tau(y^{-1}x)$$

From the definition of θ and (4) we have:

$$\theta(a^{-1}x) = sup_{ya \in \widetilde{G}} u(ya)\tau((ya)a^{-1}x) = \sup_{y \in \widetilde{G}} u(ya)\tau(yx) \leq 1/e + \sup_{y \in \widetilde{G}} u(y)\tau(yx)$$

As $u(y) = u(y^{-1})$, $\sup_{y \in \widetilde{G}} u(y)\tau(yx) = \sup_{y \in \widetilde{G}} u(y^{-1})\tau(yx) = \sup_{y \in \widetilde{G}} u(y)\tau(y^{-1}x) = \theta(x)$. Thus

$$|\theta(a^{-1}x) - \theta(x)| \leq 1/e \tag{5}$$

Let ϕ be the convolution of θ with its symmetric dual $\widehat{\theta}$ (i.e. $\widehat{\theta}(x) = \theta(x^{-1})$.)

$$\phi(x) = \theta * \widehat{\theta} = \int_y \theta(xy)\theta(y)$$

(i) $\phi(x) = 0$ unless x is a product of an element of the support of u, with an element of $supp(\tau)$. By (3), it follows that $x \in U_1^{\cdot 3}$. Since $0 \leq u, \tau \leq 1$, and the integral is normalized, it is clear that $0 \leq \phi \leq 1$.
(ii) We have $\theta \geq \tau$, using (3). So $\phi(0) \geq \int \tau(x)^2 > 0$, by (1).
(iii) $|D_a \phi| \leq |D_a \theta|$ from the definition, and using $|\theta| \leq 1$. By (5) we have $|D_a \theta| \leq 1/e$.
(iv) Again it suffices to show $|D_b \theta| \leq o(1)$. Now $|\theta(xb') - \theta(x)| \leq |\tau(xb') - \tau(x)| \in o(1)$ by (2). As $\tau(bx) = \tau(xb')$ with $b' = x^{-1}bx \in \Gamma$, the statement follows.
(v) (cf. [13], Lemma 5.1). We have $D_a(\theta * \widehat{\theta}) = D_a \theta * \widehat{\theta}$, by an easy computation:

$$D_a \phi = \phi(a^{-1}x) - \phi(x) =$$
$$= \int \theta(a^{-1}xy)\theta(y) - \int \theta(xy)\theta(y)dy$$
$$= \int [\theta(a^{-1}xy) - \theta(xy)]\theta(y) = \alpha * \widehat{\theta}(y)$$

where $\alpha = D_a \theta$. By (5) we have $|\alpha| \leq 1/e$.

Now $D_b D_a \phi = D_b(\alpha * \widehat{\theta})$, $\alpha * \widehat{\theta}(x) = \int \alpha(xy)\theta(y)dy$, so

$$D_b D_a \phi(x) = \int \alpha(b^{-1}xy)\theta(y)dy - \int \alpha(xy)\theta(y)dy$$

$$= \int \alpha(xu)\theta(x^{-1}bxu)du - \int \alpha(xu)\theta(u)du$$

$$= \int \alpha(xu)[\theta(x^{-1}bxu) - \theta(u)]du$$

$$= \int \alpha(xu)D_{x^{-1}bx}\theta(u)du$$

Now the set $b^{U^{\cdot 3}}$ of $U^{\cdot 3}$-conjugates of b is a definable set, contained in Γ. Since $|D_c\theta| \leq o(1)$ by (iii), this for each $c \in b^{U^{\cdot 3}}$, by compactness there exists $\epsilon_0 \leq o(1)$ such that $|D_c\theta| \leq \epsilon_0$ for each $c \in b^{U^{\cdot 3}}$. We have already seen that $|\alpha| \leq 1/e$, so

$$|D_b D_a \phi| \leq |\int (1/e)\epsilon_0 = \epsilon_0/e \leq o(1/e)$$

□

Fix ϕ for a moment. We will use the map $a \mapsto D_a\phi$, viewing it in a certain sense as an approximate homomorphism into the additive group of definable functions $G \to \mathbb{R}^*$. We have:

(6) $\quad D_{ab}\phi = D_a\phi + D_b\phi + D_a D_b\phi$

Define:

$$|a| = ||D_a(\phi)||$$

Note:

(7) $\quad |ab| \leq |a| + |b|, |1| = 0$

Lemma 2.6. *Let $m \in \mathbb{N}^*$, and let $a, b : \{1, \ldots, m\} \to G, i \mapsto a_i$ be definable maps, with $a_i \in Q$, $b_i = b_{i-1}a_i$, $b_0 = 1$. Assume each $b_i \in \Gamma$. Then*

$$||D_{b_m}\phi - \Sigma_{i \leq m} D_{a_i}\phi|| \in (m/e)o(1)$$

In particular, if $a \in Q \cap \mathfrak{g}(m)$ then

$$||D_{a^m}\phi - mD_a\phi|| \in (m/e)o(1)$$

So $(1/m)|a^m| - |a| \in (1/e)o(1)$.

Proof. We have, using (6) and induction on n,

$$D_{b_n}\phi - \Sigma_{i \leq n} D_{a_i}\phi = \Sigma_{i<n} Db_{i+1}Da_i\phi$$

By Lemma 2.5 (v), $||Db_{i+1}Da_i\phi|| \in 1/eo(1)$. Let $f = \max_{i<m} ||Db_{i+1}Da_i\phi||$; then $f \in 1/eo(1)$. So $||D_{b_m}\phi - \Sigma_{i \leq m} D_{a_i}\phi|| \leq mf \in (m/e)o(1)$. □

The above lemma provides the critical connection between the norm $||D_a\phi||$, with its natural submultiplicative property, and the exit norm $s_U(g)$. To use it well, we need $s_U(g)$ to be independent of U, in order of magnitude. In the NSS case, this holds automatically. In general, we can still arrange the weaker statement needed in practice, with a call to Peter-Weyl to handle the compact case.

Let $Q^{o(e)} = \cup_{\gamma \in o(e)} Q^\gamma$, $Q^{O(e)} = \cup_{\gamma \in O(e)} Q^\gamma$. Call (N, O) a *Yamabe pair* in a locally compact group L if N is a compact normal subgroup of L, and O is an open subset containing N, such that any subgroup of O is contained in N.

Lemma 2.7. *Let Q be a definable subset of Γ, and let (N, O) be a Yamabe pair of some closed subgroup $\mathbf{H} \leq \widetilde{G}/\Gamma$. Let $U \subset U'$ be two definable subsets of $\widetilde{G}$ with $\pi^{-1}N \subset U \subset U'$, while $U' \cap \pi^{-1}\mathbf{H} \subset \pi^{-1}O$. If $e = s_U(Q), e' = s_{U'}(Q)$ and $\pi Q^{e'} \subset \mathbf{H}$, then $s_U(Q) \Bumpeq s_{U'}(Q)$. (I.e. if one of the two quantities are defined, then both are, and are commensurable.)*

Proof. Suppose not; then $e = s_U(Q) << e' = s_{U'}(Q)$ (and in particular $e \in \mathbb{R}^*$.) By definition there exist $a \in Q$ and $b \in Q^e \subseteq U$ with $ab \notin U$. It follows that $\pi(ab) \notin N$ (since $\pi^{-1}N \subset U$). So $\pi(b) = \pi(ab) \notin N$. Let $H = Q^{o(e')}$. Then $\pi(H)$ is a subgroup of $\mathbf{H}$, containing an element $\pi(b)$ not in N; a contradiction. $\square$

We could quote Theorem 2.13 for the existence of Yamabe pairs; but since we wish to prove this theorem, we arrange to quote it only in the compact case.

Proposition 2.8. *Let Q be a definable subset of Γ, and let V be a definable set with $\Gamma \subset V \subset \widetilde{G}$. Then there exists a definable U, $\Gamma \subset U \subset V$, such that $s_U(Q) \Bumpeq s_{U^{\cdot 4}}(Q)$.*

Proof. If $s_V(Q) = \infty$, or $s_V(Q) \Bumpeq s_{V^{\cdot 4}}(Q)$, we can take $V = U$. Otherwise, $s_V(Q) << e'' = s_{V^{\cdot 4}}(Q)$. Let $H = Q^{o(e'')}$, and let $\mathbf{H} = \pi(H)$. Since $H \subset V^{\cdot 4}$, $\mathbf{H}$ is compact. By the *compact case* of Theorem 2.13, there exists a Yamabe pair for $\mathbf{H}$. The statement now easily follows from Lemma 2.7. $\square$

Fix a definable Q (it may as well be $G(\alpha)$) and assume U has been chosen so that $e := s_U(Q) \Bumpeq s_{U^{\cdot 4}}(Q)$. We obtain:

Corollary 2.9. *If $Q \subset \mathfrak{g}(\alpha)$, then $s_V(Q) >> \alpha$, i.e. $Q^{O(\alpha)} \subset \Gamma$.*
If $Q \subset G(\alpha)$, it is not the case that $\alpha << s_V(Q)$

Proof. Choose U as in Proposition 2.8. So $e := s_U(Q) \Bumpeq s_{U^{\cdot 4}}(Q)$. Construct ϕ as in Lemma 2.5, for Q, U, e. If $e >> \alpha$ we are done (since $e \leq s_V(Q)$.) Otherwise, by Lemma 2.6, for $a \in Q$,

$$||\frac{1}{e} D_{a^e}\phi - D_a\phi|| \leq (1/e)o(1)$$

But also, as $a^e \in \Gamma$, $||D_{a^e}\phi|| \leq o(1)$. So $||D_a\phi|| \leq 1/eo(1)$. It follows inductively from (7) that $||D_b\phi|| \leq n/eo(1)$ for $b \in Q^n$; in particular if $n \leq O(e)$, we have $||D_b\phi|| \leq o(1)$. As ϕ is supported on $U^{\cdot 4}$ and $st\phi(1) > 0$, it follows that $st\phi(b) > 0$, so $b \in U^{\cdot 4}$. Thus $Q^{O(e)} \subset U^{\cdot 4}$, so $s_{U^{\cdot 4}}(b) >> e$; contradiction.

Since this holds for any $V \supset \Gamma$, we have $Q^{O(\alpha)} \subset \Gamma$.

If $Q \subset G(\alpha)$, then $Q \subset \mathfrak{g}(\beta)$ for $\beta << \alpha$, so $Q^{O(\beta)} \subset \Gamma$ for $\beta << \alpha$, thus $s_Q(V)/\Bumpeq$ is no smaller than $\alpha/\Bumpeq$.

$\square$

As a special case, $a, b \in \mathfrak{g}(\alpha)$, then $(ab)^{\alpha/2} \in \Gamma$, so $ab \in \mathfrak{g}(\alpha/2) = \mathfrak{g}(\alpha)$. Thus $\mathfrak{g}(\alpha)$ is a subgroup.

Corollary 2.10. *$G(\alpha)$ and $\mathfrak{g}(\alpha)$ are normal subgroups of $\widetilde{G}$.*

Proof. We have just seen that $\mathfrak{g}(\alpha)$ is a subgroup; it is obviously normal, since Γ and the power map are $\widetilde{G}$- conjugation invariant. As $G(\alpha) = \cap_{\beta<<\alpha}\mathfrak{g}(\beta)$, it is also a normal subgroup of $\widetilde{G}$. $\square$

Now consider $\alpha \in \mathbb{N}^*$, $\beta = [\alpha/m]$, $m \in \mathbb{N}^*, m << \alpha$.

Theorem 2.11. *Let $\alpha \in \mathbb{N}^*$. Then:*

1. *Let $L_\alpha = G(\alpha)/\mathfrak{g}(\alpha)$. If $\beta = [\alpha/m]$, $m \in \mathbb{N}^*$, then $x \mapsto x^m$ induces a well-defined map $\underline{m} : L_\alpha \to L_\beta$, with $\underline{m}^{-1}(1) = \{1\}$.*
2. *$\underline{m}$ is a group homomorphism.*
3. *$\underline{m}$ is injective.*
4. *Γ acts trivially on L_α by conjugation. In particular, L_α is commutative. If $a \in G(\alpha)$ and $b \in \Gamma$, then $[a, b] \in \mathfrak{g}(\alpha)$.*

Proof. (1) If $x \in G(\alpha)$, it is clear that $x^m \in G(\beta)$; and $x^m \in \mathfrak{g}(\beta)$ iff $x^{m\beta} \in \Gamma$ iff $x^\alpha \in \Gamma$. We have to show that if $c \in \mathfrak{g}(\alpha)$ and $a \in G(\alpha)$, then $(ca)^m a^{-m} \in \mathfrak{g}(\beta)$. Now $(ca)^m a^{-m} = c_1 c_2 \cdots c_m$, where $c_i = a^i c a^{-i} \in \mathfrak{g}(\alpha)$. By Corollary 2.9, $(c_1 c_2 \cdots c_m)^\beta \in \Gamma$. So $c_1 c_2 \cdots c_m \in \mathfrak{g}(\beta)$, as required.

(2) Let $a, b \in G(\alpha)$. We have to show that $d := (ab)^m b^{-m} a^{-m} \in \mathfrak{g}(\beta)$, i.e. that $d^\beta \in \Gamma$; i.e. for any definable $V \supset \Gamma$, we have to show that $d^\beta \in V$.

Let $Q = \{1, a, b, a^{-1}, b^{-1}\}$. Let U be as in Proposition 2.8. Again, let $e := s_U(Q) \Bumpeq s_{U^{\cdot 4}}(Q)$, and construct ϕ as in Lemma 2.5, for Q, U, e. By Lemma 2.6, $D_d\phi \in o(4m/e) = o(m/e)$ (using $D_a\phi + D_{-a}\phi \in o(2/e)$, again by Lemma 2.6; so the additive expression cancels to 0 up to $o(8m/e)$.) We have $e \geq\Bumpeq \alpha \Bumpeq m\beta$. By (7), $|d^\beta| \in o(1)$. As in Lemma 2.9 it follows that $d^\beta \in U^{\cdot 4} \subset V$.

(3) We saw in (1) that the kernel is (1).

(4) Let $g \in \Gamma$. Then $g \in \mathfrak{g}(\beta)$ for some $\beta \in \mathbb{N}^* \setminus \mathbb{N}$, $\beta < \alpha$. Let $ad_g(x) = g^{-1}xg$. Then ad_g acts trivially on L_β. It respects the map $x \mapsto x^m$, and hence the injective induced map $\underline{m} : L_\alpha \to L_\beta$; thus ad_g must act trivially on L_α.

□

L_α can be viewed as a tangent space for $\widetilde{G}$, whose addition approximates the multiplication of $\widetilde{G}$ at scale α. In the case of an ultrapower of an NSS group, all maps $\underline{m}$ are isomorphisms, and we have essentially constructed the Lie algebra. At this point, Gleason's theorem that the group admits a Lie structure becomes readily accessible; for this, we refer the reader to [13]. As Yamabe's theorem is essential for us, we pause to prove it.

2.12. **Yamabe's theorem.** Let G be a locally compact group, G^* an ultrapower, $\widetilde{G}, \Gamma$ as above.

The proof of Theorem 2.13 relies on the *compact case* of the same statement, which can be deduced from Peter-Weyl easily; see [13] 4.2.

Theorem 2.13 (Yamabe). *If G is locally compact, for any neighborhood U of 1 there exists an open subgroup $H \leq G$ and a normal compact $K \leq H$, such that H/K is connected and NSS, and U contains the pullback of a neighborhood of 1 in H/K.*

Proof. The compact case follows from Peter-Weyl. We assume this, and prove the locally compact case.

Find an open neighbhorhood U' of 1 such that $U'U' \subset U$. If we arrange that $N \subset U'$, then $NU' \subset U$, and NU'/N is a neighborhood of 1 in H/N. Thus, replacing U by U', it suffices to find $N \subset U'$, H/N connected Lie.

Claim. For any neighborhood V of 1 there exists a neighborhood U of 1 such that the group generated by all subgroups contained in U is contained in C.

To see this, let U_n be a compact neighborhood of 1 with $U_n^{\cdot n} \subset V$. We have to show - for some $n \in \mathbb{N}$ - that if $c_1, \ldots, c_m$ are such that $c_i^{\mathbb{Z}} \subset U_n$, then $c_1 \cdots c_m \in V$. Otherwise, we will have $m \in \mathbb{N}^*$ and an internal sequence $c_1, \ldots, c_m$, with $c_i^{\mathbb{Z}^*} \subset \Gamma$, but $c_1 \cdots c_m \notin V$; contradicting Lemma 2.9.

Applying the claim instead to some V' with $cl(V') \subset V$, and with $cl(V')$ compact, we can conclude that there exists a compact subgroup C with $C \subset V$, and a neighborhood U of C, such that any group contained in U is contained in C.

Now we assume known the theorem for compact groups; applying it to C, we find an open subgroup $C' \leq C$ and a normal compact $K \leq C'$, such that C'/K is NSS, and $U \cap C$ contains the pullback W of a neighborhood of 1 in C'/K. This neighborhood of 1 may be taken to contain no nontrivial subgroups. So W contains no subgroups bigger than K. As $W \subset C'$, W normalizes K, so every subgroup contained in W is contained in K.

From this it follows that the normalizer H of K is open, since $H = \{h : hK \subset W\}$. And H/K clearly has NSS (the image of W contains no proper subgroups.)

By Lemma 2.17 below, H/K has a connected open subgroup; it has the form H'/K for an appropriate open $H' \leq H$, $K \leq H$. So replacing H by H', we can can take H/K to be connected. □

Remark 2.14. If G is connected locally compact group, then in Theorem 2.13 we must have $H = G$, so N is a normal co-Lie subgroup of G. Let P be the projective limit of G/N, over all closed normal subgroups N such that G/N is Lie. Then Theorem 2.13 implies that the natural map $h : G \to P$ is an injective, continuous group homomorphism. In fact it is an isomorphism of topological groups; [9], Theorem 18 contains a short and elementary proof, and refers to an extension by Glushkov, which also follows readily.

We conclude this section with some facts concerning NSS groups that we will need; they follow easily from the basic Lie theory or from the Pontrjagin structure theory of Abelian NSS groups, but we give direct proofs.

2.15. **The Lie algebra of an NSS group.** Let G_0 be an NSS locally compact group. Let G be an ultrapower, and $\widetilde{G}, \Gamma$ as above. For any $\alpha \in \mathbb{N}^* \setminus \mathbb{N}$, the Abelian group $L_\alpha = G(\alpha)/\mathfrak{g}(\alpha)$ is in natural bijection with the set $L(G)$ one-parameter subgroups of G_0; map c to the one-parameter group $t \mapsto st(c^{t\alpha})$. This shows that the natural maps $L_\alpha \to L_\beta$ are isomorphisms; and allows defining addition on $L(G)$, so as to be isomorphic to any L_α by the above correspondence. It also shows that $L(G)$ is an $\mathbb{R}$-vector space. Now the map $x \mapsto x^\alpha$ induces an embedding $exp : L_\alpha \to \widetilde{G}/\Gamma \cong G_0$; by definition, $(exp(x), exp(x/2), exp(x/4), \ldots)$ is injective on L_α; it follows that $G(\alpha)/\mathfrak{g}(\alpha)$ is bounded (in the model theoretic sense), hence admits a locally compact structure. It is known that a locally compact $\mathbb{R}$-space L that it must have finite dimension (indeed if U is a compact neighborhood of 1 and $U^{\cdot 2} \subset FU$ with F finite, then the $\mathbb{R}$-span $\mathbb{R}F$ must equal L; passing to the quotient we see that $L/\mathbb{R}F$ is *compact*, hence must be trivial.)

In case G is Abelian, exp is a group homomorphism; if G is connected, it is surjective.

2.16. **The dimension of an NSS group** G**.** We define $\dim(G) = \dim(L(G))$. We will only need to know that $\dim(G) > \dim(G/N)$ if $N \cong \mathbb{R}$ is a nontrivial connected central subgroup. This follows easily from the definition, and the fact that $L(N) \neq (0)$.

Lemma 2.17. *An NSS group has a connected open subgroup.*

Proof. Let H be NSS, and let H^0 be the connected component of the identity in H. Then $G_0 = H/H^0$ admits no one parameter subgroups (nontrivial homomorphisms from $\mathbb{R}$.) Moving to the ultraproduct $G, \widetilde{G}, \Gamma$ as above, it follows that $\Gamma = 1$, since any non-identity $a \in \Gamma$ gives rise to a 1-parameter subgroup $st(a^{\lambda n})$, for appropriate λ. Thus $G_0 \cong \widetilde{G}/\Gamma$ is discrete. So H^0 is open. □

Lemma 2.18. *Let H be an Abelian NSS group with no nontrivial compact connected subgroups. Then $H \cong \mathbb{R}^n \oplus B$ with B discrete.*

Proof. First assume H is connected. Let $G, \widetilde{G}, \Gamma$ be an ultrapower as above, $H \cong \widetilde{G}/\Gamma$. We have the groups $L = L_\alpha$, for large $\alpha \in \mathbb{N}^*$. They are finite-dimensional $\mathbb{R}$-spaces, and we can use induction on their dimension. In particular, if $\dim(L) = m$, it is clear that H contains no copy of $\mathbb{R}^{m+1}$. Let k be maximal such that $\mathbb{R}^k \cong H_1 \leq H$. If $H = H_1$ we are done. Otherwise, by connectedness, H_1 cannot contain a neighborhood of 1; so arbitrarily small neighborhoods have an element a_i with a_i not in H_1; fix U containing no nontrivial subgroups; we have $a_i^{n_i} \in U^{\cdot 2} \smallsetminus U$ for some n_i; and $a_i^{n_i} \notin H_1$ since H_1 is divisible and H is torsion-free. Hence there exists $a \in \Gamma$ and an infinite $\alpha \in \mathbb{N}^*$ with $a^\alpha \notin H_1^*$ and $a^\alpha \in U^{\cdot 2} \smallsetminus U$. Now the corresponding one-parameter subgroup $st(s^{O(\alpha})$ is a copy of $\mathbb{R}$ in H, not contained in H_1, contradicting the maximality of H_1. (Here we use that H has no nontrivial compact connected subgroups. The image of the homomotphism $t \mapsto st(s^{t\alpha})$, intersected with any compact set, must be compact, so the image is a closed subgroup homeomorphic to $\mathbb{R}$.)

Now in general, the result holds for the connected component H^0, so $H^0 \cong \mathbb{R}^n$. By Lemma 2.17, H/H^0 is discrete. Let B be a maximal subgroup of H disjoint from H^0. We claim that $H = H^0 \oplus B$. At any rate B is relatively divisible in H and as H^0 is divisible, $H^0 \oplus B$ is relatively divisible too. Thus $H/(H^0 \oplus B)$ is torsion-free. So if $c \in H \smallsetminus (H^0 \oplus B)$, then $\mathbb{Z}c \cap (H^0 \oplus B) = (0)$; but this contradicts the maximality of B. So $H \cong \mathbb{R}^n \oplus B$. □

Lemma 2.19. *Let L be a connected NSS group. Assume L has no nontrivial compact Abelian normal subgroups, and let Z be a central subgroup of L, with $Z \cong \mathbb{R}$. Then L/Z has no nontrivial compact Abelian normal subgroups.*

Proof. Suppose L/Z has a compact normal Abelian subgroup L'/Z. If L' is not Abelian, let $c \in L'$ be non-central; then $[L', c]$ is a nontrivial compact subgroup of Z, a contradiction. Thus L' is Abelian.

If L' has a nontrivial compact connected subgroup C, then clearly $C+C_1+\cdots+C_n$ is compact for any L-conjugates $C_1, \ldots, C_n$ of C. Each C_i will contain a one-parameter subgroup, and as above the dimension of an L_α will bound n. So the sum of all L-conjugates of C is a normal compact subgroup C' of L'. But clearly $C' \cap Z = (0)$, so C' is a subgroup of L, a contradiction.

By Lemma 2.18, $L' \cong \mathbb{R}^n + D$ with D discrete; so $\mathbb{R}^{n-1} + D$ is compact; hence $n = 1$, and D is finite. By divisibility of $\mathbb{R}$, D is a subgroup of L, so $D = (0)$. Hence $L = L'$. □

It is also easy to see that a connected NSS group has a maximal compact normal subgroup; see [8], 4.1, paragraph 2.

3. Essential nilpotence of finite approximate subgroups, and Freiman's theorem.

We aim towards the main results[2] of [3].

Let X_i be a k-approximate subgroup of a group G_i, and let $(X, G, \mathbb{Z}^*)$ be the ultraproduct. We write $\mathbb{Z}^*$ to indicate the power structure.

Let $\widetilde{G}_0$ be the $\bigvee$-definable subgroup of G generated by X, and let $\Gamma \subset X^4$ be a $\bigwedge$-definable group, such that X is commensurable with any definable subset of $\widetilde{G}_0$ containing Γ.

Theorem 3.1. *There exists a $\bigvee$–definable subgroup $\widetilde{G}$ of $\widetilde{G}_0$, containing Γ, and a definable normal subgroup K of $\widetilde{G}$, $K^{\mathbb{Z}^*} \subset K \leq \Gamma$, such that $\widetilde{G}/K$ is nilpotent. In fact $\widetilde{G}/K$ satisfies the conclusion of Theorem 3.2.*

[2] We will not consider local groups, and will omit one of the estimates on order of nilpotence, and the applications.

Proof. By Theorem 1.5 there exist an Ind-definable subgroup $\widetilde{G}$ of $\widetilde{G}_0$, and a $\bigwedge$-definable subgroup $\Gamma \supset \Gamma_0$, normal in $\widetilde{G}$, such that $\widetilde{G}/\Gamma$ is a connected NSS group with no normal compact subgroups. Since $\widetilde{G}/\Gamma$ is NSS, there exists a definable U_1 with $\Gamma \subset U_1 \subset \widetilde{G}$, and such that the image of U_1 modulo Γ does not contain any nontrivial subgroups.

Let $K = \{g : g^{\mathbb{Z}^*} \subset U_1\}$. This is a definable set. If $g \in K$, the image modulo Γ of $g^{\mathbb{Z}^*}$ is a subgroup, hence trivial; so $g \in \Gamma$ and indeed $g^{\mathbb{Z}^*} \subset \Gamma$; so $K = \{g : g^{\mathbb{Z}^*} \subset \Gamma\} = \cap_\alpha \mathfrak{g}(\alpha)$. By Lemma 2.10, K is a subgroup of Γ; it is clearly normal in $\widetilde{G}$, , and closed under $\mathbb{Z}^*$-powers. The normalizer H of K is a definable subgroup of G containing $\widetilde{G}$. It is also clear that any definable subgroup of Γ is contained in K. If we factor out K from H,$\widetilde{G}$ and Γ, we have the same situation but now $K = (1)$. Thus the conditions of Theorem 3.2 holds. In particular $\widetilde{G}/K$ is nilpotent. □

Theorem 3.2. *Assume $\widetilde{G}/\Gamma$ is connected, NSS, with no compact normal Abelian subgroups. Assume also that Γ contains no nontrivial definable subgroups.*

Then $\widetilde{G}$ is contained in a definable nilpotent group, of class $\leq \dim(\widetilde{G}/\Gamma)$. There exist elements $a_1, \ldots, a_n$ and $\lambda_1 \geq \ldots \geq \lambda_n \in \mathbb{Z}^*$, such that $A_i := a_1^{O(\lambda_1)} \cdots a_n^{O(\lambda_i)}$ is a normal subgroup of $\widetilde{G}$, A_{i+1}/A_i is central in $\widetilde{G}/A_i$, and we have:

$$\widetilde{G} = a_1^{O(\lambda_1)} \cdots a_i^{O(\lambda_n)}$$

$$\Gamma = a_1^{o(\lambda_1)} \cdots a_n^{o(\lambda_n)}$$

For any definable X containing Γ and contained in $\widetilde{G}$, for some $m \in \mathbb{N}$,

$$a_1^{\leq \lambda_1/m} \cdots a_n^{\leq \lambda_n/m} \subset X \subset a_1^{\leq m\lambda_1} \cdots a_n^{\leq m\lambda_n}$$

and $X \approx a_1^{\leq \lambda_1} \cdots a_n^{\leq \lambda_n}$.

Proof. Let $\pi : \widetilde{G} \to L := \widetilde{G}/\Gamma$ be the natural map.

Let U be a definable set containing Γ, whose image in $\widetilde{G}/\Gamma$ contains no nontrivial subgroups. Then definable subgroup of U is contained in Γ; so by assumption, any definable subgroup contained in U is trivial.

Any definable U' containing Γ generates $\widetilde{G}$: for $\pi(U')$ contains an open set, and so generates the connected group $\widetilde{G}/\Gamma$.

Recall that for $1 \neq a \in \widetilde{G}$, $s_U(a)$ is the least $n \in \mathbb{N}^*$ such that $a^{n+1} \notin U$. This is well-defined since U contains no nontrivial definable subgroups, and by the definable well-ordering property of $\mathbb{N}^*$. Moreover, if $U' \subset U$, then $s_{U'}(a) \leq s_U(a)$, but $s_{U'}(a) \approx s_U(a)$ since otherwise there will be $x \in U \smallsetminus U'$ with $x^n \in U$ for all $n \in \mathbb{N}$, contradicting the NSS assumption on $\widetilde{G}/\Gamma$. Thus the $\approx$-class of $S_U(a)$ does not depend on the choice of U.

Choose $a_1, a_2, \ldots$ inductively. Let $\lambda_i = s_U(a_i)$, $C_i = a_i^{O(\lambda_i)}$, $A_i = C_1 \cdots C_i$. Assume inductively that $a_1, \ldots, a_{i-1}$ have been chosen, and that Claims 0-3 below hold below i. If $\widetilde{G} = A_{i-1}$, skip to the lines after Claim 3 for the end of the proof. Assume now that $\widetilde{G} \neq A_{i-1}$.

Choose an element d such that $d \in U \smallsetminus A_{i-1}$, and $\lambda := s_U(d)$ is as large as possible with this constraint. This is possible with another use of the definable well-ordering property, as $U \cap A_{i-1}$ is a definable (by Claim 3.)

Claim 0. $\lambda \in \mathbb{N}^* \smallsetminus \mathbb{N}$; $d \in \Gamma$.

Proof. To see that $\lambda \in \mathbb{N}^* \smallsetminus \mathbb{N}$, it suffices to show that $\lambda \geq m$ for any $m \in \mathbb{N}$. Let U_m be a definable set containing Γ with $U_m^{\cdot m} \subset U$. If $\lambda \leq m$ then as $s_U(x) \geq m$ for $x \in U_m$, it follows that $U_m \subset A_{i-1}$. But U_m generates $\widetilde{G}$, so $\widetilde{G} = A_{i-1}$, contradicting out assumption.

The fact that $d \in \Gamma$ now follows, since the image of $d^{\mathbb{Z}}$ in $\widetilde{G}/\Gamma$ is a subgroup of the image of U, and so must be trivial. $\square$

By induction, A_{i-1} is a normal subgroup of $\widetilde{G}$. As it is locally definable, the image πA_{i-1} is a closed subgroup of $L = \widetilde{G}/\Gamma$. Let $\mathbf{L} = (\widetilde{G}/\Gamma)/\pi A_{i-1}$, $\bar{\pi} : \widetilde{G} \to \mathbf{L}$ the natural map. By hypothesis, $\widetilde{G}/\Gamma$ has no normal compact Abelian subgroups; using Lemma 2.19, this hypothesis is maintained as we successively factor out copies of $\mathbb{R}$; so $\mathbf{L}$ has no normal compact Abelian subgroups. So the center $\mathbf{Z}$ of $\mathbf{L}$ is isomorphic to $\mathbb{R}^p \oplus D$ for some $p \in \mathbb{N}$ and some discrete D.

Let C_1 be a convex, symmetric, compact neighborhood of the origin in $\mathbf{Z}$, with $C_1 \cap D = (0)$; let C_2 be the interior of $1.1C_1$; let $\mathbf{O}_1$, $\mathbf{O}_2$ be a compact (respectively open) neighborhood of the origin in $\mathbf{L}$, with $\mathbf{O}_i \cap \mathbf{Z} = C_i$, and $\mathbf{O}_1 \subset \mathbf{O}_2$; let O_1, O_2 be a compact (resp. open) neighborhood of the origin in L, with $O_i/\pi A_{i-1} = \mathbf{O}_i$. [3] Finally, let O be a definable set with $\pi^{-1}O_1 \subset O \subset \pi^{-1}O_2$. Recall that $s_U(a) \Bumpeq s_O(a)$. Since we have only used $\lambda / \Bumpeq$ so far, we may redefine λ as the maximal $s_O(a)$. Let $s(a) = s_O(a)$.

Claim 1. If $a \in \widetilde{G}$, $s(a) \Bumpeq \lambda$, then a is central in $\widetilde{G}$ modulo A_{i-1}.

Proof. Let $c \in \Gamma$. We have $[c,a] \in \Gamma \subset U$, and $s([c,a]) >> s(a) = \lambda$ by Gleason, Lemma 2.11 (2). By maximality of λ, $[c,a] \in A_{i-1}$. By compactness, $[c,a] \in A_{i-1}$ for all c in some definable set U' containing Γ. As $\widetilde{G}/\Gamma$ is connected, U' generates $\widetilde{G}$, so every element of $\widetilde{G}$ centralizes a modulo A_{i-1}. $\square$

Let $a \in \Gamma$, $a \notin A_{i-1}$. Let $f \in A_{i-1} \cap U$ be such that $s(af)$ is maximal; and define $s''(a)$ to be this maximal number. Then $s''(a) = s(af) \geq s(af')$ for any $f' \in A_{i-1}$; indeed $s(af)$ is infinite, and if $s(af')$ is infinite then $af' \in \Gamma$ so $f' \in \Gamma$, so $f' \in U$. Now if $\mu \Bumpeq s(af)$ then $\pi((af)^\mu) \notin \pi A_{i-1}$: otherwise by Claim 2 for $i-1$ we could find f' with $(aff')^\mu \in \Gamma$, so $s(aff') >> \mu \Bumpeq s(af)$, contradicting the maximality of $s(af)$. In particular if $s(a) \Bumpeq \mu \Bumpeq \lambda$, then as $s(a) \leq s(af) \leq \lambda$ we have $s(af) \Bumpeq \lambda$, so $\pi((af)^\mu) \notin \pi A_{i-1}$; but $(af)^\mu = a^\mu f'$ for some $f' \in A_{i-1}$; so $\pi(a^\mu) \notin \pi A_{i-1}$.

(8) $\qquad \lambda \Bumpeq \mu \Bumpeq s(a) \Rightarrow \pi(a^\mu) \notin \pi A_{i-1}$

Choose $a_i \in U$, $a_i \notin A_i$, with $s(a_i) = \lambda$ maximal. Then by (8), $\pi(a_i^\lambda) \notin \pi A_{i-1}$.

By Claim 1, a_i is central in $\widetilde{G}/A_{i-1}$, and in particular $\bar{\pi}(A_i) \subset Z \cong \mathbb{R}^p$. As $\pi(a_i^\lambda) \notin \pi A_{i-1}$, $\bar{\pi}(A_i) \cong \mathbb{R}$. Identify $\bar{\pi}(A_i)$ with $\mathbb{R}$, in such a way that the (convex, symmetric) image of C_1 is $[-1,1]$, so the image of C_2 is $(-1.1, 1.1)$. Then using (8) it is clear[4] that

(9) $\qquad \pi(a^\lambda) \in A_i, \bar{\pi}(a^\lambda) \neq 0 \Rightarrow |s(a)/\lambda - 1/\bar{\pi}(a^\lambda)| \leq 1.1$

Claim 2. Let $b \in \Gamma \setminus A_i$, $\mu \Bumpeq s(b)$.

(1) If $s(b) \Bumpeq \lambda$ then $\pi(b^{s(b)}) \notin \pi A_i$.
(2) If $\mu << \lambda$ and $\pi(b^\mu) \in \pi A_i$ then there exists $d \in A_i$ with $(db)^\mu \in \Gamma$.

Proof. Assume $\pi(b^\mu) \in \pi A_i$. We have $\mu \leq \lambda$, and μ is infinite. Suppose first that $\mu << \lambda$. Say $b^\mu = a^k cd$, with $c \in \Gamma$ and $d \in A_{i-1}$, and $k \in O(\lambda)$. Write $k = e\mu + r$, $r < \mu$. As μ is infinite, $e << \lambda$. Then $e, r << \lambda$. Note that $[a,b] \in A_{i-1}$, and $s([a,b]) << \mu$. We have

[3] One can let O_1, O_2 be the images under the exponential map of concentric balls of radius $1, 1.1$ in the Lie algebra; but as we promised not to use the Lie algebra, it can also be done as an elementary excercise in topology.

[4] using also the fact that any real can be written as $st(\mu/\lambda)$ for some $\mu \in \mathbb{Z}^*$

$(a^{-e}b)^{\mu} = a^r cd \in \Gamma A_{i-1}$. By Claim 2 for $i-1$, there exists $e' \in A_{i-1}$ with $(d'a^{-e}b)^{\mu} \in \Gamma$. So $d'a^{-e}$ is as required

Next suppose $\mu \not\approx \lambda$, but there exist an integer $l \in \mathbb{Z}$ with $\pi((a^l b)^{\mu}) \in \pi A_{i-1}$ Then by induction for some $d \in A_{i-1}$, $(da^l b)^{\mu} \in \Gamma$; so $s(da^l b) >> \lambda$. Seeing that $da^l b \notin A_i$, this contradicts the choice of λ.

The remaining case is $\mu \not\approx \lambda$, but no such l exists, i.e. $\bar{\pi}(b)$ is not an exact multiple of $\bar{\pi}(a)$. Note that $\bar{\pi}(a) < 1.2\bar{\pi}(b)$. As $\bar{\pi}(A_i) \cong \mathbb{R}$, there exists $l \in \mathbb{N}$ with $\bar{\pi}(a^l b) \leq (1/2)\bar{\pi}(a)$. Using (9), this implies that $s(a^l b) > s(a)$, a contradiction.

□

Claim 3. For any definable set D, $D \cap A_i$ is a definable set.

Proof. Let $a = a_i$. Define $h : \mathbb{R} \to \mathbf{L}$ by $h(t) = \bar{\pi}(a^n)$, where n is any element of $\mathbb{Z}^*$ with $st(n/\lambda_i) = t$. By Claim 2, the image of h is nontrivial; by Claim (1) it lies in the center $Z \cong \mathbb{R}^p$ of $\mathbf{L}$. The image of h is thus a 1-dimensional subspace. So for any compact subset of $\mathbf{L}$, specifically for $\bar{\pi}(D)$, there exists $\nu \in \mathbb{N}$ such that if $t \geq \nu$ then $h(t) \notin K$. Thus if $n > s\nu$ then $a_i^n \notin D$. So $D \cap A_i = D \cap a_i^{\leq s\nu} A_{i-1}$. By induction, A_{i-1} is locally definable, so $Da_i^{\leq s\nu} \cap A_{i-1}$ is contained in a definable set E. So $D \cap A_i \subset a_i^{\leq s\nu} E$, and hence $D \cap A_i = D \cap (a_i^{\leq s\nu} E)$. □

By Claim 2, $\pi A_1 < \pi A_2 < \ldots <$ is a strictly increasing sequence of closed normal subgroups of $\widetilde{G}/\Gamma$, for as long as A_i is defined; we saw the successive quotients are $\cong \mathbb{R}$; hence the inductive process must terminate at some n with $n \leq \dim(\widetilde{G}/\Gamma)$; the only reason a_{n+1} cannot be chosen, is that $U \subseteq A_n$. Since U generates $\widetilde{G}$, we have $\widetilde{G} = A_n$.

By definition of λ_i, $a_i^{o(\lambda_i)} \subset \Gamma$. Thus $a_1^{o(\lambda_1)} \cdots a_n^{o(\lambda_n)} \subset \Gamma$. It is easy to see, by induction on i, that $a_1^{O(\lambda_1)} \cdots a_n^{O(\lambda_i)} / a_1^{o(\lambda_1)} \cdots a_n^{o(\lambda_i)}$ has no nontrivial compact subgroups; in particular this holds for $i = n$, so the image of Γ in $a_1^{O(\lambda_1)} \cdots a_n^{O(\lambda_n)} / a_1^{o(\lambda_1)} \cdots a_n^{o(\lambda_n)}$ must be trivial. Hence $\Gamma = a_1^{o(\lambda_1)} \cdots a_n^{o(\lambda_n)}$.

Let X be a definable set, $\Gamma \subset X \subset \widetilde{G}$. For any element $u \in X$, for some $m \in \mathbb{N}$ we have $u \in a_1^{\leq m\lambda_1} \cdots a_n^{\leq m\lambda_n}$. By compactness, for some $m \in \mathbb{N}$, for any $u \in X$, $u \in a_1^{\leq m\lambda_1} \cdots a_n^{\leq m\lambda_n}$. Thus $X \subset a_1^{\leq m\lambda_1} \cdots a_n^{\leq m\lambda_n}$. Similarly, $a_1^{\leq \lambda_1/m} \cdots a_n^{\leq \lambda_n/m} \subset X$ for appropriate m.

At this point it is clear that $\widetilde{G}$ is n-nilpotent. To obtain an enveloping definable group with the same property, let $G_1 = C_G(a_1)$, $G_2 = \{x \in G_1 : [x, a_2] \in a_1^{\mathbb{Z}^*}\}$, etc. Then inductively, $\widetilde{G} \leq G_n$; moreover $\widetilde{G}$ is contained in $Z_n(G_n)$, the n'th element of the upper central series of G_n. So $Z_n(G_n)$ is a definable nilpotent group of class $\leq \dim(\widetilde{G}/\Gamma)$, containing $\widetilde{G}$. □

To understand the import of Claim 3, one should think of the possibility of a 1-parameter group taking an irrational angle along a two-dimensional torus, and thus entering a given open neighborhood countably many times (so not a definable set.) The assumption on no compact normal Abelian subgroups is used to show this does not happen.

Let us say that elements $a_1, \ldots, a_n$ of a group H form a *central sequence* if (letting $A_0 = 1$ and A_i be the group generated by $a_1, \ldots, a_i$), for $i \geq$, A_i is normal in A_n, and A_{i+1}/A_i is central in A_n/A_i.

Corollary 3.3. *Fix k. For some b, m, the following holds. For any group G and finite $X \subset G$, if $1 \in X = X^{-1}$ and $|X^{\cdot 3}| \leq k|X|$, there exists a subgroup H of G, a definable normal subgroup K of H, and a subset Y of H/K of the form*

$$Y = a_1^{[-d_1, d_1]} \cdots a_m^{[-d_m, d_m]}$$

with $d_i \in \mathbb{Z}$ and a_i forming a central sequence, such that $K \subset X^4$, and $X^{\cdot 2}$ is b-commensurable with the pullback to H of Y.

The corollary follows from Theorem 3.1 by the standard compactness argument. One can add that the pullback of Y to H is contained in $X^{\cdot 4}$,

Remark 3.4. As Immanuel Halupczok pointed out, the fact that $\widetilde{G}$ and Γ are subgroups in Theorem 3.2 entails some additional information about $\lambda_1, \ldots, \lambda_n$. For instance, we have $[a_2, a_k] = a_1^{\mu_k}$ for some $\mu_k \in o(\lambda_1)$; and since since $[a_2^{\lambda_2}, a_k^{\lambda_k}] \in A_1$, we must have $\lambda_2 \lambda_k \mu_k \in O(\lambda_1)$. We refrain from listing these conditions as they follow in an elementary fashion from the statement, and are not needed in the proof; the interested reader can see the definition of a "nil-sequence" in [3].

4. Order of nilpotence

As noted in [8] 4.11, 4.12, there is a canonical *maximal* choice of K (contained in X^m for some bounded m) as well as non-canonical smaller choices; one can obtain $K \subseteq X^4$. On the other hand the canonical maximal K gives precise and optimal l bounds on the degree of nilpotence. §10 of [3] includes a study of both directions; we will only cover the former in these notes.

In Theorem 3.1, let $k_m = \mu(X_0^m)/\mu(X_0)$. In order to bound the degree of nilpotence directly in terms of these doubling coefficients, we note two lemmas (cf. [8], p. 220).

Lemma 4.1. *Let X_0 be a near-subgroup of (G, μ), Let $\widetilde{G}$ be a subgroup of G, $X = X_0^{-1} X_0 \cap \widetilde{G}$, $\mathbf{k}_5 = \mu(X^5)/\mu(X)$. Then $\mathbf{k}_5 \leq k_{11}$.*

Proof. For any coset C of $\widetilde{G}$ with $C \cap X_0 \neq \emptyset$ we have $\mu(C \cap X_0^{\cdot 11}) \geq \mu(X^{\cdot 5}) = \mu(X)\mathbf{k}_5$. Indeed if $c \in C \cap X_0$, then $cX^{\cdot 5} \subset C \cap X_0^{\cdot 11}$. Also $\mu(X) \geq \mu(C \cap X_0)$ since $c^{-1}(C \cap X_0) \subset X$. Thus $\mu(C \cap X_0^{\cdot 11}) \geq (C \cap X_0)\mathbf{k}_5$. Summing over all C we obtain $\mu(X_0^{\cdot 11}) \geq \mu(X_0)\mathbf{k}5$. □

Lemma 4.2. *Let X be a Haar measurable subset of a group $\mathbb{R}^n/\mathbb{Z}^m$, $X = X^{-1}$. Let $m_2 = |\{a \in L : a^2 = 1, a \in X^{-1}X\}$. Then $\mu(X^2) \geq 2^n/m_2 \mu(X)$.*

If X is a measurable subset of a torsion-free nilpotent Lie group L of dimension n, then similarly $\mu(X^2) \geq 2^n \mu(X)$

Proof. $\mu(X^2) \geq \mu(fX)$ where $f(x) = x^2$. Now f is at most m_2-to-one, and $|\det f'| = 2^n$. □

Corollary 4.3. *Let hypotheses be as in Theorem 3.1. Let $k_m = \mu(X_0^m)/\mu(X_0)$. Then there exists a $\bigvee$–definable subgroup $\widetilde{G}$ of $\widetilde{G}_0$, and a definable normal subgroup K of $\widetilde{G}$ ($K \subset X_0^m$ for some m), such that and X_0 is contained in finitely many cosets of $\widetilde{G}$, and $\widetilde{G}/K$ is nilpotent, of class* $[\log_2(k_{11})]$.

Proof. Let $\widetilde{G}, N$ be as in Theorem 3.1, and take $X = X_0^{-1} X_0 \cap \widetilde{G}$. Let $\mathbf{k}_5 = \mu(X^5)/\mu(X)$. By Lemma 4.1, $\mathbf{k}_5 \leq k_{11}$. Let Γ be an $\bigwedge$-definable normal subgroup of $\widetilde{G}_0$ containing Γ_0, such that $\widetilde{G}_0/\Gamma$ is connected with no compact normal subgroups, as in [8]. As in Theorem 3.1 we reduce to Proposition 3.2, with $\widetilde{G}/\Gamma$ nilpotent, connected without compact normal subgroups. We then run the proof of Proposition 3.2, using the compact case of Lemma 2.19. This shows that the nilpotence class of $\widetilde{G}$ is bounded by $\dim(\widetilde{G}/\Gamma)$. But $\dim(\widetilde{G}/\Gamma) \leq log_2(\mathbf{k}_5)$ by Lemma 4.2. □

Remark 4.4. *In Theorem 3.1, the class of nilpotence of $\widetilde{G}/\Gamma$ is at most* $1 + [\log_2(k_{11})]$ (without factoring out by any K.) Indeed to pass from Theorem 3.1 to Proposition 3.2 we need merely factor out a compact Abelian (so central in the connected component) normal subgroup of $\widetilde{G}/\Gamma$, which raises the degree of nilpotence by at most 1.

Remark 4.5. In Corollary 4.3, factoring out a compact in order to obtain a torsion-free nilpotent group obliges us to weaken the assertion that $K \subset X_0^4$; in this version, we can only assert $K \subset X_0^m$ for some m.

Corollary 4.6. *Fix k, and let $m = [10 \log_2 k + 1]$. For some b, l, the following holds. For any group G and finite $X \subset G$, if $1 \in X = X^{-1}$ and $|X^{\cdot 3}| \leq k|X|$, there exists a subgroup H of G, a definable normal subgroup K of H, such that H/K is $\leq m$-step nilpotent, $K \subset X^{\cdot l}$, and $|X/H| \leq b$. Moreover, X is b-commensurable with the pullback to H of a subset of H/K of the form $a_1^{[-d_1,d_1]} \cdots a_n^{[-d_n,d_n]}$, with $a_i \in H/K$ as in Theorem 3.2, and $n \leq m$. and $d_i \in \mathbb{Z}$.*

Again this follows by the standard compactness argument.

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